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An elliptical fracture criterion for mixed mode fracture I+II emanating from notches

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Abstract. Some mixed mode fracture criterion may be converted in to elliptical or ellipsoidal formula with the aid of mathematical translation. Hence, the crack initiation in mixed mode fracture I+II emanating from notches, has been studied using notched circular ring specimens. On the basis of Irwin (1957) theory, a new criteria in mixed mode fracture I+II, based fracture elliptic criterion and notch stress intensity factors has been developed.

Keywords: notch effect; angled crack problem; maximum tangential stress criterion; effective tangential stress; effective tangential distance; relative tangential stress gradient; weight function; notch stress intensity factors $K_{I\rho}$ and $K_{II\rho}$.

1. Introduction

In the special case of mode I loading, fracture toughness methodology is well established. How ever, fracture under combined modes loading is sometimes found in technical applications, and it is important to develop an understanding of cracking behaviour in this situation.

Different test conditions have been used for most of the establishing experiments made until now and numerous types of specimen have been proposed in recent years for investigating fracture toughness, many mixed mode fracture criterion, measuring fatigue threshold values and determining the crack growth laws applicable to multi-axial stress situations. Some of these specimens have

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been presented and critically analysed by Richard (1989).

Recent investigations (Tamine *et al.* 1996, Tamine 1994) showed that the pre-cracked circular ring specimens were well suited for mixed mode experiments. However the used pre-cracked specimens present two inconveniences: These specimens are regarded as time consuming and expensive. For very brittle materials as ceramics and high strength steels it is practically impossible to pre-crack the specimens and the use of notched specimen *is preferred.*

The analytical solution to determine the stress intensity factors, for notch tip distribution is Creager's solution (Creager and Paris 1967). This one consists in adding a geometrical correction factor to Irwin's solution (Irwin 1957). It has been proposed recently (Pluvinage 1997, 1999, El Minor 2002, El Minor *et al.* 2002) to characterise fracture conditions for a notched specimen by using the actual stress gradient at the notch root. This stress gradient can be characterized by a relationship different from the crack tip stress gradient. This method has been used in the present work to determine the fracture resistance in applied mixed mode fracture (I+II) using notched circular ring specimens.

2. Material and specimen

2.1 Material

The material studied is a high strength steel named 45CDS6 according to French standard. Mechanical properties are listed in Table 1. The microanalysis of the material gives*: 0.45% of C, 1.60 of Si, 0.60% of Mn, 0.60% of Cr and 0.25% of Mo.* The chemical composition is given in atomic percentage.

