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A new approach for finite element analysis of delaminated composite beam, allowing for fast and simple change of geometric characteristics of the delaminated area

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Abstract. In this work, a new approach is developed for dynamic analysis of a composite beam with an interply crack, based on finite element solution of partial differential equations with the use of the COMSOL Multiphysics package, allowing for fast and simple change of geometric characteristics of the delaminated area. The use of COMSOL Multiphysics package facilitates automatic mesh generation, which is needed if the problem has to be solved many times with different crack lengths. In the model, a physically impossible interpenetration of the crack faces is prevented by imposing a special constraint, leading to taking account of a force of contact interaction of the crack faces and to nonlinearity of the formulated boundary value problem. The model is based on the first order shear deformation theory, i.e., the longitudinal displacement is assumed to vary linearly through the beam's thickness. The shear deformation and rotary inertia terms are included into the formulation, to achieve better accuracy. Nonlinear partial differential equations of motion with boundary conditions are developed and written in the format acceptable by the COMSOL Multiphysics package. An example problem of a clamped-free beam with a piezoelectric actuator is considered, and its finite element solution is obtained. A noticeable difference of forced vibrations of the delaminated and undelaminated beams due to the contact interaction of the crack's faces is predicted by the developed model.

Keywords: composite delaminated beam; contact of crack faces; shear deformation theory; nonlinear partial differential equations; nonlinear finite element analysis; COMSOL Multiphysics package; automatic mesh generation.

1. Introduction

The task of evaluating dynamic response of delaminated composite panels to external loading with the use of the commercial finite element codes requires much users' time for preparation of the input data, especially if many damage cases (different locations and lengths of cracks) need to be considered: the preparation of the input data for commercial finite element codes involves creating a shape of the delaminated area with the use of the graphical user interface, or specifying coordinates of nodes of finite element mesh with account of presence of the delamination. To overcome this

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difficulty, the author developed a software package that facilitates preparation of the input data, allowing its users to specify only coordinates of the crack tips. This software package is run from the MATLAB command window and interacts with a commercial finite element code COMSOL Multiphysics, using its mesh generation functions and solvers. It allows its users to evaluate the changed frequencies, mode shapes and transient response of composite beams with delamination cracks in fast and practical engineering manner and allows for automatic change of the coordinates of the crack tips by external functions. This capability can be used, for example, to combine the developed analysis software with the MATLAB optimization tools to find effective stiffness characteristics of the damaged composite panels, or to perform model-aided experimental crack detection. The optimization procedures are not described in this paper. The developed analysis method is based on finite element solution of partial differential equations of motion of delaminated composite beam with the use of the COMSOL Multiphysics software package. These partial differential equations are based on the first-order shear deformation beam theory and take into account nonlinear effects due to the force of contact interaction of the delamination crack faces. This force appears in the partial differential equations due to introducing a non-interpenetration constraint for the crack faces with the use of the Heaviside function.

Several types of models of delaminated beams have been proposed in the literature. In some models, for example Ramkumar *et al.* (1979) and Wang *et al.* (1982), the contact force between the delaminated parts is not taken into account, and the physically impossible mutual penetration of the delaminated parts is allowed. In other models, for example Mujumdar and Suryanarayan (1988), the delaminated parts are constrained to have the same transverse displacement, excluding the possibility of the delamination crack opening during the vibration. In the Luo and Hanagud (2000), the interaction between the delaminated parts is modeled with the use of a nonlinear (piecewise-linear) spring between the surfaces of the delaminated parts. Stiffness of the spring depends on the difference of displacements of the lower and upper delaminated parts. If the delamination crack is open, the stiffness of the spring is set equal to zero, making the distributed contact force equal to zero. When the delamination crack is closed, the stiffness of the spring is set either to infinity, or to some finite constant value. The authors set the spring stiffness equal to a constant (either zero, or 0.1, or infinity) before solving the problem, thus assuming that the crack remains either open or closed all the time during the vibration. So, the possibility for the crack to be open in some time intervals and closed in other time intervals during the vibration is not foreseen in this model.

In the paper Wang and Tong (2002), the contact force between the delaminated sublaminates is introduced as a function of the relative transverse displacement of the sublaminates, in such a way that the contact force automatically turns out to be zero, when the delamination crack is open, and takes on a non-zero value, if the crack is closed. So, this model does not require to specify in advance if the crack is open or closed, and allows for contact and separation of the crack faces during the vibration. However, the physically impossible interpenetration of the crack faces is not always prevented in this model. The interpenetration occurs because a constraint, preventing this phenomenon, is not introduced.

In the model of the delaminated composite beam, presented by the author in the Perel (2005a), the constraint, preventing the mutual penetration (interpenetration, overlapping) of the delaminated sublaminates (of the crack's faces), was introduced with the use of the Heaviside function and the penalty function method Reddy (1984), which was the main novelty in solving dynamic problems for beams with cracks. The longitudinal force resultants in the delaminated sublaminates and rotary inertia terms were taken into account also. The use of the constraint, which prevented the interpenetration of the crack faces, and taking account of the longitudinal force resultants led to nonlinear partial differential equations of motion, in which a force of contact interaction of the crack faces was taken into account.

But the model, presented in Perel (2005a), did not take the shear strain energy into account, and, therefore, produced sufficiently accurate results only for thin beams. To model thicker beams with delamination, one needs to use a beam theory, based on simplifying assumptions, which do not lead to vanishing of the shear strains. The first order shear deformation theory Reddy (1984), based on assumed linear variation of a longitudinal displacement in the thickness direction, is the simplest approach that satisfies the requirement of a non-zero shear strain. This approach is used in the present paper for modeling a composite delaminated beam with a piezoelectric actuator. In this model, the interpenetration of the crack faces is prevented by a method similar to the one, which was used in Perel (2005a): by imposing a constraint, written with the use of the Heaviside function in one of its analytical forms, leading to taking account of a force of contact interaction of the crack faces and to nonlinearity of the formulated boundary value problem.

