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On procedures for reliability assessment of mechanical systems and structures

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Abstract. In this paper a brief overview of methods to assess the reliability of mechanical systems and structures is presented. A selection of computational procedures, stochastic structural dynamics, stochastic fatigue crack growth and reliability based optimization are discussed. It is shown that reliability based methods may form the basis for a rational decision making.

Keywords: structural reliability; uncertainty analysis; computational methods.

1. Introduction

The realistic modelling of structures and the loading conditions to which they are expected to be exposed during their design life as well as the mechanisms of their possible deterioration with time emerged to be one of the major issues in structural mechanics. It has been recognized already at an early stage - i.e. in the mid twentieth of the last century - that this requires for most cases also the quantitative consideration of the statistical uncertainties of the events, mechanisms and consequently the parameters involved. The treatment of uncertainties has always been part of structural mechanics and analysis, however, in a more intuitive way, e.g. by selecting maximum, minimum and mean values for loading, material strength and geometric values respectively. A more rational treatment of the uncertainties is accomplished by applying probabilistic and stochastic methods. From this the fields of Stochastic Structural Mechanics and Structural Reliability have emerged. As the modelling of the parameters requires the processing of considerably more information than when carrying out traditional deterministic analyses, appropriate procedures are required to accomplish this task. While for smaller types of problems as well as for certain conditions analytical and semi-analytical methods may suffice, problems of larger and more general type require generally the use of numerical procedures. Hence stochastic computational procedures should be also compatible with discretization techniques as currently used, e.g. Finite and Boundary Element procedures respectively.

In this work an attempt is made to survey the development of some selected procedures in

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stochastic structural mechanics and reliability. For reasons as stated, emphasis will be put on computational procedures. It is well understood that procedures dealing with stochastic systems are computationally considerably more involved than their deterministic counterparts, because the parameter set assumes a (finite or infinite) number of values in contrast to a single point in the parameter space. Hence, in order to be competitive and tractable in practical applications, the computational efficiency of the procedures used is a crucial issue. This applies to methods of representing stochastic processes to model adequately random fluctuations in time and space - also by random fields and wavelets - as well as to methods of assessing the stochastic response of structural systems. Typical examples of engineering interest of the first aspect are the ground motion of earthquakes, the sea state, wind turbulence, imperfections of shells, fluctuating properties of random media, etc., while the second aspect is mainly concerned with the properties of the structural model, i.e. whether it is assumed to be deterministic or stochastic, linear or non-linear, as well as the number of degrees of freedom involved, etc.. For problems where the stochasticity of the mechanical system becomes important, the procedures of Stochastic Finite Elements (SFE) are generally applied. For example in static analysis the stiffness matrix might be random due to unpredictable variation of some material properties, random coupling strength between components, uncertain boundary conditions, etc.. For buckling analysis, shape imperfections of the structure may have an important effect on the buckling load. In dynamic analysis, in addition to the stiffness, the damping properties and sometimes also the mass matrix might not be predictable with certainty. This leads directly to the problem of random eigenvalues. However, contrary to the theory of deterministic systems subjected to stochastic excitation the corresponding theories of uncertain linear and non-linear systems respectively are still somewhat less developed. This in part is also the case for non-Gaussian properties of the excitation. The methods of stochastic structural mechanics form the basis for assessing the reliability of mechanical systems and structures. For this purpose various procedures, i.e. of exact, accurate and approximate characteristics have been developed and applied so far. The advantages and limitations have already been discussed at length elsewhere, however, remarks concerning their properties w.r.t. their capabilities to solve problems of the engineering practice will be made. This applies particularly to dynamical systems with high numbers of degrees of freedom.

As structures and mechanical systems are generally designed to serve for a particular lifetime, the deterioration effects, e.g. due to cyclic loading, corrosion and other effects of similar nature, also have to be considered in a credible reliability analysis. Methods of fatigue, fracture as well as damage analysis are used for this purpose.

Finally the rational treatment of uncertainties provides a sound basis for Reliability Based Optimization (RBO). Although the field of structural optimization is already quite developed, RBO is still in its infancy, particularly in view of the possibility to solve problems of higher dimensions. It is shown that for most problem areas of stochastic structural mechanics and reliability the Monte Carlo simulation procedure, either in its direct or advanced forms proves to be not only generally applicable but also both accurate and in its advanced form also computationally efficient. As indicated above, in the following - due to space limitations - only a selected number of procedures of stochastic structural mechanics and reliability can be discussed. For a more detailed and extensive review of a wider scope of the current developments in this field it is referred to e.g. Schuëller (1997, 2005a, 2005b, 2006), Schuëller and Spanos (2001), Kleiber (2005), Ghanem and Spanos (1991), Schenk and Schuëller (2005).