Table 1 Mechanical properties

M. V. (kg/m^3)	ν	A%	E (MPa)	$R_{p0,2}$ (MPa)	R_m (MPa)	K_{IC} (MPa m$^{0.5}$)
7800	0,28	2.8%	210065	1463	1662	97

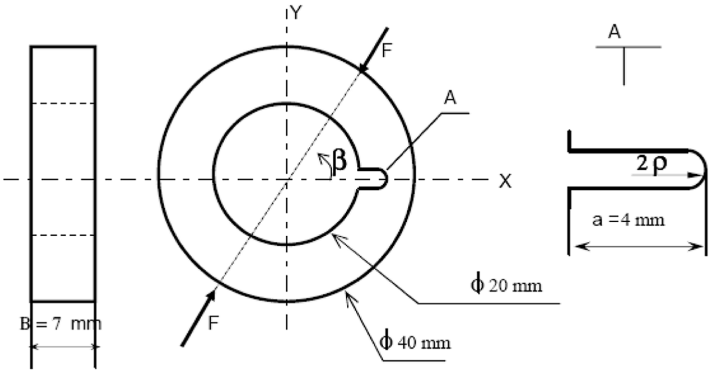

Fig. 1 U-notched circular ring specimen

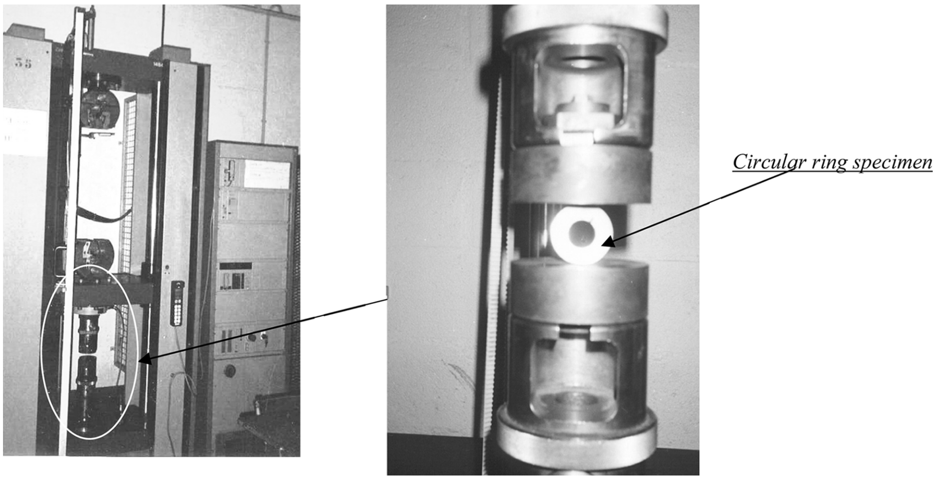

Fig. 2 Mechanical testing

2.2 Specimens

Tests are performed using U-notched circular ring specimens (Fig. 1), with external radius $R_e = 20$ mm, internal radius $R_i = 10$ mm, thickness $B = 7$ mm and notch length $a = 4$ mm. Different notch radius are introduced using a wire-cut EDM (Electrical Discharge Machine) and using wires of different diameter. The notch root radius was measured using a profile projector. Five notch radius values are used: $\rho = 0.15$, 0.3, 0.5, 1 and 2 mm.

Mode of fracture

$\beta = 0^\circ$: Mode I (Jones 1974)

$\beta = 33^\circ$: Mode II (Tamine *et al.* 1996, Tamine 1994)

$0^\circ < \beta < 33^\circ$: Mixed mode fracture I+II.

2.3 Mechanical testing

The mechanical testing using to notched specimens subjected to compressive loading is reported to Fig. 2. The critical loading and fracture angle has been registered.

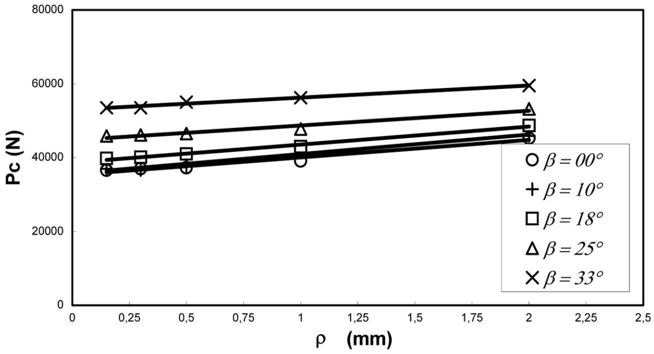

Fig. 3 Critical load P_C versus notch radius ρ for different inclination angle β

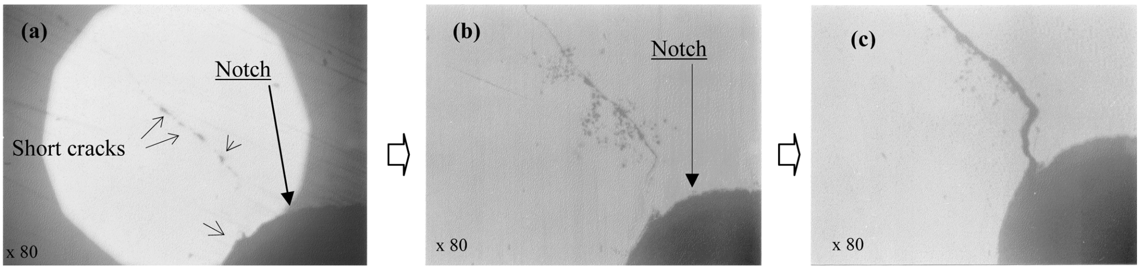

Fig. 4 Three steps crack mechanisms emanating from the notch of a ring notched specimen loaded in mixed mode fracture I+II. Fracture micro-mechanism: (a) Short cracks mechanism, (b) Evolution of micro-cracks, (c) Crack growth

3. Experimental and results

3.1 Critical load P_C

The critical loading has been registered. Fig. 3 shows the experimental critical load P_C evolution versus notch radius ρ. We noticed that the critical load P_C increases linearly with notch radius ρ.