Besides, in Perel (2005a), the solution was obtained by the Ritz method in the form of a series in terms of eigenfunctions of an eigenvalue problem, associated with the linearized partial differential equations and linearized natural boundary conditions. This series converged rapidly, providing high accuracy of the solution. But the process of constructing the system of the eigenfunctions for each particular crack length involved solving a nonlinear algebraic eigenvalue problem by an iterative method, which required good initial approximations for each of the frequencies. This caused difficulty in achieving a complete automatization of the process of constructing the eigenfunctions and, therefore, required much time, if the problem had to be solved many times with different crack lengths. This difficulty led to the need of developing a finite element solution of the formulated problem (in conjunction with the first order shear deformation theory, as mentioned above) and the computer program with automatic mesh generation, which became the subject of the present paper. The model is developed to include it, later, into computational procedures for model-aided detection of cracks, with the use of methods presented in Liu and Han (2003). These procedures involve giving small increments to crack lengths at each step of the search algorithm for the crack detection, as a result of which the crack tip does not coincide with the nodes of the initial finite element mesh after each increment of the crack length. This leads to the need of fast and automatic construction of the new finite element mesh after each increment of the crack length, and this task is achieved with the use of the capabilities of the COMSOL Multiphysics package. In this paper, the COMSOL Multiphysics is used to solve the partial differential equations derived by the author in Perel (2005b).

So, the main novelty of the model of the delaminated composite beam, presented in this paper, as compared to the author's model in Perel (2005a), is that the method of taking account of force of contact interaction of the crack faces, presented in the Perel (2005a), is combined here with the first order shear deformation theory and the finite element method, with automatic re-meshing after each increment of the crack length. This improvement of the model, as compared to the model in Perel (2005a), leads to higher accuracy of solutions and allows for full automatization of the solution process.

2. Partial differential equations with boundary conditions

The partial differential equations, based on the first-order shear deformation theory (Reddy 1984),

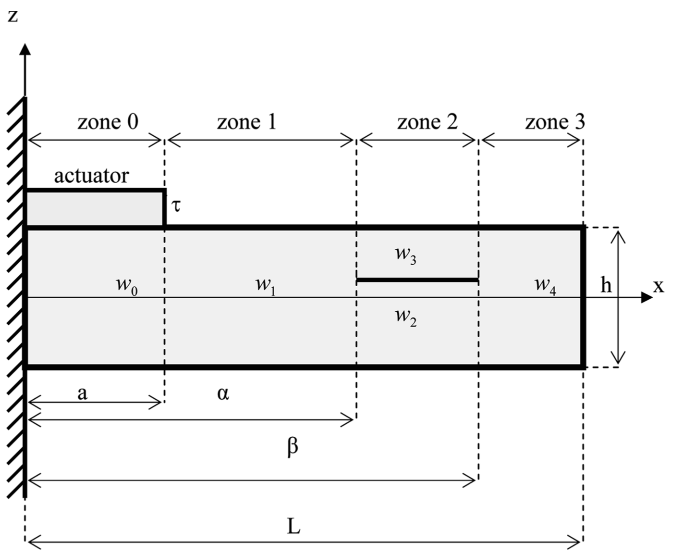

Fig. 1 Cantilever beam with delamination and piezoelectric actuator. a is length of the actuator, α is x-coordinate of the left crack tip, β is x-coordinate of the right crack tip, γ is z-coordinate of the crack (distance from x-axis to crack), τ is thickness of the actuator, w_0 is transverse displacement of zone 0, w_1 is transverse displacement of zone 1, w_2 is transverse displacement of lower part of zone 2 (under the crack), w_3 is transverse displacement of upper part of zone 2 (above the crack), w_4 is transverse displacement of zone 3

describing vibration of delaminated clamped-free beam with piezoelectric actuator (Fig. 1) and with account of contact of the crack faces, are derived by the author in Perel (2005) and have the following form.

Partial differential equations

for Zone 0 (Part 0)

$$KG_0(w_0'' + \phi_0') - B_0\ddot{w}_0 = 0 \quad \text{in} \quad x \in [0, a] \tag{1}$$

$$A_0\phi_0'' - KG_0(w_0' + \phi_0) - C_0\ddot{\phi}_0 = I_pV' \quad \text{in} \quad x \in [0, a] \tag{2}$$

for Zone 1 (Part 1)

$$KG_1(w_1'' + \phi_1') - B_1\ddot{w}_1 = 0 \quad \text{in} \quad x \in [a, \alpha] \tag{3}$$

$$A_1\phi_1'' - KG_1(w_1' + \phi_1) - C_1\ddot{\phi}_1 = 0 \quad \text{in} \quad x \in [a, \alpha] \tag{4}$$

for Zone 2 (Part 2 and Part 3)

$$KG_2(w_2'' + \phi_2') - B_2\ddot{w}_2 - \chi(w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\varepsilon}\right) = 0 \quad \text{in} \quad x \in [\alpha, \beta] \tag{5}$$

$$A_2\phi_2'' - KG_2(w_2' + \phi_2) - C_2\ddot{\phi}_2 = 0 \quad \text{in} \quad x \in [\alpha, \beta] \tag{6}$$

$$KG_3(w_3'' + \phi_3') - B_3\ddot{w}_3 + \chi(w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\varepsilon}\right) = 0 \quad \text{in} \quad x \in [\alpha, \beta] \tag{7}$$

$$A_3\phi_3'' - KG_3(w_3' + \phi_3) - C_3\ddot{\phi}_3 = 0 \quad \text{in} \;\; x \in [\alpha, \beta] \tag{8}$$

for Zone 3 (Part 4)

$$KG_4(w_4'' + \phi_4') - B_4\ddot{w}_4 = 0 \quad \text{in} \;\; x \in [\beta, L] \tag{9}$$

$$A_4\phi_4'' - KG_4(w_4' + \phi_4) - C_4\ddot{\phi}_4 = 0 \quad \text{in} \;\; x \in [\beta, L] \tag{10}$$

Essential boundary conditions

$$R_i(t) = 0 \quad (i = 1, 2, \ldots, 12) \tag{11a}$$

where

$$\begin{aligned} R_1 &\equiv w_0(0, t), \quad & R_2 &\equiv \phi_0(0, t) \\ R_3 &\equiv w_0(a, t) - w_1(a, t), \quad & R_4 &\equiv \phi_0(a, t) - \phi_1(a, t) \\ R_5 &\equiv w_1(\alpha, t) - w_2(\alpha, t), \quad & R_6 &\equiv \phi_1(\alpha, t) - \phi_2(\alpha, t) \\ R_7 &\equiv w_1(\alpha, t) - w_3(\alpha, t), \quad & R_8 &\equiv \phi_1(\alpha, t) - \phi_3(\alpha, t) \\ R_9 &\equiv w_2(\beta, t) - w_4(\beta, t), \quad & R_{10} &\equiv \phi_2(\beta, t) - \phi_4(\beta, t) \\ R_{11} &\equiv w_3(\beta, t) - w_4(\beta, t), \quad & R_{12} &\equiv \phi_3(\beta, t) - \phi_4(\beta, t) \end{aligned} \tag{11b}$$