There are a number of computational aspects receiving currently considerable attention. This

applies particularly to the procedures based on the mentioned Monte Carlo simulation (MCS) which address on one hand the questions of generating statistical samples that must be compatible with prescribed stochastic information such as spectral density, correlation, distributions etc. for the purpose of e.g. unconditional simulation of stochastic processes, fields and wavelets, conditional simulation compatible with observations and a priori statistical information, etc.. On the other hand the assessment of the stochastic response for mathematical models with prescribed statistics (random loading/system parameters) of the parameters is in the focus of interest. This implies, for example, the indiscriminate (blind) generation of samples, numerical integration of SDE's, representative samples for reliability assessment by generating adverse rare events by variance reduction techniques, the realization of RV's, controlling the evolution of sampling functions in time, etc.. For structural response calculations a distinction between the structural model properties involved is essential, i.e. whether it is deterministic or stochastic, linear or non-linear, number of degrees of freedom (DOF) involved, etc..

Two aspects seem to be of major importance for development of the field, i.e.

(1) Similarly as for deterministic structural analysis, the stochastic counterpart should be also adjusted to meet the new respective requirements posed by the computational facilities.

(2) The expansion of the stochastic analysis to treat higher dimensional problems is a necessary requirement for their application in the engineering practice.

2. Representation of random processes and fields

A quite general spectral representation utilized for Gaussian random processes and fields is the Karhunen Loève (KL) expansion of the covariance function (see e.g. Ghanem and Spanos 1991). This representation is applicable for stationary (homogeneous) as well as for non-stationary (inhomogeneous) stochastic processes (fields). The expansion of a stochastic process (field) $u(\mathbf{x}, \theta)$ takes the form,

$$u(\mathbf{x}, \theta) = \overline{u}(\mathbf{x}) + \sum_{i=1}^{\infty} \xi_i(\theta) \sqrt{\lambda_i}\, \phi_i(\mathbf{x}) \tag{1}$$

where the symbol θ indicates the random nature of the corresponding quantity and where $\overline{u}(\mathbf{x})$ denotes the mean, $\phi_i(\mathbf{x})$ are the eigenfunctions and λ_i the eigenvalues of the covariance function, respectively. $\{\xi_i(\theta)\}$ is a set of orthogonal (uncorrelated) zero mean random variables with unit variance. The KL expansion is mean square convergent irrespective of its probabilistic nature, provided it possesses a finite variance. For the important special case of a Gaussian process or field the random variables $\{\xi_i(\theta)\}$ are independent standard normal random variables. In many practical applications where the random quantities vary smoothly with respect to time or space, only few terms are necessary to capture the major part of the random fluctuation of the process. Its major advantage is the reduction from a large number of correlated random variables to few most important uncorrelated ones. Hence this representation is especially suitable for band limited colored excitation and stochastic finite element (SFE) representation of random media where random variables are usually strongly correlated. It might also be utilized to represent the correlated stochastic response of MDOF-systems by a small subset of the most important variables and hence achieving a space reduction.

As already mentioned above, one could distinguish between two main categories of computational procedures treating the response of stochastic systems. The first is based on MCS and the second provides numerical solutions of analytical procedures for obtaining quantitative results. Regarding the numerical solutions of analytical procedures, a clear distinction between dynamical system-models and FE models should be made. Most recent research efforts in stochastic dynamics focus to a large extent on dynamical-system-models while there are few new numerical approaches concerning the evaluation of the stochastic dynamic response of e.g. FE models (see e.g. Schuëller 2006).

3. Monte Carlo simulation based procedures

Several aspects favor procedures based on MCS in engineering applications (see e.g. Schuëller 1997, 2005a, Schuëller and Spanos 2001):

(1) Considerably smaller growth rate of the computational efforts with dimensionality than analytical procedures.
(2) Generally applicable, well suited for parallel processing and computationally straightforward.
(3) Non-linear complex behavior does not complicate the basic procedure.
(4) Manageable efforts for complex systems.

Contrary to numerical solutions of analytical procedures, the employed structural model and the type of stochastic loading do for MCS not play a decisive role. For this reason, MCS procedures might be structured according to their purpose i.e. where sample functions are generated either for the estimation of the overall distribution or for generating rare adverse events for an efficient reliability assessment. In the former case, the probability space is covered uniformly by an indiscriminate generation of sample functions representing the random quantities. Basically, a set of random variables will be generated by a pseudo random number generator followed by a deterministic structural analysis. Based on generated random numbers realizations of random processes, fields, etc. are constructed and utilized without any further modification in the following structural analysis. The situation may not be considered to be straightforward, however, in case of a discriminate MCS for the reliability estimation of structures, where rare events, contributing considerably to the failure probability, should be generated. Since the effectiveness of direct indiscriminate MCS is not satisfactory for producing a statistically relevant number of low probability realizations in the failure domain, the generation of samples is restricted or guided in some way. Most important is the class of variance reduction techniques which operate on the probability of realizations of random variables. The most widely used representative of this class in structural reliability assessment is importance sampling where a suitable sampling distribution controls the generation of realizations in the probability space. The challenge e.g. in importance sampling is the construction of a suitable sampling distribution which depends in general on the specific structural system and on the failure domain (see e.g. Schuëller and Stix 1987). Hence, the generation of sample functions is no longer independent of the structural system and failure criterion as for indiscriminate direct MCS. Due to these dependencies, computational procedures for an automated establishment of sampling distributions are urgently needed.

