Structural Engineering and Mechanics, Vol. 25, No. 2 (2007) 201-217
DOI: http://dx.doi.org/10.12989/sem.2007.25.2.201

Nonlinear dynamic response of MDOF systems by the method of harmonic differential quadrature (HDQ)

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(Received April 12, 2005, Accepted August 18, 2006)

Abstract. A harmonic type differential quadrature approach for nonlinear dynamic analysis of multi-degree-of-freedom systems has been developed. A series of numerical examples is conducted to assess the performance of the HDQ method in linear and nonlinear dynamic analysis problems. Results are compared with the existing solutions available from other analytical and numerical methods. In all cases, the results obtained are quite accurate.

Keywords: structural dynamics; harmonic differential quadrature; nonlinear analysis.

1. Introduction

An important part of engineering applications is the analysis and prediction of the dynamic behavior of structures. The analysis of the response of structures to dynamic loads is a difficult work, especially if the response is nonlinear. For most practical problems, nonlinear analysis can be carried out only by numerical analysis techniques. In their well-known book, Dynamics of Structures, Clough and Penzien (1975) stated that the most powerful and practical method for nonlinear analysis is the step-by-step integration procedure. In this technique, the process is continued step-by-step from the known initial instant to any desired time approximating the nonlinear behavior as a sequence of successively converting to the linear systems. Furthermore, many other algorithms such as, Newmark, Houbolt, Wilson, and Runge-Kutta methods are available for the numerical integration (Newmark 1959, Houbolt 1965, Bathe and Wilson 1973, Runge 1895, Kutta 1901). Each of these numerical approaches employes difference equivalents to develop recurrence relations which may be used in step-by-step computation to obtain the dynamic response of a structure. In general, the critical parameter in each of these techniques is the largest value of the time step that may be used to provide sufficiently accurate results, as this is directly related to the computational time of the analysis. The use of temporal finite elements, i.e., finite elements in the time domain, with displacements and their derivatives as nodal parameters has been suggested by, among others, Zienkiewicz (1977). A very detailed mathematical treatment is given in the book by Wood (1990), by Chopra (1995) and Bathe (1982). A survey of explicit and implicit methods and recent developments of the computational structural dynamics are given by Dokainish and Subbaraj (1989, 1989a), Bert and Stricklin (1988), Zienkiewicz and Lewis (1973), and Senjanovic

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(1984). DQ method is recently proposed to solve some initial value problems in the time domain. Shu *et al*. (2002) developed block-marching technique in both the spatial and temporal directions by DQ method for time-dependent problems. Recently, Fung (2001, 2001a, 2002), Tanaka and Chen (2001, 2001a) and Wu and Liu (2000) stated that the DQ method is unconditionally stable higher-order time steps integration algorithms. Unconditionally stable Hermittian time finite elements and time step integration methods with complex time step are proposed by Fung (1996, 1998).

In this paper, an improved version of DQ method, called the HDQ method, is used to study the linear and nonlinear dynamic response of structures. Since the dynamic response of a structure may be expanded into the harmonic series, it seems that the best approximation of the response may be obtained also by the harmonic test function in the DQ approach. The problem to be considered in this paper is limited to structures with linear inertia and damping but with nonlinear spring forces.

2. Differential quadrature (DQ) method

The basic idea of the DQ method is that the derivative of a function, with respect to a space variable at a given sampling point, is approximated as a weighted linear sum of the sampling points in the domain of that variable (Bert *et al.* 1987). As with other numerical analysis techniques, such as finite element or finite difference methods, the DQM also transforms the given differential equation into a set of analogous algebraic equations in terms of the unknown function values at the reselected sampling points in the field domain. Considering a function $f(x)$ with N grid points, we have

$$\frac{\partial^r f(x_i)}{\partial x^r} \cong \sum_{j=1}^{N} A_{ij}^{(r)} f(x_j); \qquad j = 1, 2, \ldots, N \tag{1}$$

where x_j are the co-ordinates of grid points in the variable domain. $f(x_j)$ and $A_{ij}^{(r)}$ are the function values at grid points and related weighting coefficients, respectively. To determine the weighting coefficients for the first order derivative (r = 1), the function $f(x)$ is represented by a test function, such as polynomial:

$$f(x) = x^{k-1}; \quad k = 1, 2, 3, \ldots, N \tag{2}$$

Substituting Eq. (2) into Eq. (1) for the first order derivative, one obtains

$$(k-1)x_i^{k-2} = \sum_{j=1}^{N} A_{ij} x_j^{k-1} \tag{3}$$

which represents N sets of N linear algebraic equations. A recently approach the original differential quadrature approximation called the Harmonic differential quadrature (HDQ) has been proposed by Striz *et al.* (1995) and Shu and Xue (1997). Unlike the differential quadrature that uses the polynomial functions, such as Lagrange interpolated, and Legendre polynomials as the test functions, harmonic differential quadrature uses harmonic or trigonometric functions as the test functions. As the name of the test function suggested, this method is called the HDQ method. The HDQ method has been successfully applied to solve various types of engineering problems, including the static, buckling and free vibration analysis of beams (Civalek 2004, Civalek and Ülker