Natural boundary conditions

$$KG_0(\phi_0 + w_0') + \lambda_3 = 0 \quad \text{at} \quad x = a \tag{12}$$

$$A_0\phi_0' - I_pV(t) + \lambda_4 = 0 \quad \text{at} \quad x = a \tag{13}$$

$$KG_1(\phi_1 + w_1') + \lambda_3 = 0 \quad \text{at} \quad x = a \tag{14}$$

$$A_1\phi_1' + \lambda_4 = 0 \quad \text{at} \quad x = a \tag{15}$$

$$KG_1(\phi_1 + w_1') + \lambda_5 + \lambda_7 = 0 \quad \text{at} \quad x = \alpha \tag{16}$$

$$A_1\phi_1' + \lambda_6 + \lambda_8 = 0 \quad \text{at} \quad x = \alpha \tag{17}$$

$$KG_2(\phi_2 + w_2') + \lambda_5 = 0 \quad \text{at} \quad x = \alpha \tag{18}$$

$$A_2\phi_2' + \lambda_6 = 0 \quad \text{at} \quad x = \alpha \tag{19}$$

$$KG_3(\phi_3 + w_3') + \lambda_7 = 0 \quad \text{at} \quad x = \alpha \tag{20}$$

$$A_3\phi_3' + \lambda_8 = 0 \quad \text{at} \quad x = \alpha \tag{21}$$

$$KG_2(\phi_2 + w_2') + \lambda_9 = 0 \quad \text{at} \quad x = \beta \tag{22}$$

$$A_2\phi_2' + \lambda_{10} = 0 \quad \text{at} \quad x = \beta \tag{23}$$

$$KG_3(\phi_3 + w_3') + \lambda_{11} = 0 \quad \text{at} \quad x = \beta \tag{24}$$

$$A_3\phi_3' + \lambda_{12} = 0 \quad \text{at} \quad x = \beta \tag{25}$$

$$KG_4(\phi_4 + w_4') + \lambda_9 + \lambda_{11} = 0 \quad \text{at} \quad x = \beta \tag{26}$$

$$A_4\phi_4' + \lambda_{10} + \lambda_{12} = 0 \quad \text{at} \quad x = \beta \tag{27}$$

$$KG_4(\phi_4 + w_4') = 0 \quad \text{at} \quad x = L \tag{28}$$

$$A_4\phi_4' = 0 \quad \text{at} \quad x = L \tag{29}$$

In the following text it will be assumed that the voltage $V(x, t)$, applied to the piezoelectric actuator, is distributed uniformly over the length of the actuator (over the interval $x \in [0, a]$) and depends on time as $V(x, t) = V(t) = V_0\sin(\Omega t)$. Therefore, the spatial derivative $V' \equiv \partial V(x, t)/\partial x$, in the right-hand side of the differential Eq. (2) will be considered equal to zero in the subsequent text, and the boundary condition (13) will be written as

$$A_0\phi_0' - I_p V_0 \sin(\Omega t) + \lambda_4 = 0 \quad \text{at} \quad x = a \tag{30}$$

3. Formulation in a form convenient for COMSOL multiphysics implementation

Unknown functions w_0, w_1, w_2, w_3, w_4, ϕ_0, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are defined only in the beam's parts, which are indicated by the function's subscripts (Fig. 1). So, the functions with subscript 0 are defined only in Part 0 (Zone 0); the functions with subscript 1 are defined only in Part 1 (Zone 1); the functions with subscripts 2 and 3 are defined in Part 2 (Zone 2) and Part 3 (Zone 2) respectively; the functions with subscript 4 are defined in Part 4 (Zone 3). But for convenience of using the COMSOL Multiphysics package, one needs to give some definitions to functions w_1, w_2, w_3, w_4, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 in Zone 0; to functions w_0, w_1, w_2, w_3, w_4, ϕ_0, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 in Zone 1; to functions w_0, w_1, w_4, ϕ_0, ϕ_1 and ϕ_4 in Zone 2; and to functions w_0, w_1, w_2, w_3, w_4, ϕ_0, ϕ_1, ϕ_2 and ϕ_3 in Zone 3. These definitions must not contradict the essential boundary conditions (30). Therefore, the following definitions are introduced

For Zone 0 (Part 0), i.e. $0 \le x \le a$

$$\begin{gathered} w_1 \equiv w_0, \;\; w_2 \equiv w_0, \;\; w_3 \equiv w_0, \;\; w_4 \equiv w_0 \\ \phi_1 \equiv \phi_0, \;\; \phi_2 \equiv \phi_0, \;\; \phi_3 \equiv \phi_0, \;\; \phi_4 \equiv \phi_0 \end{gathered} \tag{31}$$

For Zone 1 (Part 1), i.e. in $a \le x \le \alpha$

$$\begin{gathered} w_0 \equiv w_1, \;\; w_2 \equiv w_1, \;\; w_3 \equiv w_1, \;\; w_4 \equiv w_1 \\ \phi_0 \equiv \phi_1, \;\; \phi_2 \equiv \phi_1, \;\; \phi_3 \equiv \phi_1, \;\; \phi_4 \equiv \phi_1 \end{gathered} \tag{32}$$

For Zone 2 (Part 2 and Part 3), i.e. in $\alpha \le x \le \beta$

$$\begin{gathered} w_0 \equiv w_2,\ \ w_1 \equiv w_2,\ \ w_4 \equiv w_2 \\ \phi_0 \equiv \phi_2,\ \ \phi_1 \equiv \phi_2,\ \ \phi_4 \equiv \phi_2 \end{gathered} \tag{33}$$

For Zone 3 (Part 4), i.e. in $\beta \le x \le L$

$$\begin{gathered} w_0 \equiv w_4,\ \ w_1 \equiv w_4,\ \ w_2 \equiv w_4,\ \ w_3 \equiv w_4 \\ \phi_0 \equiv \phi_4,\ \ \phi_1 \equiv \phi_4,\ \ \phi_2 \equiv \phi_4,\ \ \phi_3 \equiv \phi_4 \end{gathered} \tag{34}$$