Recent challenges in computational stochastic structural mechanics refer to reliability procedures for problems with high dimension (see e.g. Schuëller *et al.* 2004). The respective applicability and

computational efficiency of the various methods available (see Beck *et al.* 2006, Pradlwarter *et al.* 2006, Katafygiotis and Cheung 2006, Jensen and Valdebenito 2006, Ghanem *et al.* 2006) are compared in terms of a Benchmark study (Schuëller *et al.* 2005). The results show that the so called subset simulation procedures proved to be most versatile.

4. Stochastic Finite Elements (SFEs)

With regard to stochastic system properties it should be pointed out that part of these uncertainties can be quantified by SFEs, which are structural models represented by FEs involving randomness. In static analysis, the stiffness matrix might be random due to unpredictable variation of some material properties, random coupling strength between structural components, uncertain boundary conditions, etc.. For buckling analysis, shape imperfections of the structures have an additional important effect on the buckling load (see e.g. Schuëller *et al.* 2004). Considering structural dynamics, in addition to the stiffness matrix, the damping properties and sometimes also the mass matrix might not be predictable with certainty. Discussing numerical SFE procedures, two categories should be distinguished clearly. The first is the representation of SFEs and their global assemblage as random structural matrices. The second category addresses the evaluation of the stochastic response of the FE model due to its randomness. Focussing first on the FE representation, several representations such as the mid-point method, the interpolation method, the local average method and more recently the weighted-integral method have been developed to describe spatial random fluctuations within the element. After assembling all FEs, the random structural stiffness matrix $\mathbf{K}$, taken as representative example, assumes the form,

$$\mathbf{K}(\alpha) = \overline{\mathbf{K}} + \sum_{i=1}^{n} \mathbf{K}_i^I \alpha_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{K}_{ij}^{II} \alpha_i \alpha_j \tag{2}$$

where $\overline{\mathbf{K}}$ is the mean of the matrix, $\mathbf{K}_i^I$ and $\mathbf{K}_{ij}^{II}$ denote the deterministic first and second rate of change with respect to the zero mean random variables α_i and α_j and n is the total number of random variables. For a normally distributed set of random variables $\{\alpha_i\}$, the correlated set can be represented advantageously by the KL expansion (Ghanem and Spanos 1991).

It is well known, that the physical relevance of the results of the SFE analysis is based on measured data, i.e. in terms of linking micromechanics of materials with SFEs. In this context a recent numerical study has shown (Charmpis and Schuëller 2005) that it is of major importance to gather data on the correlation length, the probability distribution function, the mean and the variance of stochastic fields for various material properties. Moreover, based on engineering reasoning, it is of major interest to explore the assumptions of statistical homogeneity and isotropy usually made for random fields. SFE would largely benefit from micromechanical information gathered for the spatial variation characteristics of various structural material types.

5. Random eigenvalues

The solution of a number of engineering problems depends on a previous solution of an eigenvalue problem, e.g. the first eigenvalue is of interest for stability considerations, the first few

eigenvectors can be utilized to reduce the overall system size for a linear dynamic analysis, etc.. Taking into consideration that parameters of the systems may be uncertain and described in probabilistic terms, the eigenvectors and eigenvalues are of course random as well. A brief overview on some techniques for solving these random eigenvalue problems is given in the following. Although there are a number of cases where random eigenvalues of continuous systems are of interest from the point of view of structural engineering, the solution of random eigenvalues of discrete(ized) systems are of more immediate concern. The eigenvalue problem to be solved is generally of the type

$$\mathbf{A}\phi_i = \lambda_i \mathbf{B}\phi_i \tag{3}$$

where **A**, **B** are the matrices describing the system with random parameters/fields (e.g. stiffness and mass matrices in case of a linear dynamic analysis) and λ_i, ϕ_i are the eigenvalues and eigenvectors respectively. Most of the available techniques for the analysis of this random eigenvalue problems focus on the solution for their first two moments (see e.g. Székely and Schuëller 2001). In this respect one can distinguish between: (1) perturbation method, (2) iteration method, (3) methods to calculate eigenvalue bounds and (4) crossing theory method.