2004a, 2004b), the three-dimensional static and vibration analysis plate problems by Liew and his co-workers (1998, 1999, 1999a, 2001). The harmonic test function $h_k(x)$ used in the HDQ method is defined as (Shu and Xue 1997):

$$h_k(x) = \frac{\sin\frac{(x-x_0)\pi}{2}\ldots\sin\frac{(x-x_{k-1})\pi}{2}\sin\frac{(x-x_{k+1})\pi}{2}\ldots\sin\frac{(x-x_N)\pi}{2}}{\sin\frac{(x_k-x_0)\pi}{2}\ldots\sin\frac{(x_k-x_{k-1})\pi}{2}\sin\frac{(x_k-x_{k+1})\pi}{2}\ldots\sin\frac{(x_k-x_N)\pi}{2}} \tag{4}$$

For simplicity, the following new variables are introduced

$$M(x) = \prod_{k=o}^{N}\sin\frac{x-x_k}{2}\pi = N(x,x_k)\sin\frac{x-x_k}{2}\pi \tag{5}$$

Thus, the weighting coefficients are given in the following formulas

$$A_{ij} = N^{(1)}(x_i,x_j)/P(x_k) \tag{6a}$$

$$B_{ij} = N^{(2)}(x_i,x_j)/P(x_k) \tag{6b}$$

where

$$P(x_i) = \prod_{j=1,j\neq i}^{N}\sin\left(\frac{x_i-x_j}{2}\pi\right) \qquad \text{for} \quad j = 1,2,3,\ldots,N \tag{7}$$

in which the $N^{(1)}(x_i,x_j)$ and $N^{(2)}(x_i,x_j)$ take the following values

$$N^{(1)}(x_i,x_j) = [\pi P(x_i)]/\left[2\sin\frac{x_i-x_j}{2}\pi\right] \qquad \text{for} \quad i\neq j \tag{8a}$$

$$N^{(1)}(x_i,x_i) = M^{(2)}(x_i)/\pi \qquad \text{for} \quad i = j \tag{8b}$$

$$N^{(2)}(x_i,x_j) = M^{(2)}(x_i) - \left[\pi N^{(1)}(x_i,x_j)\cos\frac{x_i-x_j}{2}\pi\right]/\sin\frac{x_i-x_j}{2}\pi \qquad \text{for} \quad i\neq j \tag{8c}$$

$$N^{(2)}(x_i,x_i) = \frac{2}{3\pi}\left[M^{(3)}(x_i) + \frac{\pi^3}{8}N(x_i,x_i)\right] \qquad \text{for} \quad i = j \tag{8d}$$

Consequently, the weighting coefficients of the first-order derivatives A_{ij} for $i \neq j$ can be obtained by using the following formula:

$$A_{ij} = \frac{(\pi/2)P(x_i)}{P(x_j)\sin[(x_i-x_j)/2]\pi} \qquad i,j = 1,2,3,\ldots,N \tag{9}$$

The weighting coefficients of the second-order derivatives B_{ij} for $i \neq j$ can be obtained using following formula:

$$B_{ij} = A_{ij}\left[2A_{ii}^{(1)} - \pi ctg\left(\frac{x_i-x_j}{2}\right)\pi\right] \qquad i,j = 1,2,3,\ldots,N \tag{10}$$

The weighting coefficients of the first-order and second-order derivatives $A_{ij}^{(p)}$ for $i = j$ are given as

$$A_{ii}^{(p)} = -\prod_{j=1, j \neq i}^{N} A_{ij}^{(p)}, \quad p = 1 \text{ or } 2 \text{ ; and for } i = 1, 2, \ldots, N \tag{11}$$

A decisive factor to the accuracy of the all type differential quadrature solutions is the choice of the sampling or grid points. It should be mentioned that in the differential quadrature solutions, the sampling points in the various coordinate directions may be different in number as well as in their type. A natural, an often convenient, choice for sampling points is that of equally spaced point. It was also reported (Bert and Malik 1996, Liew *et al.* 1996, 2002, Du *et al.* 1996, Civalek 2002) that the Chebyshev-Gauss-Lobatto or non-equally sampling grid (NE-SG) points for spatial and temporal discretization as;

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\right)\pi\right]; \quad i = 1, 2, \ldots, N \tag{12a}$$

$$t_j = \frac{1}{2}\left[1 - \cos\left(\frac{j-1}{N-1}\right)\pi\right]; \quad j = 1, 2, \ldots, N \tag{12b}$$

performed consistently better than the equally spaced. The equally sampling gird (E-SG) points are given for spatial and temporal discretization as;

$$x_i = \frac{i-1}{N-1} \quad \text{for} \quad i = 1, 2, \ldots, N \tag{13a}$$

$$t_j = \frac{j-1}{N-1} \quad \text{for} \quad j = 1, 2, \ldots, N \tag{13b}$$

3. Method for solving the equation of motion

In this study, we investigated the linear and nonlinear dynamic response of discrete-parameter systems, such as SDOF systems and MDOF systems by the method of HDQ. For simplicity, a SDOF system is considered in detail first, and the extension to MDOF systems is then discussed and the related formulations are given. Lower case letters are used for all matrices in the SDOF case, which are replaced by upper case equivalents when treating MDOF systems.