In the further presentation, to create a formulation that complies with the format, required by the COMSOL Multiphysics package, the following notations will be introduced for the Lagrange multipliers

$$\begin{gathered} \hat{\lambda}_1 \equiv \lambda_3, \quad \hat{\lambda}_2 \equiv \lambda_4 \\ \tilde{\lambda}_1 \equiv \lambda_5, \quad \tilde{\lambda}_2 \equiv \lambda_7, \quad \tilde{\lambda}_3 \equiv \lambda_6, \quad \tilde{\lambda}_4 \equiv \lambda_8 \\ \bar{\lambda}_1 \equiv \lambda_9, \quad \bar{\lambda}_2 \equiv \lambda_{11}, \quad \bar{\lambda}_3 \equiv \lambda_{10}, \quad \bar{\lambda}_4 \equiv \lambda_{12} \end{gathered} \tag{35}$$

In view of definitions (31)-(34), and in view of the notations (35), the partial differential equations and boundary conditions take the form presented below. To comply with the terminology of COMSOL Multiphysics, the zones will be called subdomains. The Zone 0 will be called Subdomain 1, the Zone 1 will be called Subdomain 2, the Zone 2 will be called Subdomain 3, the Zone 3 will be called Subdomain 4.

Partial differential equations

For Zone 0 (Subdomain 1), i.e. in the interval $x \in [0, a]$

$$-B_0\ddot{w}_0 + KG_0(w_0'' + \phi_0') = 0 \quad \text{in} \quad x \in [0, a] \tag{36}$$

$$-C_0\ddot{\phi}_0 + (A_0\phi_0'' - KG_0 w_0') = KG_0\phi_0 \quad \text{in} \quad x \in [0, a] \tag{37}$$

$$0 = w_0 - w_1 \quad \text{in} \quad x \in [0, a] \tag{38}$$

$$0 = w_0 - w_2 \quad \text{in} \quad x \in [0, a] \tag{39}$$

$$0 = w_0 - w_3 \quad \text{in} \quad x \in [0, a] \tag{40}$$

$$0 = w_0 - w_4 \quad \text{in} \quad x \in [0, a] \tag{41}$$

$$0 = \phi_0 - \phi_1 \quad \text{in} \quad x \in [0, a] \tag{42}$$

$$0 = \phi_0 - \phi_2 \quad \text{in} \quad x \in [0, a] \tag{43}$$

$$0 = \phi_0 - \phi_3 \quad \text{in} \quad x \in [0, a] \tag{44}$$

$$0 = \phi_0 - \phi_4 \quad \text{in} \quad x \in [0, a] \tag{45}$$

For Zone 1 (Subdomain 2), i.e. in the interval $x \in [a, \alpha]$

$$-B_1\ddot{w}_1 + KG_1(w_1'' + \phi_1') = 0 \quad \text{in} \quad x \in [a, \alpha] \tag{46}$$

$$-C_1\ddot{\phi}_1 + (A_1\phi_1'' - KG_1w_1') = KG_1\phi_1 \quad \text{in} \quad x \in [a, \alpha] \tag{47}$$

$$0 = w_1 - w_0 \quad \text{in} \quad x \in [a, \alpha] \tag{48}$$

$$0 = w_1 - w_2 \quad \text{in} \quad x \in [a, \alpha] \tag{49}$$

$$0 = w_1 - w_3 \quad \text{in} \quad x \in [a, \alpha] \tag{50}$$

$$0 = w_1 - w_4 \quad \text{in} \quad x \in [a, \alpha] \tag{51}$$

$$0 = \phi_1 - \phi_0 \quad \text{in} \quad x \in [a, \alpha] \tag{52}$$

$$0 = \phi_1 - \phi_2 \quad \text{in} \quad x \in [a, \alpha] \tag{53}$$

$$0 = \phi_1 - \phi_3 \quad \text{in} \quad x \in [a, \alpha] \tag{54}$$

$$0 = \phi_1 - \phi_4 \quad \text{in} \quad x \in [a, \alpha] \tag{55}$$

For Zone 2 (Subdomain 3), i.e. in the interval $x \in [\alpha, \beta]$

$$-B_2\ddot{w}_2 + KG_2(w_2'' + \phi_2') = \chi(w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\varepsilon}\right) \quad \text{in} \quad x \in [\alpha, \beta] \tag{56}$$

$$-C_2\ddot{\phi}_2 + (A_2\phi_2'' - KG_2w_2') = KG_2\phi_2 \quad \text{in} \quad x \in [\alpha, \beta] \tag{57}$$

$$-B_3\ddot{w}_3 + KG_3(w_3'' + \phi_3') = -\chi(w_3 - w_2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{w_3 - w_2}{\varepsilon}\right) \quad \text{in} \quad x \in [\alpha, \beta] \tag{58}$$

$$-C_3\ddot{\phi}_3 + (A_3\phi_3'' - KG_3w_3') = KG_3\phi_3 \quad \text{in} \quad x \in [\alpha, \beta] \tag{59}$$

$$0 = w_2 - w_0 \quad \text{in} \quad x \in [\alpha, \beta] \tag{60}$$

$$0 = w_2 - w_1 \quad \text{in} \quad x \in [\alpha, \beta] \tag{61}$$

$$0 = w_2 - w_4 \quad \text{in} \quad x \in [\alpha, \beta] \tag{62}$$

$$0 = \phi_2 - \phi_0 \quad \text{in} \quad x \in [\alpha, \beta] \tag{63}$$

$$0 = \phi_2 - \phi_1 \quad \text{in} \quad x \in [\alpha, \beta] \tag{64}$$

$$0 = \phi_2 - \phi_4 \quad \text{in} \quad x \in [\alpha, \beta] \tag{65}$$

For Zone 3 (Subdomain 4) i.e. in the interval $x \in [\beta, L]$

$$-B_4\ddot{w}_4 + KG_4(w_4'' + \phi_4') = 0 \quad \text{in} \quad x \in [\beta, L] \tag{66}$$

$$-C_4\ddot{\phi}_4 + (A_4\phi_4'' - KG_4w_4') = KG_4\phi_4 \quad \text{in} \quad x \in [\beta, L] \tag{67}$$