From the point of view of reliability engineering, the tail regions of the distributions are most important. MCS methods however are available for solving this class of problems. For systems with low dimensions and also when using fast computers these methods in their direct form prove to be quite efficient. However, once the system size becomes large, the necessary effort for solving the eigenvalue problem increases such that the method becomes intractable. Basically, there are two possibilities to improve the efficiency. Firstly the sample size may be reduced, e.g. by importance sampling procedures, etc.. Secondly the efficiency of computing each sample based on available prior knowledge may be improved. For example, for large structures with random parameters and linear behavior one can take advantage of the fact that the difference between the eigenvectors (mode shapes) of the realizations of a structure is not much pronounced. This approach leads to a considerable reduction of computational efforts, but without the possibility of error quantification. However, if a further step is carried out by using the eigenvectors as start-vectors for subspace iteration - which is only an extension of the previous developments - and completes the iteration, the obtained results are then numerically sufficiently accurate and the computational efforts are still considerably smaller when compared to the straight forward scheme (see e.g. Pradlwarter *et al.* 2002). For strategies to increase computational efficiency in stochastic mechanics, e.g. by phase space reduction, parallel processing, etc. it is referred e.g. to Schuëller (2001, to appear, 2006).

6. Stochastic structural dynamics

Stochastic dynamics in the time domain has been applied until recently mainly to small systems, so called multi-degree-of-freedom systems, of moderate size, i.e. less than 100 degrees of freedom (DOF). These approaches usually represent the equation of motion in a first order differential equation denoted as state vector representation. The stochastic excitation is then either represented by white noise or filtered white noise, where the intensity might vary with respect to time in order to model non-stationary excitation. It should be noted that these stochastic excitation models however, are not easily adjusted to measured data, which might have strongly nonstationary

characteristics regarding the amplitudes and frequency content. Measured data allow in general, to derive an estimate for the covariance matrix and therefore, the excitation should reflect the estimated covariance matrix as closely as possible. An additional difficulty to deal with FE models, which have in general thousands of DOF, is the computation of the covariance matrix of the displacement and velocity response, respectively. Due to its size (storage) and computations involved, the full symmetric covariance matrix cannot be computed efficiently. An alternative is the use of size reduction procedures. One of these approaches is based on complex modal analysis. Since the integration is performed in a reduced (truncated) modal space, the efficiency during integration is virtually independent of the number of DOF. However, this requires first the solution of the complex eigenvalue problem. Complex modal analysis in context of large FE models might be acceptable, as long as the eigenvalue problem needs not to be solved repeatedly. Another possibility in avoiding storage and direct integration of the full covariance matrix is to work directly with its KL representation, i.e. integrating a finite set of deterministic KL vectors.

For the case where the stochastic loading is described by a finite set of deterministic KL vectors, which in fact is considered, the standard FE procedures as developed for deterministic dynamic analysis can be employed. Clearly, for linear analysis and Gaussian excitation, the stochastic response can also be described uniquely by the KL expansion, where to each deterministic KL vector of the loading corresponds to a vector of the response. Treating nonlinear systems by equivalent linearization, the response of the linearized system can still be described by a KL representation. The linearized system is then a function of the stochastic response, which in turn is specified by all KL vectors. This leads in general to a time varying linear system. Very similar as for a linear analysis, to each KL vector of the loading, the resulting response vector can be computed in a deterministic manner. To enhance the computational efficiency, constant structural matrices are used. All nonlinear restoring forces are considered at the loading side of the equation of motion, where mutual dependencies are resolved by iteration. Nonlinear elements are linearized and their evolution in time is integrated independently of each other in each time step. This allows to avoid a state space representation as traditionally performed in stochastic dynamics, and hence leads to a procedure well adjusted to the stochastic response evaluation of FE system of large size, e.g. with thousands of DOF.

Most nonlinear structural dynamic FE problems can be cast into the following equation of motion

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{r}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{q}) = \mathbf{f}(t) \tag{4}$$

where $\mathbf{u}$ denotes the generalized displacement vector, a dot ' ˙ ' derivatives w.r.t. time t, $\mathbf{f}(t)$, the external force vector and $\mathbf{r}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{q})$ the nonlinear restoring force. The mass matrix $\mathbf{M}$ is assumed to be constant. The internal restoring force vector $\mathbf{r}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{q})$ is generally a function of the displacements $\mathbf{u}$, its velocities $\dot{\mathbf{u}}$ and of a set of variables $\mathbf{q}$ which describe the respective state of the nonlinear elements. The evolution of these variables may be described by a first order differential equation, i.e.,

$$\dot{\mathbf{q}} = \mathbf{g}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{q}) \tag{5}$$

where the vector function $\mathbf{g}$ is generally nonlinear. The equation of motion (4) is not well suited for treating systems with a large number of DOF. However, the response can be computed quite efficiently for FE systems with constant matrices. Therefore, (4) is reformulated by splitting the

nonlinear restoring force vector $\mathbf{r}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{q})$ into a linear and non-linear part using a constant matrix $\mathbf{R}$, i.e.,

$$\mathbf{r}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{q}) = \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} + \mathbf{R}\mathbf{q}(\mathbf{u}, \dot{\mathbf{u}}) \tag{6}$$

where $\mathbf{C}$ and $\mathbf{K}$ denote the damping and stiffness matrix, respectively, and it is assumed that the variables $\mathbf{q}$ are selected such that (6) holds, yielding