3.1 Single-degree-of-freedom (SDOF) systems

The differential equation of motion of a typical Single-Degree-Of-Freedom (SDOF) system is

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = f(t) \tag{14}$$

where m, c and k are the mass, damping coefficient and stiffness of the system, $\ddot{u}$, $\dot{u}$, and u are acceleration, velocity and displacement, $f(t)$ is a prescribed external force. In this equation a dot superscript denotes differentiation with respect to time t. In the linear systems, these coefficients namely, m, c, and k are the constant. However, in many important engineering applications k is a function of u and the problem is nonlinear. If any structure modeled as a SDOF system is allowed

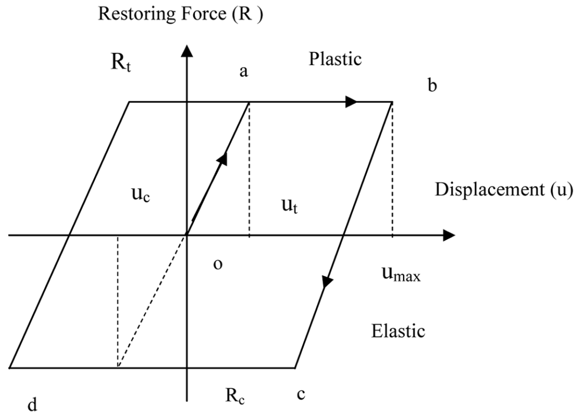


Fig. 1 Elasto-plastic behavior for force-displacement curve

to yield plastically, then the restoring force exerted is likely to be of the form shown in Fig. 1.

There is a portion of the curve in which linear elastic behavior occurs, whereupon, for any further deformation, plastic yielding takes place. When the structure is unloaded, the behavior is again elastic until further reverse loading produces compressive plastic yielding (Chopra 1995). In this Figure R_t and R_c are the respective values of the forces that produce yielding in tension and compression and k is the elastic stiffness of the structure. In this case the governing equation of motion and the initial conditions are given

$$m\ddot{u}(t) + c\dot{u}(t) + fs = f(t); \quad u(0) = u_0 \quad \text{and} \quad \dot{u}(0) = \dot{u}_0 \tag{15,16}$$

Thus the force f_s corresponding to displacements u are not single-valued and depend on the history of the displacements. This force is defined for each part of the curve as below;

$$f_s = \begin{cases} ku_i & \text{for segment } oa \\ R_t & \text{for segment } ob \\ R_t - k(u_{\max} - u_i) & \text{for segment } bc \\ R_c & \text{for segment } cd \\ etc. & \end{cases}$$

After some rearrangements, we obtain

$$\frac{m}{(T)^2}\ddot{u} + \frac{c}{T}\dot{u} + f_s = f(\tau T) \tag{17a}$$

$$u(0) = u_0 \quad \text{and} \quad \frac{du}{d\tau} = (T)_{\dot{u}_0} \tag{17b}$$

where $\tau = t/T$, T is the time length of solution domain. The time domain $t \subset [0, T]$ is normalized to $\tau \subset [0, 1]$ and then divided into $N - 1$ sections. Applying the HDQ approximation to (12) at each discrete point on the grid, we have (Civalek 2003);

$$\frac{m}{(T)^2}\sum_{j=0}^{N} B_{ij}u(\tau_j) + \frac{c}{(T)}\sum_{j=0}^{N} A_{ij}u(\tau_j) + f_s(\tau_j) = f(\tau_j T); \quad i = 1, 2, \ldots, N \tag{18}$$

and applying the HDQ approximation to the initial conditions

$$u(0) = u_0 \quad \text{and} \quad \frac{du}{d\tau} = T\sum_{j=0}^{N} A_{i0}u(\tau_j) \tag{19a,19b}$$

Eq. (18) is given in matrix form as (Civalek 2003)

$$\frac{m}{(T)^2}\begin{bmatrix} B_{10} & B_{11} & \ldots & B_{1N} \\ B_{20} & B_{21} & \ldots & B_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ B_{N0} & B_{N1} & \ldots & B_{NN} \end{bmatrix}\begin{Bmatrix} u(\tau_0) \\ u(\tau_1) \\ \cdot \\ \cdot \\ \cdot \\ u(\tau_N) \end{Bmatrix} + \frac{c}{(T)}\begin{bmatrix} A_{10} & A_{11} & \ldots & A_{1N} \\ A_{20} & A_{21} & \ldots & A_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_{N0} & A_{N1} & \ldots & A_{NN} \end{bmatrix}\begin{Bmatrix} u(\tau_0) \\ u(\tau_1) \\ \cdot \\ \cdot \\ \cdot \\ u(\tau_N) \end{Bmatrix} + \begin{Bmatrix} f_s(\tau_0) \\ f_s(\tau_1) \\ \cdot \\ \cdot \\ \cdot \\ f_s(\tau_N) \end{Bmatrix} = \begin{Bmatrix} f(\tau_0 T) \\ f(\tau_1 T) \\ \cdot \\ \cdot \\ \cdot \\ f(\tau_N T) \end{Bmatrix} \tag{20}$$

In order to introduce the given initial conditions into Eq. (20), Eq. (19) are used to solve $u(\tau_1)$ in terms of the variables $u(\tau_2)$, $u(\tau_3)$, ..., $u(\tau_N)$. The expressions for $u(\tau_1)$ in terms of the variables $u(\tau_2)$, $u(\tau_3)$, ..., $u(\tau_N)$ are then substituted into Eq. (20) to eliminates the variables $u(\tau_1)$ only the discredited equations at the time points $j = 2, 3, 4, \ldots, N$ are to be used in Eq. (15). Thus, $N - 1$ analog equations of the governing differential equation at $N - 1$ sampling points τ_2, τ_3, ..., τ_N can be written as

$$\frac{m}{(T)^2}\sum_{j=2}^{N} \overline{B}_{ij}u(\tau_j) + \frac{c}{(T)}\sum_{j=2}^{N} \overline{A}_{ij}u(\tau_j) + f_s(\tau_j) = f(\tau_j T) \tag{21}$$