3.2 Fracture angle

Experimental results of fracture angle θ_o (Fig. 2) measured by optical microscope, is compared with the numerical values predicted by the maximum tangential stress criterion. We conclude that the angled crack θ_o according to the maximum tangential stress criterion agrees well with the experimental data and can be used to predict bifurcation angles for cracks emanating from notches.

3.3 Mechanisms of crack initiation: “Volumetric approach”

According to an engineering approach, initiation exists whatever a high stress concentration and is defined as a short crack detectable with a magnification of ×80.

As shown in Fig. 4(a), the blunted notches give rise to a short crack mechanism. The Figs. 4(b) and (c) show the evolution of micro-cracks and crack growth from the blunted notches.

The probability of crack initiation is proportional to the process volume where the probability to find an initiation site is assumed to be uniform.

The crack mechanisms at notches can be explained by the volumic approach. We consider that, this role is played by the effective stress acting in the *crack process volume*. In this volume this stress is high enough to promote crack initiation.

4. An elliptical fracture criterion for mixed mode fractures I+II, emanating from notches

This criterion assumes that the initiation crack in mixed mode fracture I+II emanating from notches, is governed by the tangential stress.

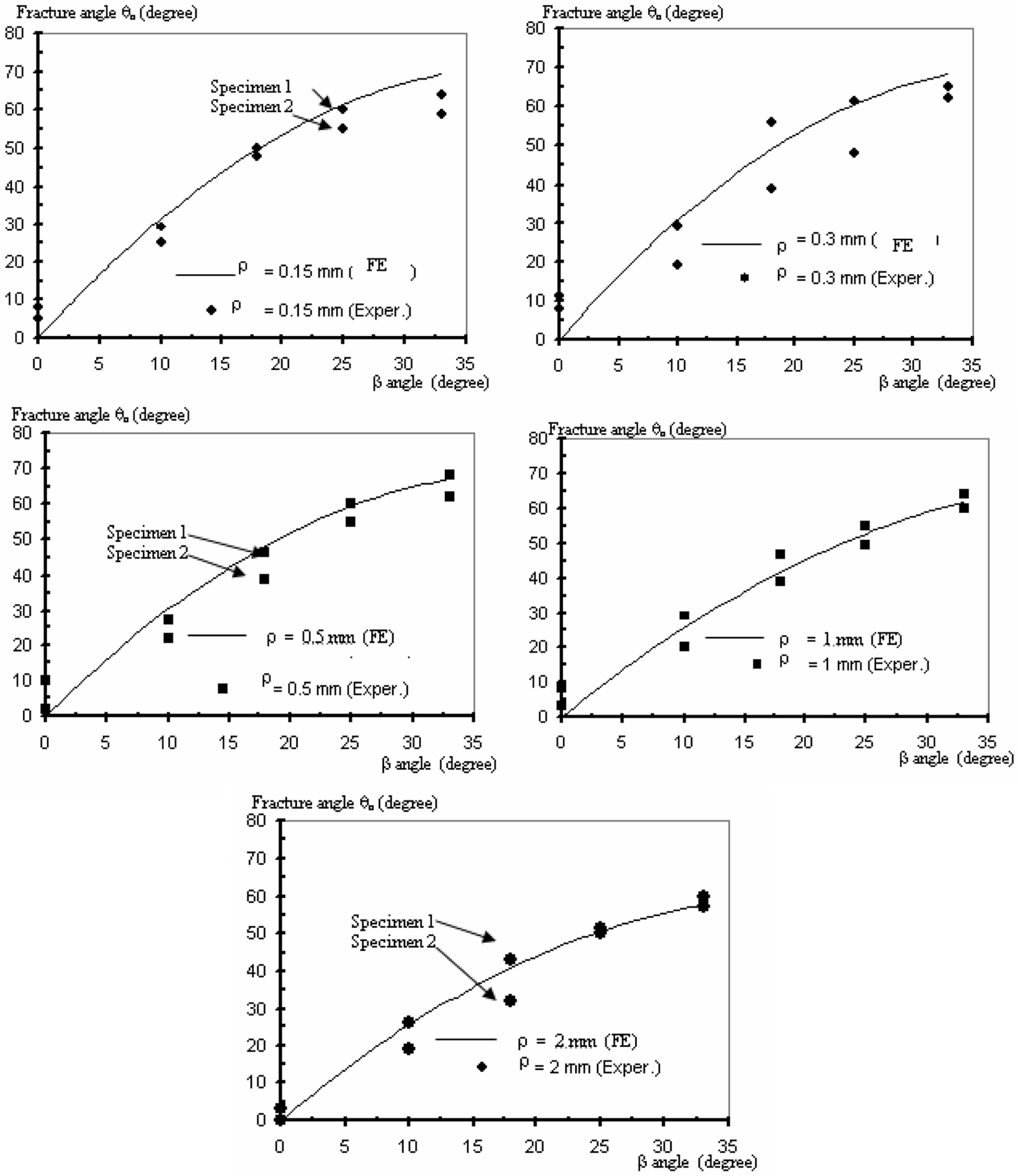