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t) - \mathbf{R}\mathbf{q}(\mathbf{u}, \dot{\mathbf{u}}) \tag{7}$$

Since the right hand side of (7) is now a function of the response $(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{q})$ iterations procedures will be required to satisfy the equilibrium conditions. The smaller the term $\mathbf{Rq}$ the faster the convergence will be. Hence, the constant matrices $\mathbf{C}$ and $\mathbf{K}$ should be selected such that the vector $\mathbf{Rq}$ remains small over a reasonable long time span. It can be seen from (7) that this formulation is especially well suited for cases where most of the elements of the structures remain linear and only a small part reveals nonlinear behavior. In most of these cases, the structural matrices $\mathbf{C}$ and $\mathbf{K}$ do not have to be updated with respect to time in order to ensure convergence of the proposed procedure. For systems where a larger number of non-linear elements are involved, the proposed procedure can still be applied but with increasing computational efforts due to possibly necessary updating of $\mathbf{C}$ and $\mathbf{K}$ with respect to time.

The nonlinear restoring force is most suitable and straight forward modeled in local element specific coordinates, with a minimal number of coordinates

$$\bar{\mathbf{r}}_L^{(e)} = -\mathbf{R}_L^{(e)}\mathbf{q}_L^{(e)}(\mathbf{u}_L^{(e)}\dot{\mathbf{u}}_L^{(e)}) \tag{8}$$

where the superscript '$^{(e)}$' denote the nonlinear element and the subscript '$_L$' indicates local coordinates (for global coordinates the subscript '$_G$' has been omitted where the meaning is clear). The local displacements $\mathbf{u}_L^{(e)}$ and velocities $\dot{\mathbf{u}}_L^{(e)}$ respectively, are related to the global displacements $\mathbf{u}$ and velocities $\dot{\mathbf{u}}$ by a linear transformation $\mathbf{T}^{(e)}$:

$$\mathbf{u}_L^{(e)} = \mathbf{T}^{(e)}\mathbf{u} \qquad \dot{\mathbf{u}}_L^{(e)} = \mathbf{T}^{(e)}\dot{\mathbf{u}} \tag{9}$$

The element specific quantity $\mathbf{q}^{(e)}$, of course, does not require a distinction between global and local, i.e.,

$$\mathbf{q}_L^{(e)} = \mathbf{q}_G^{(e)} = \mathbf{q}^{(e)} \tag{10}$$

Using the fundamental transformation laws, the forcing in global coordinates can be determined according to

$$\bar{\mathbf{r}}_G^{(e)} = [\mathbf{T}^{(e)}]^T \bar{\mathbf{r}}_L^{(e)} = -\mathbf{R}_G^{(e)}\mathbf{q}^{(e)} \tag{11}$$

where

$$\mathbf{R}_G^{(e)} = [\mathbf{T}^{(e)}]^T \mathbf{R}_L^{(e)} \tag{12}$$

All matrices $\mathbf{R}_G^{(e)}$ of the elements are finally assembled (added) into a matrix $\mathbf{R}$, yielding the pseudo load vector

$$\bar{\mathbf{r}}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{q}) = -\mathbf{R}\mathbf{q}(\mathbf{u}, \dot{\mathbf{u}}) \tag{13}$$

The stochastic loading vector can be represented also by the KL representation,

$$\mathbf{f}(t) \approx \mathbf{f}^{(0)}(t) + \sum_{j=1}^{3m} \bar{\xi}_i \mathbf{f}^{(j)}(t) \tag{14}$$

$$\approx -\mathbf{M}\mathbf{I}_\alpha \left(\alpha^0(t) + \sum_{j=1}^{m} \bar{\xi}_i \alpha^{(j)}(t) \right) \tag{15}$$

where, $\mathbf{f}^{(0)}(t)$ and $\mathbf{f}^{(j)}(t)$ denote the mean function and the j-th KL vector of $\mathbf{f}(t)$, respectively, $\mathbf{I}_\alpha$ is a displacement transformation matrix and $\alpha(t)$ e.g. a stochastic earthquake acceleration vector.