$$0 = w_4 - w_0 \quad \text{in} \quad x \in [\beta, L] \tag{68}$$

$$0 = w_4 - w_1 \quad \text{in} \quad x \in [\beta, L] \tag{69}$$

$$0 = w_4 - w_2 \quad \text{in} \quad x \in [\beta, L] \tag{70}$$

$$0 = w_4 - w_3 \quad \text{in} \quad x \in [\beta, L] \tag{71}$$

$$0 = \phi_4 - \phi_0 \quad \text{in} \quad x \in [\beta, L] \tag{72}$$

$$0 = \phi_4 - \phi_1 \quad \text{in} \quad x \in [\beta, L] \tag{73}$$

$$0 = \phi_4 - \phi_2 \quad \text{in} \quad x \in [\beta, L] \tag{74}$$

$$0 = \phi_4 - \phi_3 \quad \text{in} \quad x \in [\beta, L] \tag{75}$$

Boundary conditions
Boundary 1, i.e. $x = 0$

$$w_0 = 0 \quad \text{at} \quad x = 0 \text{ (essential BC)} \tag{76}$$

$$\phi_0 = 0 \quad \text{at} \quad x = 0 \text{ (essential BC)} \tag{77}$$

Boundary 2, i.e. $x = a$

$$w_0 - w_1 = 0 \quad \text{at} \quad x = a \text{ (essential BC)} \tag{78}$$

$$\phi_0 - \phi_1 = 0 \quad \text{at} \quad x = a \text{ (essential BC)} \tag{79}$$

$$KG_0(w_0' + \phi_0) = -\hat{\lambda}_1 \quad \text{at} \quad x = a \text{ (natural BC)} \tag{80}$$

$$KG_1(w_1' + \phi_1) = -\hat{\lambda}_1 \quad \text{at} \quad x = a \text{ (natural BC)} \tag{81}$$

$$A_0\phi_0' - I_pV_0\sin(\Omega t) = -\hat{\lambda}_2 \quad \text{at} \quad x = a \text{ (natural BC)} \tag{82}$$

$$A_1\phi_1' = -\hat{\lambda}_2 \quad \text{at} \quad x = a \text{ (natural BC)} \tag{83}$$

Boundary 3, i.e. $x = \alpha$

$$w_1 - w_2 = 0 \quad \text{at} \quad x = \alpha \text{ (essential BC)} \tag{84}$$

$$w_1 - w_3 = 0 \quad \text{at} \quad x = \alpha \text{ (essential BC)} \tag{85}$$

$$\phi_1 - \phi_2 = 0 \quad \text{at} \quad x = \alpha \text{ (essential BC)} \tag{86}$$

$$\phi_1 - \phi_3 = 0 \quad \text{at} \quad x = \alpha \text{ (essential BC)} \tag{87}$$

$$KG_1(\phi_1 + w_1') = -\tilde{\lambda}_1 - \tilde{\lambda}_2 \quad \text{at} \quad x = \alpha \text{ (natural BC)} \tag{88}$$

$$KG_2(\phi_2 + w_2') = -\tilde{\lambda}_1 \quad \text{at} \quad x = \alpha \text{ (natural BC)} \tag{89}$$

$$KG_3(\phi_3 + w_3') = -\tilde{\lambda}_2 \quad \text{at} \quad x = \alpha \text{ (natural BC)} \tag{90}$$

$$A_1\phi_1' = -\tilde{\lambda}_3 - \tilde{\lambda}_4 \quad \text{at} \quad x = \alpha \text{ (natural BC)} \tag{91}$$

$$A_2\phi_2' = -\tilde{\lambda}_3 \quad \text{at} \quad x = \alpha \text{ (natural BC)} \tag{92}$$

$$A_3\phi_3' = -\tilde{\lambda}_4 \quad \text{at} \quad x = \alpha \text{ (natural BC)} \tag{93}$$

Boundary 4, i.e. $x = \beta$

$$w_2 - w_4 = 0 \quad \text{at} \quad x = \beta \text{ (essential BC)} \tag{94}$$

$$w_3 - w_4 = 0 \quad \text{at} \quad x = \beta \text{ (essential BC)} \tag{95}$$

$$\phi_2 - \phi_4 = 0 \quad \text{at} \quad x = \beta \text{ (essential BC)} \tag{96}$$

$$\phi_3 - \phi_4 = 0 \quad \text{at} \quad x = \beta \text{ (essential BC)} \tag{97}$$

$$KG_4(\phi_4 + w_4') = -\bar{\lambda}_1 - \bar{\lambda}_2 \quad \text{at} \quad x = \beta \text{ (natural BC)} \tag{98}$$

$$KG_2(\phi_2 + w_2') = -\bar{\lambda}_1 \quad \text{at} \quad x = \beta \text{ (natural BC)} \tag{99}$$

$$KG_3(\phi_3 + w_3') = -\bar{\lambda}_2 \quad \text{at} \quad x = \beta \text{ (natural BC)} \tag{100}$$

$$A_4\phi_4' = -\bar{\lambda}_3 - \bar{\lambda}_4 \quad \text{at} \quad x = \beta \text{ (natural BC)} \tag{101}$$

$$A_2\phi_2' = -\bar{\lambda}_3 \quad \text{at} \quad x = \beta \text{ (natural BC)} \tag{102}$$

$$A_3\phi_3' = -\bar{\lambda}_4 \quad \text{at} \quad x = \beta \text{ (natural BC)} \tag{103}$$

Boundary 5, i.e. $x = L$

$$KG_4(\phi_4 + w_4') = 0 \quad \text{at} \quad x = L \text{ (natural BC)} \tag{104}$$

$$A_4\phi_4' = 0 \quad \text{at} \quad x = L \text{ (natural BC)} \tag{105}$$

In the COMSOL Multiphysics terminology, natural boundary conditions are called the Neumann boundary conditions, essential boundary conditions are called the Dirichlet boundary conditions, and the mixed boundary conditions (both essential and natural conditions at the same boundary) are called the Dirichlet boundary conditions also. With the use of this terminology, the boundary conditions (76)-(103) at boundaries $x = 0$, $x = a$, $x = \alpha$ and $x = \beta$ are the Dirichlet boundary conditions, and the boundary conditions (104) and (105) at the boundary $x = L$ are the Neumann boundary conditions.

4. Standard form of representation of equations in COMSOL multiphysics for one-dimensional problems

In COMSOL Multiphysics, in case of N unknown functions $u_k(x, t)$ $(k = 1, 2, ..., N)$ of one spacial coordinate x and time t, the partial differential equations of the second order and the boundary conditions are written in the following form (summation over repeated indices is implied).