where $\overline{A}_{ij}$ and $\overline{B}_{ij}$ are yielded by removing the related coefficients of the original DQM weighting coefficients matrices A_{ij} and B_{ij} in Eq. (21). These new modified weighting coefficients matrices $\overline{A}_{ij}$ and $\overline{B}_{ij}$ are $(N - 1) \times (N - 1)$ dimension. It is noticed that the known initial conditions specified in Eq. (20) have been built into the modified weighting coefficients matrices $\overline{A}_{ij}$ and $\overline{B}_{ij}$. This scheme is an analogy with the before developed technique in applying boundary conditions for the DQM solution of high-order boundary value problems presented by Wang and Bert (1993). The more detailed information for the imposition of the boundary conditions for the DQ method can be found in (Wang and Bert 1993, Fung 2003). Recently, Fung (2001, 2001a, 2002, 2002a, 2003a) and Tanaka and Chen (2001, 2001a) had proposed several methods to incorporate the initial conditions by modifying the elements in the weighting coefficient matrices. Eq. (21) can be written as

$$\frac{m}{(T)^2}[\overline{B}_{ij}]\begin{Bmatrix} u_2 \\ u_3 \\ \cdot \\ \cdot \\ \cdot \\ u_N \end{Bmatrix} + \frac{c}{(T)}[\overline{A}_{ij}]\begin{Bmatrix} u_2 \\ u_3 \\ \cdot \\ \cdot \\ \cdot \\ u_N \end{Bmatrix} + \begin{Bmatrix} f_s(\tau_2) \\ f_s(\tau_3) \\ \cdot \\ \cdot \\ \cdot \\ f_s(\tau_N) \end{Bmatrix} = \begin{Bmatrix} f(\tau_2 T) \\ f(\tau_3 T) \\ \cdot \\ \cdot \\ \cdot \\ f(\tau_N T) \end{Bmatrix} \tag{22}$$

Solving the algebraic Eq. (22), the displacement u_j ($j = 2, 3, \ldots, N$) at various grid points can be obtained. After the displacements are found, the velocities and accelerations can be obtained as (Civalek 2003):

$$\begin{Bmatrix} \dot{u}_0(t) \\ \dot{u}_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{u}_N(t) \end{Bmatrix} = (T)\sum_{j=0}^{N} A_{ij}u(\tau_j) = (T)\begin{bmatrix} A_{10} & A_{11} & \cdot & \cdot & A_{1N} \\ A_{20} & A_{21} & \cdot & \cdot & A_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{N0} & A_{N1} & \cdot & \cdot & A_{NN} \end{bmatrix}\begin{Bmatrix} u_0(\tau_0 T) \\ u_1(\tau_1 T) \\ \cdot \\ \cdot \\ \cdot \\ u_N(\tau_N T) \end{Bmatrix} \tag{23a}$$

$$\begin{Bmatrix} \ddot{u}_0(t) \\ \ddot{u}_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \ddot{u}_N(t) \end{Bmatrix} = (T)^2\sum_{j=0}^{N} B_{ij}u(\tau_j) = (T)^2\begin{bmatrix} B_{10} & B_{11} & \cdot & \cdot & B_{1N} \\ B_{20} & B_{21} & \cdot & \cdot & B_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ B_{N0} & B_{N1} & \cdot & \cdot & B_{NN} \end{bmatrix}\begin{Bmatrix} u_0(\tau_0 T) \\ u_1(\tau_1 T) \\ \cdot \\ \cdot \\ \cdot \\ u_N(\tau_N T) \end{Bmatrix} \tag{23b}$$

In Eq. (23a), u_0 is given and $\dot{u}_0 = v_0\Delta t$. Thus, the first equation in Eq. (23a) is given by (Fung 2002)

$$\dot{u}_0 = v_0\Delta t = \sum_{j=0}^{N} A_{0j}u_j \tag{24}$$

By using Eq. (24), u_N can be given as

$$u_N = \frac{1}{A_{0,N}}\left(-A_{00}u_0 + v_0\Delta t - \sum_{j=1}^{N-1} A_{0j}u_j\right) \tag{25}$$

Hence, using Eq. (25), $\dot{u}_1, \ldots, \dot{u}_{N-1}$ in Eq. (23a) and $\ddot{u}_1, \ldots, \ddot{u}_{N-1}$ in Eq. (23b) can be expressed in terms of $u_0, v_0\Delta t, u_1, \ldots, u_{N-1}$ after eliminating u_N.

3.2 Multi-degree-of-freedom (MDOF) systems

Consider a multi-degree-of-freedom system with mass, damping, and stiffness properties $[M]$, $[C]$ and $[K]$, respectively. The equation of motion and the initial conditions for this system are given by

$$[M]\{\ddot{U}(t_j)\} + [C]\{\dot{U}(t_j)\} + [K]\{U(t_j)\} = \{F(t_j)\} \tag{26}$$

$$\{U(0)\} = U_0 \quad \text{and} \quad \{\dot{U}(0)\} = \dot{U}_0; \quad j = 1, 2, 3, \ldots, R \qquad \text{(27a, 27b)}$$