It should be noted, that so far, no procedure is available for an exact determination of the covariance matrix of a general nonlinear system with many DOF. An exact solution exists, however, for linear systems. Therefore, the nonlinear elements are linearized by statistical equivalent linearization - which in fact has developed from the deterministic equivalent linearization procedure - and the well developed procedures, see e.g. Pradlwarter and Li (1991), to compute the covariances of the response are then applied. Linearizing the nonlinear vector function (5) leads to a linear form. The first two moments of the linearized stochastic response are then

$$\mu_\chi(t) = \mathbf{E}\{\chi(t)\} = \chi^{(0)}(t) \tag{16}$$

and

$$\Gamma_{\chi\chi}(t_1, t_2) = \mathbf{E}\{(\chi(t_1) - \mu_\chi(t_1))(\chi(t_2) - \mu_\chi(t_2))^T\} = \sum_{j=1}^{3m} \chi^{(j)}(t_1)\chi^{(j)T}(t_2) \tag{17}$$

For the computational evaluation of Eqs. (16) and (17) it is referred to Schenk *et al.* (2004).

7. Stochastic fatigue crack

The strong randomness of fatigue crack growth is not only manifested by experimental evidence, but also by large safety factors used for fatigue life predictions (e.g. up to 4.0 in aerospace). Hence, a reliable analysis of fatigue prone structures can only be accomplished if this randomness inherent in the physical process is taken into account in a quantitative manner, e.g. by stochastic analysis.

Hence in the following the practical use of already proposed fatigue crack growth models is assessed (for a more detailed analysis see e.g. Rocha and Schuëller 1996). In engineering practice, there is often insufficient financial means as well as time available to carry out fatigue tests with a greater number of load cycles. As a consequence, there is a need to extrapolate the fatigue crack

growth prediction by means of stochastic models from data that were obtained from a limited number of load cycles.

In this context the requirements in complexity for a stochastic model of fatigue crack growth in order to approximately extrapolate the statistical characteristics of fatigue crack growth are also to be noted.

Elementary fatigue crack growth laws assume a relationship between the stress state at the crack tip and the crack growth velocity. Several deterministic crack growth laws have been proposed for fitting the mean values of experimentally measured data. These laws have then been extended to account for the random phenomena inherent in crack growth.

The extended crack growth law

$$\frac{d\alpha}{d\mathbf{N}} = C\frac{(1-\mathbf{R})^m \Delta\mathbf{K}^n}{((1-\mathbf{R})\mathbf{K}_c - \Delta\mathbf{K})^q} \tag{18}$$

where α is the crack length, $\mathbf{N}$ the number of load cycles, $\Delta\mathbf{K}$ the increment of the stress intensity, $\mathbf{K_c}$ the critical stress intensity factor and $\mathbf{R} = \sigma_{\max}/\sigma_{\min}$ the stress ratio summarizes Paris' law ($m = q = 0$), Forman's law ($m = 0$, $q = 1$) and Walker's law ($q = 0$). Paris' law is a straight line in the $ln(d\alpha/d\mathbf{N}) - ln(\Delta\mathbf{K})$ representation. This is a good approximation for stable crack growth, but does not account for the different behavior of small cracks and in the range of instable crack growth. The typically S-shaped crack growth curve can only be approximated by a nonlinear relationship between the logarithm of the crack growth velocity and the logarithm of the stress intensity factor, respectively. For example, a cubic polynomial approximation would lead to

$$ln\left(\frac{d\alpha}{d\mathbf{N}}\right) = \mathbf{C}_0 + \mathbf{C}_1 ln(\Delta\mathbf{K}) + \mathbf{C}_2 (ln(\Delta\mathbf{K}))^2 + \mathbf{C}_3 (ln(\Delta\mathbf{K}))^3 \tag{19}$$

where dependence on $\mathbf{R}$ can be accounted for by substituting Elber's effective stress intensity factor $\Delta\mathbf{K}_{eff} = (\mathbf{A} + \mathbf{B})\Delta\mathbf{K}$, where $\mathbf{A}$ and $\mathbf{B}$ are constants that have to be determined from experiments, for $\Delta\mathbf{K}$. A similar approximation can be obtained from the hyperbolic sine law

$$\frac{d\alpha}{d\mathbf{N}} = 10^{(\mathbf{C}_1 sinh(\mathbf{C}_2 (ln(\Delta\mathbf{K}) + \mathbf{C}_3)) + \mathbf{C}_4)} \tag{20}$$

Relatively simple stochastic crack growth models assume that the uncertainties in one or several parameters of the deterministic crack growth laws are modelled as random variables. These models, however, can not explain the variability of the crack growth rate during the crack growth process. Models based on stochastic differential equations, in fact, are suited to account for this type of variability. Crack growth equations with a time-correlated stochastic process and models with a jump process have been also suggested.

If the random variations of the crack growth process of a single specimen are attributed to material inhomogeneity, the correlation of the stochastic process should rather be attributed to the spatial dimension, a model could be of the form

$$\frac{d\alpha}{d\mathbf{N}} = \mathbf{f}(\Delta\mathbf{K})\mathbf{Z}(\alpha) \tag{21}$$

where $log\ \mathbf{Z}(\alpha)$ is a Gaussian process whose correlation depends on the crack length.