Partial differential equations

$$M_{mk}\ddot{u}_k + \Gamma'_m = F_m \ (k, m = 1, \ldots, N) \ \text{ in subdomains of } x \tag{106}$$

Neumann boundary conditions at *external* boundaries

$$n_x\Gamma_m = -G_m \ \text{(natural BC)} \tag{107}$$

Dirichlet boundary conditions at *external* boundaries

$$R_m = 0 \ \text{(essential BC)} \tag{108a}$$

and

$$n_x\Gamma_m + \lambda_n\frac{\partial R_n}{\partial u_m} = -G_m \ \text{(natural BC)} \tag{108b}$$

where

$$\begin{aligned} \Gamma_m &\equiv -c_{mk}u'_k - \alpha_{mk}u_k + \gamma_m \\ F_m &\equiv f_m - a_{mk}u_k \\ G_m &\equiv g_m - q_{mk}u_k \\ R_m &\equiv h_{mk}u_k - r_m \end{aligned} \tag{109}$$

and coefficients c_{mk}, α_{mk}, γ_m, f_m, a_{mk}, g_m, q_{mk}, h_{mk}, r_m are, generally, some known functions of the coordinate x and time t. Of course, these coefficients can be functions of coordinates only, time only, or constants. The quantity n_x is an x-component of the subdomain's boundary's outward unit normal vector. In case of one-dimensional problems, as the one considered here, $n_x = 1$ at right edges of subdomains, and $n_x = 1$ at left edges of subdomains, if the x-axis is directed from left to right, as in Fig. 1.

If boundary conditions are specified at *internal* boundaries, i.e. at the boundaries between two adjacent subdomains (e.g. Subdomain 1 and Subdomain 2), then the Neumann boundary conditions take the form

$$\underbrace{n_x^{(1)}}_{1}\Gamma_m^{(1)} + \underbrace{n_x^{(2)}}_{-1}\Gamma_m^{(2)} = -G_m \text{ (natural BC)} \tag{110}$$

and the Dirichlet boundary conditions take the form

$$R_m = 0 \text{ (essential BC)}$$

and

$$\underbrace{n_x^{(1)}}_{1}\Gamma_m^{(1)} + \underbrace{n_x^{(2)}}_{-1}\Gamma_m^{(2)} + \frac{\partial R_k}{\partial u_m}\lambda_k = -G_m \text{ (natural BC)} \tag{111}$$

Either the Neumann or Dirichlet boundary conditions must be chosen at each boundary. If only natural boundary conditions are specified on a boundary of a subdomain, then such boundary conditions have the form of Neumann boundary conditions. If both essential and natural boundary conditions are specified at a boundary, then such boundary conditions have the form of Dirichlet boundary conditions.

Eqs. (106)-(108) can be written in matrix form as follows.

Partial differential equations

$$\underset{(N\times N)}{[M]}\frac{\partial^2}{\partial t^2}\underset{(N\times 1)}{\{u\}} + \frac{\partial}{\partial x}\underset{(N\times 1)}{\{\Gamma\}} = \underset{(N\times 1)}{\{F\}} \tag{112}$$

Neumann boundary conditions

$$n_x\underset{(N\times 1)}{\{\Gamma\}} = -\underset{(N\times 1)}{\{G\}} \tag{113}$$

Dirichlet boundary conditions

$$\underset{(N\times 1)}{\{R\}} = \underset{(N\times 1)}{\{0\}} \tag{114a}$$

and

$$n_x\begin{Bmatrix}\Gamma_1\\ \Gamma_2\\ \vdots\\ \Gamma_N\end{Bmatrix} + \begin{bmatrix}\frac{\partial R_1}{\partial u_1} & \frac{\partial R_2}{\partial u_1} & \cdots & \frac{\partial R_N}{\partial u_1}\\ \frac{\partial R_1}{\partial u_2} & \frac{\partial R_2}{\partial u_2} & \cdots & \frac{\partial R_N}{\partial u_2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial R_1}{\partial u_N} & \frac{\partial R_2}{\partial u_N} & \cdots & \frac{\partial R_N}{\partial u_N}\end{bmatrix}\begin{Bmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_3\end{Bmatrix} = -\begin{Bmatrix}G_1\\ G_2\\ \vdots\\ G_N\end{Bmatrix} \tag{114b}$$

Similarly, the boundary conditions (110) and (111) at an internal boundary, being written in the matrix form, are

Neumann boundary conditions

$$\underset{(N\times 1)}{\{\Gamma\}}^{(1)} - \underset{(N\times 1)}{\{\Gamma\}}^{(2)} = -\underset{(N\times 1)}{\{G\}} \tag{115}$$

Dirichlet boundary conditions

$$\underset{(N\times 1)}{\{R\}} = \underset{(N\times 1)}{\{0\}} \tag{116a}$$

and

$$\underset{(N\times 1)}{\{\Gamma\}}^{(1)} - \underset{(N\times 1)}{\{\Gamma\}}^{(2)} + \left[\frac{\partial R_m}{\partial u_k}\right]^T \underset{(N\times 1)}{\{\lambda\}} = -\underset{(N\times 1)}{\{G\}} \tag{116b}$$

5. Subdomain and boundary settings of the problem

To comply with the COMSOL Multiphysics's requirements for notations, the following *alternative notations* are introduced for the unknown functions of the present problem:

$$\underset{(10\times 1)}{\{u\}} \equiv \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{Bmatrix} \equiv \begin{Bmatrix} w0 \\ p0 \\ w1 \\ p1 \\ w2 \\ p2 \\ w3 \\ p3 \\ w4 \\ p4 \end{Bmatrix} \equiv \begin{Bmatrix} w_0 \\ \phi_0 \\ w_1 \\ \phi_1 \\ w_2 \\ \phi_2 \\ w_3 \\ \phi_3 \\ w_4 \\ \phi_4 \end{Bmatrix} \tag{117}$$

for the spatial derivatives of the unknown functions

$$w0x \equiv w_0', \quad p0x \equiv \phi_0', \ldots \tag{118}$$

for the constants and for matrix $[M]$ in Eq. (112)

$$A0 \equiv A_0, \quad B0 \equiv B_0, \ldots, \quad Omega \equiv \Omega, \quad [d_a] \equiv [M] \tag{119}$$

and all kinds of notations will be used interchangeably in the subsequent text.
Partial differential Eqs. (36)-(45) for Zone 0 (Subdomain 1), i.e. for $x \in [0, a]$ can be written in matrix form as

$$\underset{(10\times 10)}{[M]^{(1)}}\frac{\partial^2}{\partial t^2}\underset{(10\times 1)}{\{u\}}+\frac{\partial}{\partial x}\underset{(10\times 1)}{\{\Gamma\}^{(1)}}=\{F\}^{(1)} \tag{120a}$$

where

$$[M]^{(1)}=\begin{bmatrix} -B0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -C0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{120b}$$

$$\{\Gamma\}^{(1)}=\begin{Bmatrix} K*G0*(w0x+p0) \\ A0*p0x-K*G0*w0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix},\quad \{F\}^{(1)}=\begin{Bmatrix} 0 \\ K*G0*p0 \\ w0-w1 \\ w0-w2 \\ w0-w3 \\ w0-w4 \\ p0-p1 \\ p0-p2 \\ p0-p3 \\ p0-p4 \end{Bmatrix} \tag{120c}$$