Fig. 5 Fracture angle (θ_o) vs the loading angle β

4.1 Maximum tangential stress direction at notch tip

Finite element calculations have been used to determine the maximum tangential stress on all points of the notch contour. We notice that the direction according to this stress varies linearly versus the notch radius. This variation is given by the following fitted relations

$$\theta_o = A(\beta) \cdot \frac{\rho}{a} + B(\beta) \quad (degree) \tag{1}$$

Where

$$A(\beta) = -0.0222\beta^2 + 1.4983\beta \quad (2)$$

$$B(\beta) = -0.0456\beta^2 + 3.6178\beta \quad (3)$$

When $\beta = 33^\circ$ (mode II) and for $\rho = 0$ (crack case), $\theta_o \approx 70.5^\circ$ fracture angle according by Sih and Erdogan (1963).

The numerical results of Eq. (1) (for $\rho = 0.15$, 0.3, 0.5, 1, 2 mm) are shown in Fig. 5 which is plot of θ_o-β according to the maximum tangential stress criterion are compared to experimental values. For differents radius notchs we can noted that fracture angle θ_o agree with experimental data.

4.2 Tangential stress distributions at notch tip

The Fig. 6 shows that the tangential stress $\sigma_{\theta\theta}$ is distinctly superior to σ_{rr}, σ_{zz} and $\tau_{r\theta}$. We assume that the mixed mode fracture I+II, emanating from notch is governed by the tangential stress. *In the*

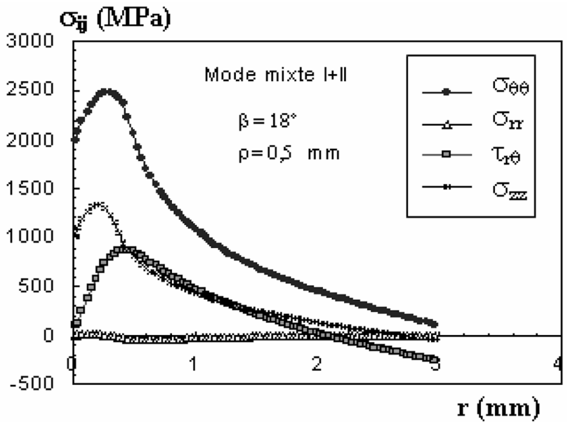

Fig. 6 Stress distributions $\sigma_{\theta\theta}$, σ_{rr} , σ_{ZZ} et $\tau_{r\theta}$ at notch tip