Markov chain models reflect the fact that the load process is often discretized into independent events and that an absorbing state - failure - can be introduced. Markov chain models can be directly fitted to experimental data. However, this makes predictions for other load conditions or geometrical configurations a difficult task. This problem can be circumvented by using a suitable stochastic crack growth model for the determination of the transition probabilities.

It should be noted that the stochastic analysis of fatigue crack growth allows a rational modelling of NDE-techniques for crack inspection, etc., as well as the performance of quality assurance (see e.g. Rocha and Schuëller 2005).

Recently, particularly in aircraft engineering, crack initiation models play an increasingly important role. This is partially due to the high performance initial crack detection techniques, by which initial cracks can be excluded. Besides microscale models, particularly for practical applications, cohesive zone elements are being used for this purpose. The effect of randomness in the cohesive strength may be taken into account by using an extrapolation scheme (see e.g. Koutsourelakis *et al.* 2006).

8. Reliability Based Optimization (RBO)

One of the major goals of structural reliability theory is that the design of structural systems can be optimized based on reliability concepts. For this purpose the design problem has to be formulated following a well defined procedure by transcribing the verbal description of the RBO problem into an appropriate mathematical statement. This formulation contains the identification of deterministic design variables and the definition of the criterion needed to judge the qualities of the design alternatives. For the assessment of the adequacy of a newly designed structure the structural reliability or the probability of failure may be utilized. The feasibility of the system design may be defined by a given set of constraints which include the limitations on member sizes, structural response, risk of failure, etc.. In the literature several reliability-based optimization approaches have been suggested. In most cases, however, decision criteria formulated as constrained single objective functions aimed at the minimization of the structural weight or total lifetime costs, by taking into account reliability constraints, are considered. For these types of problems one basic difficulty is the specification of the constraints, e.g. the definition of the acceptable probability of failure. Therefore, a meaningful structural optimization can be performed only within a framework of potential consequences of structural failure, where the economic probability of failure makes the sum of all anticipated costs a minimum.

The reliability-based optimization problem can generally be formulated as follows: Determine a design vector $\mathbf{x}$ such that

$$\mathbf{x} = arg\ min\, f(\mathbf{x}, \mathbf{r}(\mathbf{x})) \tag{22}$$

subject to

$$h_j(\mathbf{x}, \mathbf{r}(\mathbf{x})) \leq 0; \quad j = 1,..., m \tag{23}$$

where f represents the design objective, $\mathbf{r}$ denotes the vector of considered reliability measures, and m is the number of constraints h_j, in (23) the side constraints of the design variables are included.

The vector **r** contains either safety indices or probabilities of failure with respect to specific serviceability or total failure criteria. For example, a rather often used RBO approach is to minimize simply the weight of the structure under investigation taking into account failure or safety index constraints. This idea to consider the structural weight as objective function is derived from pure deterministic design aims in civil engineering.

In the simplest form such a RBO problem may write, e.g. for a truss or frame structure

$$\mathbf{x} = arg\ min\ W(\mathbf{x}), \quad W(\mathbf{x}) = \sum_{j=1}^{m} A_j(\mathbf{x}) l_j \rho_j \tag{24}$$

subject to

$$p_{fj}(\mathbf{x}) \le p_{fj}^{accept}; \quad j = 1, ..., m \tag{25}$$

where the objective function W represents the weight, A_j is the cross-sectional area of element j of the length l_j, ρ_j is the constant unit weight, p_{fj} is the probability of failure for component or structural system j, and p_{fj}^{accept} is the acceptable prescribed failure probability or target reliability for component or system j. The design vector **x** contains for example cross-sectional dimensions whereas the global topology remains unchanged.

As already mentioned before, an improved decision making becomes possible when the risk of failure in terms of expected failure consequence costs are balanced with the structural costs. For this purpose an objective function has to include basically three types of costs: the initial construction costs and the expected consequences related to partial and total system failure respectively. The general reliability-based optimization problem formulation meeting these requirements may write as follows:

$$E[C(\mathbf{x}, \mathbf{Y})] = c_I(\mathbf{x}) + \sum_{j=1}^{m} E[C_{f,j}(\mathbf{x}, \mathbf{Y})] \rightarrow min \tag{26}$$

where $C(\mathbf{x}, \mathbf{Y})$ represents the total costs, $c_I(\mathbf{x})$ reflect the initial structural costs respectively the costs of safety, and $C_{f,j}(\mathbf{x}, \mathbf{Y})$ are the expected costs due to failure with, respectively regard to criterion j, i.e. partial damage, loss of serviceability or system collapse, respectively. The considered random variables describing the statistical uncertainties of the structural system are contained in the vector **Y**. Constraints (25) can be taken into account additionally, if a specific level of reliability has to be achieved. However, for the quality of the decision making this has no significant influence, because failure consequence costs are included in the objective function (26). For the simplified case of taking into account only one single partial failure criterion related to a specified structural damage state and a small total failure probability, the cost function (26) can be rewritten:

$$\mathbf{E}[C(\mathbf{x})] \approx c_I(\mathbf{x}) + c_s p_{fs}(\mathbf{x}) + c_c\, p_{fc}(\mathbf{x}) \rightarrow min \tag{27}$$

where p_{fs} and p_{fc} are probabilities of partial and total failure respectively. The corresponding consequence costs are represented by the factors c_s and c_c.