Similarly, one can write partial differential equations for other zones in the COMSOL Multiphysics standard form

Partial differential Eqs. (46)-(55) for Zone 1 (Subdomain 2), i.e. for $x \in [a, \alpha]$

$$\underset{(10\times 10)}{[M]^{(2)}}\frac{\partial^2}{\partial t^2}\underset{(10\times 1)}{\{u\}}+\frac{\partial}{\partial x}\underset{(10\times 1)}{\{\Gamma\}^{(2)}}=\{F\}^{(2)} \tag{121}$$

Partial differential Eqs. (56)-(65) for Zone 2 (Subdomain 3), i.e. for $x \in [\alpha, \beta]$

$$\underset{(10\times 10)}{[M]^{(3)}}\frac{\partial^2}{\partial t^2}\underset{(10\times 1)}{\{u\}}+\frac{\partial}{\partial x}\underset{(10\times 1)}{\{\Gamma\}^{(3)}}=\{F\}^{(3)} \tag{122}$$

Partial differential Eqs. (66)-(75) for Zone 3 (Subdomain 4), i.e. for $x \in [\beta, L]$

$$\underset{(10\times 10)}{[M]^{(4)}}\frac{\partial^2}{\partial t^2}\underset{(10\times 1)}{\{u\}}+\frac{\partial}{\partial x}\underset{(10\times 1)}{\{\Gamma\}^{(4)}}=\{F\}^{(4)} \tag{123}$$

Matrices, which enter into Eqs. (121)-(123) are not written here explicitly for brevity.

The Dirichlet boundary conditions (76)-(77) at Boundary 1, i.e. at $x = 0$, written in COMSOL Multiphysics standard form, are

$$\underset{(10\times 1)}{\{R\}^{(1)}}=\underset{(10\times 1)}{\{0\}}\ \text{and}\ -\underset{(10\times 1)}{\{\Gamma\}^{(1)}}+\underset{(10\times 10)}{\left[\frac{\partial R_m^{(1)}}{\partial u_k}\right]^T}\underset{(10\times 1)}{\{\lambda\}}=-\underset{(10\times 1)}{\{G\}^{(1)}} \tag{124a}$$

where the column-matrix $\{\Gamma\}^{(1)}$ is defined by formula (120c),

$$\underset{(10\times 1)}{\{R\}^{(1)}}\equiv[w0\ \ p0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0] \tag{124b}$$

$$\underset{(10\times 1)}{\{G\}^{(1)}}=\underset{(10\times 1)}{\{0\}} \tag{124c}$$

and

$$\underset{(10\times 10)}{\left[\frac{\partial R_m^{(1)}}{\partial u_k}\right]^T}=\begin{bmatrix}1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix} \tag{124d}$$

but the matrix $[\partial R_m^{(1)}/\partial u_k]^T$ need not be defined by a user of COMSOL Multiphysics.

Similarly, one can write boundary conditions for all other external and internal boundaries in the COMSOL Multiphysics standard form.

The *Dirichlet boundary conditions (78)-(83) at an internal Boundary 2,* i.e. at $x = a$

$$\underset{(10\times 1)}{\{R\}^{(2)}}=\underset{(10\times 1)}{\{0\}}\ \text{and}\ \underset{(10\times 1)}{\{\Gamma\}^{(1)}}-\underset{(10\times 1)}{\{\Gamma\}^{(2)}}+\underset{(10\times 10)}{\left[\frac{\partial R_m^{(2)}}{\partial u_k}\right]^T}\underset{(10\times 1)}{\{\hat{\lambda}\}}=-\underset{(10\times 1)}{\{G\}^{(2)}} \tag{125}$$

The *Dirichlet boundary conditions (84)–(93) at an internal Boundary 3,* i.e. at $x = \alpha$

$$\underset{(10\times 1)}{\{R\}^{(3)}}=\underset{(10\times 1)}{\{0\}}\ \text{and}\ \underset{(10\times 1)}{\{\Gamma\}^{(2)}}-\underset{(10\times 1)}{\{\Gamma\}^{(3)}}+\underset{(10\times 10)}{\left[\frac{\partial R_m^{(3)}}{\partial u_k}\right]^T}\underset{(10\times 1)}{\{\hat{\lambda}\}}=-\underset{(10\times 1)}{\{G\}^{(3)}} \tag{126}$$

The *Dirichlet boundary conditions (94)–(103) at an internal Boundary 4,* i.e. at $x = \beta$

$$\underset{(10\times 1)}{\{R\}}^{(4)} = \underset{(10\times 1)}{\{0\}} \text{ and } \underset{(10\times 1)}{\{\Gamma\}}^{(3)} - \underset{(10\times 1)}{\{\Gamma\}}^{(4)} + \underset{(10\times 10)}{\left[\frac{\partial R_m^{(4)}}{\partial u_k}\right]^T} \underset{(10\times 1)}{\{\bar{\lambda}\}} = -\underset{(10\times 1)}{\{G\}}^{(4)} \tag{127}$$

The *Nuemann boundary conditions (104) and (105) at an external Boundary 5*, i.e. at $x = L$, written in COMSOL Multiphysics standard form, are

$$\underset{(N\times 1)}{\{\Gamma\}}^{(4)} = -\underset{(N\times 1)}{\{G\}}^{(5)} \tag{128}$$

Matrices, which enter into Eqs. (125)-(128), are not written here explicitly for brevity.