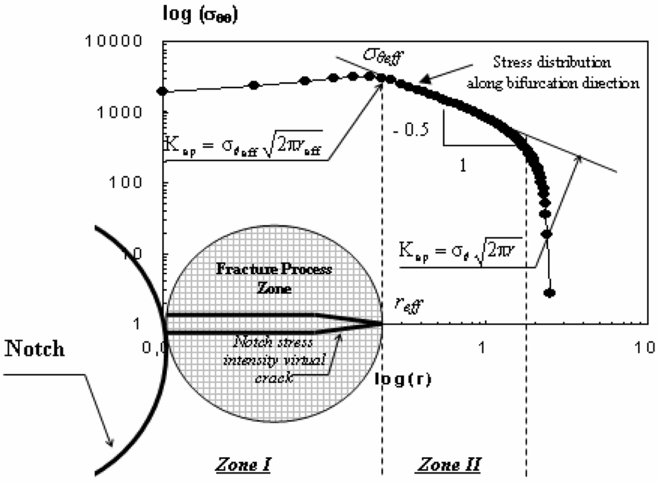

Fig. 7 Distribution of tangential stress at notch tip

following, we have studied this stress distribution according to maximum tangential stress direction.

According to procedure described by Pluvinage (1997, 1999), Kadi (1997), El Minor (2002), Weixing (1995), Qylafku *et al.* (1999), El Minor *et al.* (2002) we plot the tangential stress distribution in a bilogarithmic graph (Fig. 7).

This distribution can be divided in three zones

Zone I: ($r \leq r_{ef}$) a « High stress » region.

Zone II: ($r \geq r_{ef}$) "Pseudo singularity" stress distribution governs by notch stress intensity factors. This distribution can be written as follows

$$\sigma_{\theta\theta} = K_{e\rho}/(2\pi r)^{0.5}$$

$$\sigma_{\theta\theta} = \frac{K_{e\rho}}{\sqrt{2\pi r}} \tag{4}$$

Where

$\sigma_{\theta\theta}$: Tangential stress (MPa))

r : distance at notch tip (mm) ($r \geq r_{eff}$ (Tangential effective distance defined in El Minor (2002), El Minor *et al.* (2002)

$K_{e\rho}$: Equivalent notch stress intensity factor (MPa(m)$^{0.5}$ El Minor (2002), El Minor *et al.* (2002)).

In the "Pseudo singularity" zone, the tangential stress distribution can be also written versus notch stress intensity factor $K_{I\rho}$ and $K_{II\rho}$ as follows

$$\sigma_{\theta\theta} = \frac{1}{(2\pi r)^{0.5}} \cdot \cos\frac{\theta}{2}\left[K_{I\rho}\cos^2\frac{\theta}{2} - \frac{3}{2}K_{II\rho}\sin\theta\right] \tag{5}$$

With $K_{I\rho}$ $K_{II\rho}$: Notch stress intensity factor (MPa(m)$^{0.5}$)

Zone III: *far region*

4.3 Mixed mode I+II criterion

The tangential stress distribution at notch tip can lead to two types of fracture criteria: *global and local*. In the case of notch, there is no stress singularity at crack tip, but a maximum tangential stress followed by a pseudo singularity in which the stress distribution is governed by notch stress intensity factor.

4.3.1 Global fracture criterion

It is assumed that crack initiation occurs

i. in perpendicular direction to tangential stress when it reaches its maximum value : $\frac{\partial\sigma_{\theta\theta}}{\partial\theta} = 0$

ii. when equivalent notch stress intensity factor $K_{e\rho}$ reaches a critical value: $K_{e\rho}^C = K_{I\rho}^C = K_{IC}$

4.3.2 Local fracture criterion (volumetric approach) El Minor (2002), El Minor et al. (2002)

The local fracture criterion is based on the following consideration

For physical reason, fracture process needs a given volume called "effective volume". In this volume, the effective tangential stress can be considered as an average stress tangential weight which takes in to account the tangential stress distribution. This process volume can be described by the distance r_{eff}, so called “effective tangential distance”, considering that specimen thickness is constant and process volume is cylindrical.

The crack initiation is assumed to occur when the effective tangential stress $\sigma_{\theta eff}$ and the effective tangential distance reaches critical values.