Although both optimization problems defined in (24) to (27) look rather simple, their solution appears to be an extensive problem, particularly in case of real world structures exhibiting complex response behavior. It should be noted that probabilities of failure used in the optimization problem

formulations are dependent on the probability structure of the uncertainties **Y** and the structural design **x**. This requires the repeated determination of the probabilities of failure which consequently requires the time-consuming calculation of multi-dimensional probability integrals. Hence, efficient reliability analysis and optimization methods have to be developed for solving such involved optimization problems. In this context, approximations within the scope of reliability analyses and optimization procedures are almost unavoidable for reduction of the computational effort, but they must provide sufficient accuracy to obtain reliable results.

For reliability based structural optimization several strategies may be pursued. Probably the most general approach to RBO is the direct connection of structural analysis, reliability analysis, and mathematical programming. The straightforward strategy has been applied quite frequently, because it can be carried out almost without any problems when limit state conditions can be mathematically formulated, i.e. in case of truss or simple frame structures. The main feature of this approach is that the calculation of the reliability measures have to be carried out repeatedly during the iterative process. Every time, when the design variables are changed due to directives of the optimization algorithm, the limit state functions and further the reliability measures have to be reevaluated. At this point the main problem occurs. For real complex structures the evaluation of limit state conditions (highly nonlinear functions) is very time-consuming which - in final consequence - becomes impractical for engineering applications. Therefore, additional approximation strategies have to be developed to reach the efficiency required for practical real world applications. An approximation of the reliability measures introduced in the problem formulations is proposed for this purpose. The intention of this concept is to reduce the computational effort due to repeated reliability analyses during the reliability-based optimization process and sensitivity analyses with respect to the design variables. Keeping in mind the response surface methodology, where the limit state function is approximated by an interpolating function, one can represent reliability measures using approximate functions, i.e. depending on the design variables. The advantage is that during the optimization procedure the reliability measure **r** in (22) and (23) can be approximated by the explicit function $\overline{\mathbf{r}}$.

The use of exponential functions to represent the probability of rare events is used here. It was found that for small values the probability of failure can be approximated with sufficient accuracy applying this approach. Provided that the probability of failure p_f is small over the complete design space, the following approximation is thus proposed:

$$\overline{\overline{p_f}} \approx p_f(\mathbf{x}) \tag{28}$$

where

$$\overline{\overline{p_f}} = e^{a + \mathbf{b}^T\mathbf{x} + \mathbf{x}^T\mathbf{C}\mathbf{x}} \tag{29}$$

The parameters a, **b** and **C** are the polynomial coefficients which have to be determined by solving a linear equation system. The equation system is formulated by estimating failure probabilities for specific design variable combinations and interpolating the polynomial function in the exponent of (29). The interpolation points for the approximate probability function are determined as follows:

$$ln(\overline{\overline{p_f}}(\mathbf{x}_j)) = ln(\overline{p_f}(\mathbf{x}_j)), \quad j = 1, ..., k \tag{30}$$

where by definition

$$\overline{p}_f(\mathbf{x}_j) = p[\overline{g}_f(\mathbf{x}_j, \mathbf{Y}) \leq 0] \tag{31}$$

In (31) $\overline{g}_f(\mathbf{x}_j, \mathbf{Y})$ is defined as the response surface for the actual design status $\mathbf{x}_j$. The probabilities $\overline{p}_f(\mathbf{x}_j)$ are the estimated failure probabilities using the RSM and e.g. importance sampling. For numerical examples it is referred to Gasser and Schuëller (2002).

9. Conclusions and outlook

Although in this paper - due to severe space limitations - only very limited aspects of procedures for reliability assessment of mechanical systems and structures could be discussed, (for a wider scope it is referred to the mentioned literature) the following conclusions apply to the entire field:

- In order to provide meaningful results, structural reliability should be well rooted in procedures of stochastic structural mechanics and analysis.
- Reliability may provide quantitative information on the risk of failure of structural and mechanical components and hence form the basis for rational decision making.
- Reliability based analysis and design concepts should include aspects of damage estimation, maintenance and repair, etc., so costs can be minimized while maintaining a particular target reliability (e.g. by optimization of life cycle costs).

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