If the system experiences non-linear deformations while subjected to an external force, the equations of motion can be written as

$$[M]\{\ddot{U}(t_j)\} + [C]\{\dot{U}(t_j)\} + \{F^S(t_j)\} = \{F(t_j)\} \tag{28}$$

where $U, \dot{U}$ are $\ddot{U}$ the displacement, velocity and acceleration vectors of each mass, $\{F^S(t_j)\}$ is the resisting force, which may be a function of displacement. Upon normalization of (28) we have

$$[M]\frac{1}{(T)^2}\{\ddot{U}(\tau_j T)\} + [C]\frac{1}{T}\{\dot{U}(\tau_j T)\} + \{F^S(\tau_j T)\} = \{F(\tau_j T)\} \tag{29}$$

where $\tau = t/T$, T is the time length of solution domain. The time domain $t \subset [0, T]$ is normalized to $\tau \subset [0, 1]$. Applying the DQ approximation to (29) at each discrete time on the grid, we have (Civalek 2003):

$$[M]\frac{1}{(T)^2}\left\{\sum_{j=0}^{N} B_{ij}U_k(\tau_j)\right\} + [C]\frac{1}{T}\left\{\sum_{j=0}^{N} A_{ij}U_k(\tau_j)\right\} + \{F_k^S(\tau_j)\} = \{F_k(\tau_j)\} \quad k = 1, 2, 3, \ldots, R \tag{30}$$

This equation can be written in matrix form as

$$[M]\frac{1}{(T)^2}\begin{Bmatrix} [B_{ij}]\{U_1(\tau_j)\} \\ [B_{ij}]\{U_2(\tau_j)\} \\ \cdot \\ \cdot \\ \cdot \\ [B_{ij}]\{U_R(\tau_j)\} \end{Bmatrix} + [C]\frac{1}{T}\begin{Bmatrix} [A_{ij}]\{U_1(\tau_j)\} \\ [A_{ij}]\{U_2(\tau_j)\} \\ \cdot \\ \cdot \\ \cdot \\ [A_{ij}]\{U_R(\tau_j)\} \end{Bmatrix} + \begin{Bmatrix} \{F_1^S(\tau_j)\} \\ \{F_2^S(\tau_j)\} \\ \cdot \\ \cdot \\ \cdot \\ \{F_R^S(\tau_j)\} \end{Bmatrix} = \begin{Bmatrix} \{F_1(\tau_j)\} \\ \{F_2(\tau_j)\} \\ \cdot \\ \cdot \\ \cdot \\ \{F_R(\tau_j)\} \end{Bmatrix} \tag{31}$$

and applying the DQ approximation to the initial conditions;

$$\{U_k(\tau_0)\} = \{U_k(0)\} \quad \text{and} \quad \{\dot{U}_k(\tau_0)\} = T\sum_{j=0}^{N} A_{i0}U_k(\tau_0) \quad i = 1, 2, 3, \ldots, N \tag{32}$$

Thus, we obtain the displacement responses in each direction of degree of freedom $\{U_k\,(k = 1, 2, ..., R)$ for each of considered time step $\{\tau_j\,(j = 0, 1, 2, ..., N)\}$. This displacement vector is given as

$$\langle U_k(\tau_j)\rangle^T = \langle \{U_1(\tau_j)\}, \{U_2(\tau_j)\}, \ldots, \{U_N(\tau_j)\}\rangle^T \tag{33}$$

After the displacements are found, the velocities and accelerations can be obtained by (Civalek 2003):

$$\begin{Bmatrix} \{\dot{U}_1(\tau_j)\} \\ \{\dot{U}_2(\tau_j)\} \\ \cdot \\ \cdot \\ \cdot \\ \{\dot{U}_R(\tau_j)\} \end{Bmatrix} = (T)\begin{Bmatrix} [A_{ij}]\{U_1(\tau_j)\} \\ [A_{ij}]\{U_2(\tau_j)\} \\ \cdot \\ \cdot \\ \cdot \\ [A_{ij}]\{U_R(\tau_j)\} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \{\ddot{U}_1(\tau_j)\} \\ \{\ddot{U}_2(\tau_j)\} \\ \cdot \\ \cdot \\ \cdot \\ \{\ddot{U}_R(\tau_j)\} \end{Bmatrix} = (T)^2\begin{Bmatrix} [B_{ij}]\{U_1(\tau_j)\} \\ [B_{ij}]\{U_2(\tau_j)\} \\ \cdot \\ \cdot \\ \cdot \\ [B_{ij}]\{U_R(\tau_j)\} \end{Bmatrix} \tag{34a, 34b}$$