6. Solution of example problems

As an example problem, a clamped-free wooden beam with the following characteristics (Fig. 1) is considered: length $L = 20 \times 10^{-2}$ m, width $b = 2.76 \times 10^{-2}$ m, thickness $h = 0.99 \times 10^{-2}$ m, wood density $\rho^{(0)} = 418.02$ kg/m^3, Young's modulus of the wood in the direction of fibers $E_1^{(0)} = 1.0897 \times 10^{10}$ N/m^2. The piezoelectric actuator is QP10W (Active Control Experts). Thickness of the actuator is $\tau = 3.81 \times 10^{-4}$ m, its length is $a = 5.08 \times 10^{-2}$ m, the piezoelectric constant in the range of applied voltage (from 0 to 200V) is $\bar{d}_{31} \approx -1.05 \times 10^{-9}$ m/V, the Young's modulus of the actuator with its packaging is $E_1^{(p)} = 2.57 \times 10^{10}$ N/m^2, mass density of the actuator with its packaging is $\rho^{(p)} = 6151.1$ kg/m^3. The voltage $V(t)$, applied to the piezoelectric actuator, is distributed uniformly along the length of the actuator and varies with time as

$$V(t) = V_a \sin(\Omega t)$$

where $V_a = 200$ V, $\Omega = 600$ 1/s. The wooden beam is cut along its fibers, so that the angle θ in the formula (6) is equal to zero, and, therefore, the elastic compliance coefficient $\bar{S}_{11}$ for the wood is equal to $\bar{S}_{11}^{(0)} = 1/E_1^{(0)} = 9.1768 \times 10^{-11}$ m$^2/N$. For the piezoelectric actuator, the material coordinate system coincides with the problem coordinate system, so that the elastic compliance coefficient $\bar{S}_{11}$ for the material of the piezo-actuator is $\bar{S}_{11}^{(p)} = 1/E_1^{(p)} = 3.8911 \times 10^{-11}$ m$^2/N$. Coordinates of the crack tips are: $\alpha = 10 \times 10^{-2}$ m, $\beta = 15 \times 10^{-2}$ m, $\gamma = 0.66 \times 10^{-2} - h/2 = 1.65 \times 10^{-3}$ m. Then the constants, entering into the variational formulation and the differential equations of the problem, have the following values in SI units (Perel 2005): $A_0 = 31.463$, $B_0 = 0.1789$, $C_0 = 2.6429 \times 10^{-6}$, $G_0 = 1.29910 \times 10^6$, $A_1 = 24.319$, $B_1 = 0.11422$, $C_1 = 9.3289 \times 10^{-7}$, $G_1 = 1.190999 \times 10^6$, $A_2 = 12.61$, $B_2 = 7.6147 \times 10^{-2}$, $C_2 = 4.8372 \times 10^{-7}$, $G_2 = 7.93999 \times 10^5$, $A_3 = 11.709$, $B_3 = 3.8073 \times 10^{-2}$, $C_3 = 4.4917 \times 10^{-7}$, $G_3 = 3.969995 \times 10^5$, $A_4 = 24.319$, $B_4 = 0.11422$, $C_4 = 9.3289 \times 10^{-7}$, $G_4 = 1.190999 \times 10^6$, $I_p = -3.8285 \times 10^{-3}$, $a = 5.08 \times 10^{-2}$, $V_a = 200$, $\Omega = 600$, $\alpha = 10 \times 10^{-2}$, $\beta = 15 \times 10^{-2}$, $\gamma = 1.65 \times 10^{-3}$, $b = 2.76 \times 10^{-2}$, $h = 0.99 \times 10^{-2}$. The small constant and the large constant χ in Eqs. (5) and (6) are chosen to be $\varepsilon = 1 \times 10^{-3}$ and $\chi = 1 \times 10^6$. The shear correction factor K in expressions for strain energy is set to $K = 5/6$.

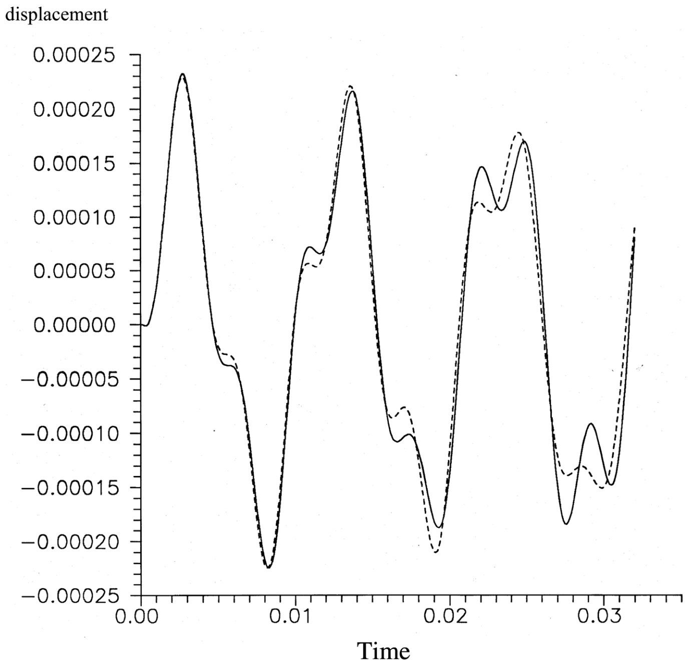

Fig. 2 Transverse displacement of free end of delaminated beam (solid line) and undelaminated beam (dashed line). Coordinates of the crack tips of the delaminated beam are $\alpha = 1.0$ m, $\beta = 15.0$ m, $\gamma = 65.1 \times 10^{-3}$ m

7. Time-domain response to dynamic excitation

A system of ordinary differential equations of a global (assembled) semi-discrete finite element model has the form

$$[M]\{\ddot{\Theta}\} + [M]\{\Theta\} + \{R\}_{\text{nonlin}} = \{F\} \tag{129}$$

In the last equation, $\{R\}_{\text{nonlin}}$ is a column-matrix, which contains components that depend nonlinearly on the unknown nodal parameters Θ_i. Transverse displacements as functions of time at free ends of delaminated and undelaminated beams, obtained by solving Eq. (129), are shown in graphs of Fig. 2. These graphs are noticeably different. Numerical experiments show that this difference is mainly due to the mutual impact of the crack faces during the vibration.

So, taking account of nonlinearity of the forced response of the delaminated beam due to the contact interaction of the crack faces can be important for model-aided detection of cracks in composite beams.

8. Conclusions

The method of analyzing vibration of the composite delaminated beam, presented in this paper, is suitable for studying the effects of the crack length and the crack location on the vibration of the beam, because this method allows for automatic finite element mesh generation after each increment of the coordinates of the crack tips. Besides, the model presented in this paper has an increased accuracy due to taking account of the effect of contact of the crack faces on the vibration.

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