4.4 An elliptic fracture criterion in mixed mode fracture I+II

The mixed mode criterion can be writed as the following

i. $$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \tag{6}$$

Where

$$\sigma_{\theta\theta} = \frac{1}{(2\pi r)^{0.5}} \cdot \cos\frac{\theta}{2}\left[K_{I\rho}\cos^2\frac{\theta}{2} - \frac{3}{2}K_{II\rho}\sin\theta\right] \tag{7}$$

ii. $$K_{e\rho} = K_{I\rho}^{C}$$

$$K_{e\rho} = \sigma_{\theta eff}\sqrt{2\pi r_{eff}} = K_{I\rho}^{C} \tag{8}$$

Where

$$\sigma_{\theta eff} = \frac{1}{(2\pi r_{eff})^{0.5}} \cdot \cos\frac{\theta}{2}\left[K_{I\rho}\cos^2\frac{\theta}{2} - \frac{3}{2}K_{II\rho}\sin\theta\right] \tag{9}$$

Using the Eqs. (6) and (8), $\frac{K_{I\rho}}{K_{I\rho}^{C}}$ and $\frac{K_{II\rho}}{K_{I\rho}^{C}}$ can be written as following

i. $$\frac{K_{I\rho}}{K_{I\rho}^{C}} = \frac{3\cos(\theta_o) - 1}{\cos\left(\frac{\theta_o}{2}\right) \cdot \left[\cos^2\left(\frac{\theta_o}{2}\right) \cdot (3\cos(\theta_o) - 1) + \frac{3}{2}\sin\theta_o)\right]} \tag{10}$$

ii. $$\frac{K_{II\rho}}{K_{I\rho}^{C}} = \frac{-\sin(\theta_o)}{\cos\left(\frac{\theta_o}{2}\right) \cdot \left[\cos^2\left(\frac{\theta_o}{2}\right) \cdot (3\cos(\theta_o) - 1) + \frac{3}{2}\sin\theta_o)\right]} \tag{11}$$