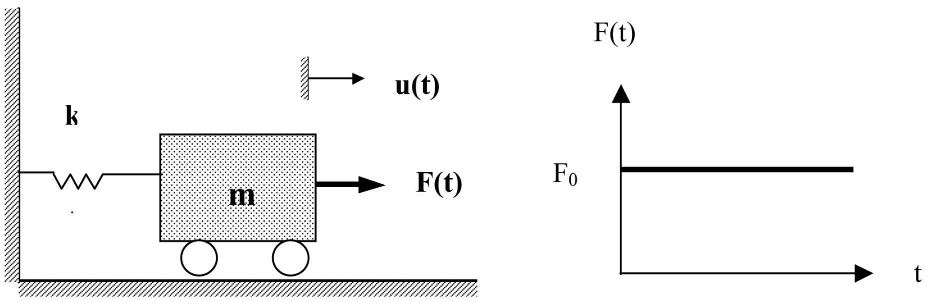


Fig. 2 SDOF system and applied constant step load

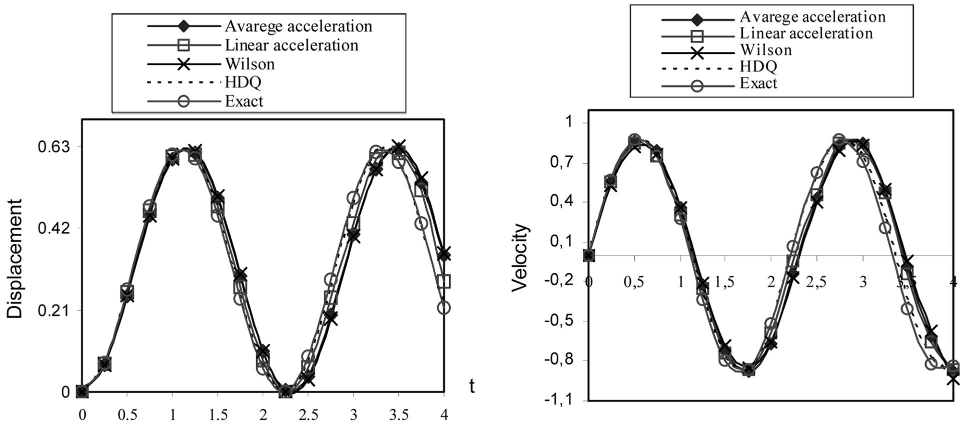
4. Numerical applications and results

To demonstrate the effectiveness, characteristics and merits of HDQ in the analysis of structural dynamics, several application problems are presented. Four examples are presented in this section. The first one dealing with a simple problem for which detailed results are available shows the validity of the present formulation. After this, some linear and nonlinear numerical examples for SDOF and MDOF systems are presented and compared with the results reported in the literature to demonstrate the accuracy, stability, and efficiency of the proposed method. For simplicity, all variables and computing parameters are assumed dimensionless in the following examples. In order to evaluate the proposed method, the error of the solution may be defined as the relative discrepancies between the proposed and exact (or numerical) solution. The results presented in this section attempt to illustrate the effect of time step on the numerical accuracy. During the study, N is taken as 15. For this purpose, the relative percentage error is defined as

$$\%\text{Error} = \left|\left(\frac{Exact\ (or\ numerical)\ value - HDQ\ solution\ value}{Exact\ (or\ numerical)\ value}\right) \times 100\right| \tag{35}$$

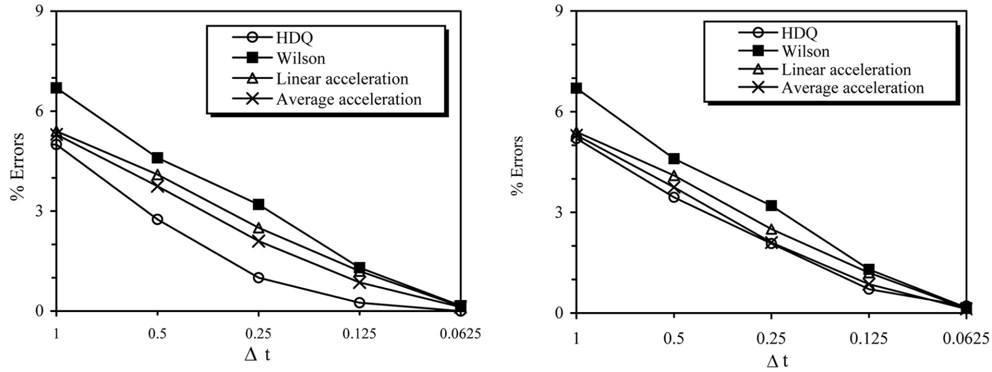
Example 1: Analysis of a single-degree-of-freedom (Fig. 2) system was performed to verify predicted rates-of-convergence of the HDQ algorithm presented herein. This simple example will study the forced vibration of a SDOF system with the following parameters: m = 2, c = 0, k = 16, and F_0 = 5.

The initial displacement and velocity of the system are assumed to be zero. The calculated displacement, velocity and acceleration response of this system are shown in Fig. 3(a) and Fig. 3(b) (for Δt = 0.2) using HDQ method. For comparison purposes, results obtained by some other numerical integration methods with Δt = 0.1 are also shown in the figures. It can be seen that for larger time steps the HDQ method gives more precise results than the Newmark's and Wilson's method. It was found that as the step sizes increases, the relative error also increases. This can be seen clearly in Figs. 4(a) and 4(b), which show the relative error of deflection versus Δt for equally sampling grid (E-SG) points and for non-equally sampling grid (NE-SG) points. From Fig. 4, we can see that the optimal convergence could be achieved with the NE-SG of Δt = 0.25. For E-SG points, the reasonable accurate results are obtained for Δt = 0.125. The Wilson method is the least efficient. Wilson method requires considerably smaller time steps to achieve low error.



(a) (b)

Fig. 3 Calculated responses for SDOF system: (a) Displacement, (b) Velocity

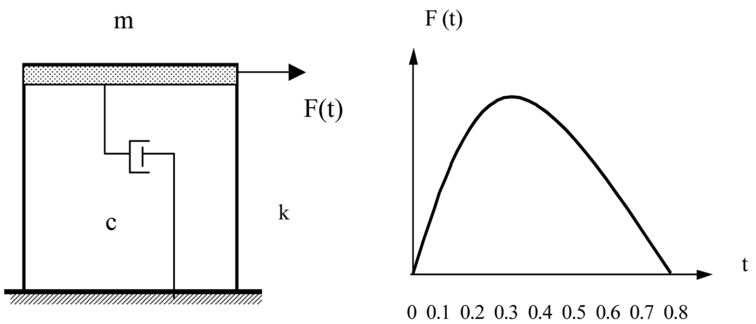


(a) (b)

Fig. 4 Variation of %Errors with time steps for displacements: (a) (NE-SG), (b) (E-SG)

Example 2: Non-linear elastic SDOF system: In order to demonstrate the computation procedure and to measure the accuracy and the range of applicability of the proposed method, the forced vibration response of a non-linear SDOF system given by Clough and Penzien (1975), shown in Fig. 5, to the loading history indicated has been calculated. This system has the following properties: m = 0.1, c = 0. 2, k = 5, and the initial conditions are; $u(0) = 0$ and $\dot{u}(0) = 0$. Nonlinear load-displacement curve is given in Fig. 6.

The dynamic response calculated with the proposed method for time intervals, Δt = 0.20 is plotted in Fig. 7 in dashed lines. The results obtained using Newmark's average acceleration method for $\Delta t = 0.1$ also shown in this figures. It is seen that the HDQ results are in good agreement with the results of Clough and Penzien and the Newmark's results. When the present HDQ solution is compared with the step-by-step integration solution (Clough and Penzien 1975), the greatest deviation is 5.7% for E-SG points using Δt = 0.5. However, this deviation is 4.82% for NE-SG points of the same time interval (i.e. Δt = 0.5). It is observed that the Chebyshev-Gauss-Lobatto (NE-SG) grid points have the most rapid converging speed in this figure.



t	F(t)
0	0
0.1	5
0.2	8
0.3	7
0.4	5
0.5	3
0.6	2
0.7	1

Fig. 5 SDOF system and load history

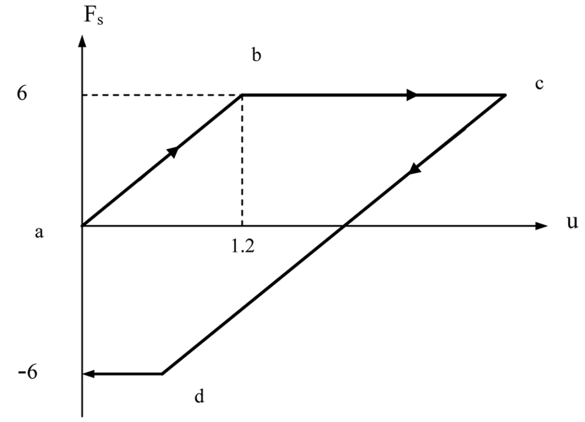


Fig. 6 Non-linear stiffness

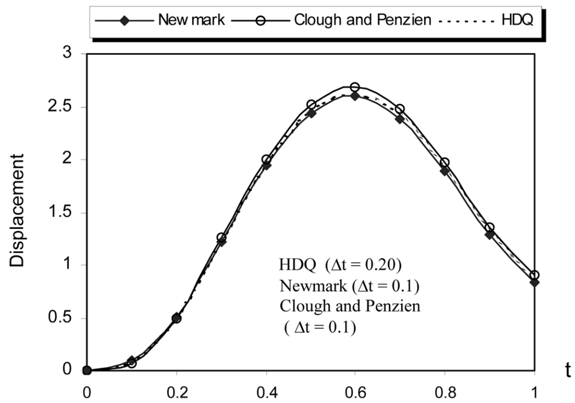


Fig. 7 Displacement response for nonlinear SDOF system

For small value of Δt, the HDQ solutions with the stretched Chebyshev-Gauss-Lobatto grids are much more accurate than those with the conventional equally spaced sampling grid points. This means that the equally spaced grid points are not reliable in the HDQ solution of dynamic problems. The results in Fig. 8 that, to obtain accurate HDQ solutions for dynamic problems, the equally sampling grid points are not suitable for large value of Δt. So, for large value of time step Δt, we have to use non-equally sampling (NE-SG) grid stretching to get accurate and efficient HDQ results.

Example 3: Linear Dynamic response of a 2-DOF system. Consider a two-degree-of-freedom system governed by

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \end{Bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

with initial conditions

$$\begin{Bmatrix} U_1(0) \\ U_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dot{U}_1(0) \\ \dot{U}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

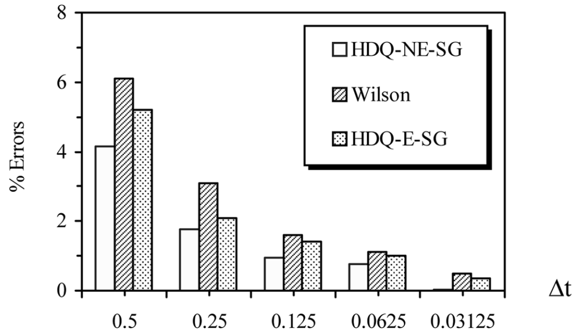


Fig. 8 Error in the displacements at various values of Δt for E-SG and NE-SG points

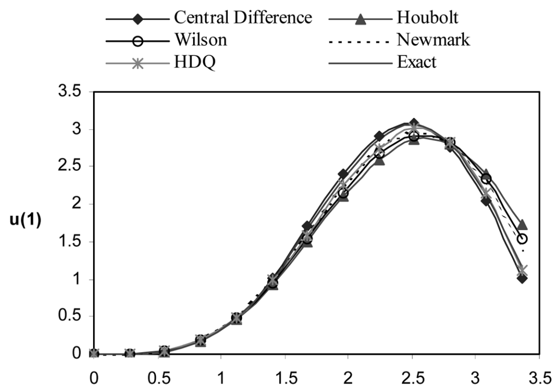


Fig. 9 Displacement time-history for a 2-DOF system (first mass)