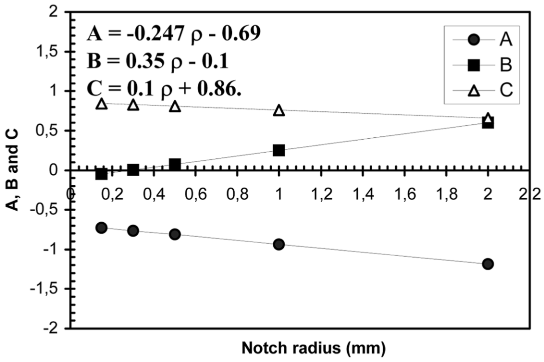

Fig. 8 Evolution of A, B and C versus the notch tip ρ

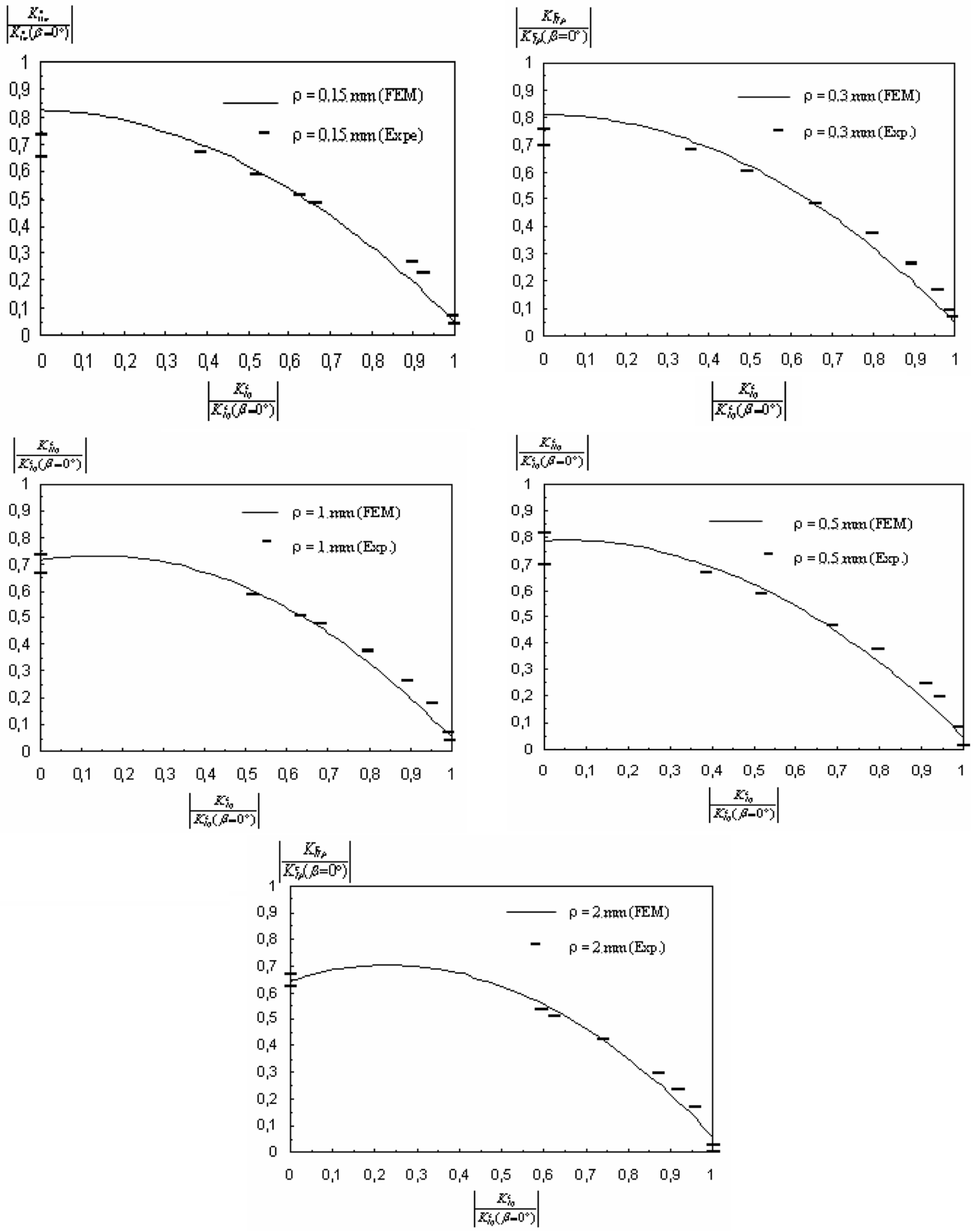

Fig. 9 Comparison of numerical predicted and test results for mixed mode fracture I+II

With
$$\theta_o = -A(\beta)\cdot\frac{\rho}{a} + B(\beta) \quad (degree)$$

The numerical values of $\frac{K_{I\rho}}{K_{I\rho}^{C}}$ is reported in Fig. 9 versus $\frac{K_{II\rho}}{K_{I\rho}^{C}}$ and compared to experimental results.

The numerical result is as follows

$$\frac{K_{II\rho}}{K_{I\rho}^{c}} = A \cdot \left(\frac{K_{I\rho}}{K_{I\rho}^{c}}\right)^{2} + B \cdot \left(\frac{K_{I\rho}}{K_{I\rho}^{c}}\right) + C \tag{12}$$

Where A, B and C vary linearly whit notch radius ρ. This variation is reported in the Fig. 8.

We notice that, when $K_{I\rho} = 0$ (mode II) and for $\rho = 0$ (crack case) $K_{II\rho} = 0.86\ K_{I\rho c}$ (this value depends in fact of Poisson's ratio). This value is given by the criterion of maximum tangential stress of Erdogan and Sih (1963).

5. Conclusions

A new test for measuring fracture toughness in pure mixed mode fracture I+II is proposed. This test uses a notch ring specimen load in compression.

- The use of notch specimen is preferable when pre cracking is impossible i.e. for brittle materials. However fracture toughness has to be is expressed by the critical notch stress intensity factor.
- The maximum tangential stress criterion based an elliptic fracture criterion and notch stress intensity factors agree well with the experimental data and can be used in the angled crack problem emanating from *notches*.
- The volumetric approach considers that the crack process needs a physical volume to develop and is governed by the maximum tangential stress (called the effective volume).
- The results of this works and other (Pluvinage 1997, Kadi 1997, El Minor 2002, Weixing 1995, Qylafku *et al.* 1999, El Minor *et al.* 2002) indicates that use the notch stress intensity factor gives a relative good description of notch effect.
- The Linear Fracture Mechanics can be considered a particular case of the Notch Fracture Mechanics.

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