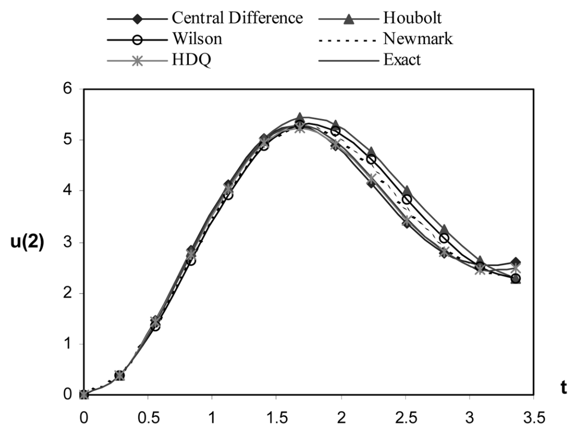


Fig. 10 Displacement time-history for a 2-DOF system (second mass)

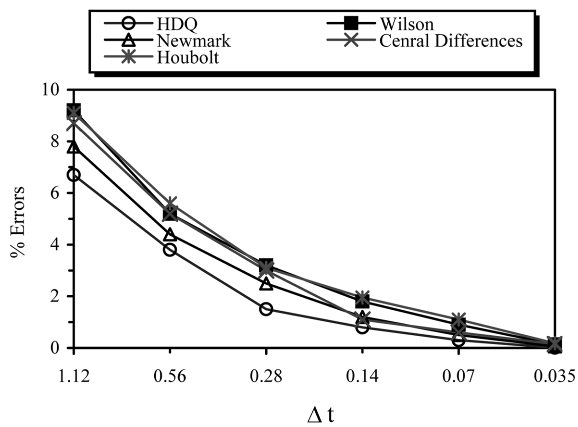


Fig. 11 Error in displacement (first mass) at various values of Δt for example 3

Analytical solutions of the problem as well as the results obtained by the most commonly used integration methods are given by Bathe (1982). All the numerical solutions are obtained by using the time step of 0.3 s for HDQ method. The results are shown in Figs. 9 and 10.

For comparison purposes, some results produced by Bathe (1982) for Δt = 0.28 and the exact solutions are also shown in the figures. The results obtained from HDQ are in excellent agreement with the exact solutions. The percentage errors of the displacements for displacements of first mass are displayed in Fig. 11. The results in Fig. 11 that, the error increases as the time step is increased. The Houbolt, central differences and Wilson methods are the least efficient; they require considerably smaller time steps to obtain the reasonable accurate results. The HDQ and Newmark methods give very similar results. However, it can be also seen that for larger time step the HDQ method gives more accurate results than the Newmark's average acceleration method.

Example 4: As the last example, consider a non-linear dynamic response of a two-degree-of-freedom spring-mass system. Non-linear governing equation of motion of this system is given by (Sun *et al.* 1991) in matrix form as

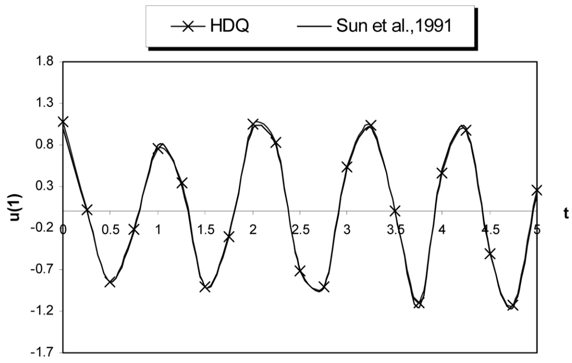


Fig. 12 Calculated displacement for unequally sampling grid distributions-first mass

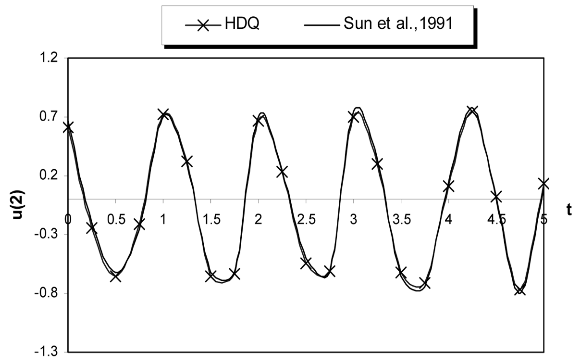


Fig. 13 Calculated displacement for unequally sampling grid distributions-second mass

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \varepsilon(u_1 - u_2)^3 = \begin{Bmatrix} F\cos(\omega t) \\ 0 \end{Bmatrix}$$

The above matrix equation describes the motion of a two-degree-of-freedom system with mass (m_2) connected to the ground by the linear elastic spring k_2 and mass (m_1) connected to mass (m_2) by a non-linear elastic spring with restoring force

$$F_s = k_1(u_1 - u_2) + \varepsilon(u_1 - u_2)^3$$

With initial conditions

$$\begin{Bmatrix} u_1(0) \\ u_2(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.6667 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dot{u}_1(0) \\ \dot{u}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

As with the other examples, all variables and computing parameters are assumed dimensionless for simplicity. The calculated responses of this nonlinear system are shown in Figs. 12 and 13 for Δt = 0.05. For comparison purposes, results produced by Sun *et al.* (1991) with Δt = 0.03 are also shown in the figures. The results obtained from HDQ are in excellent agreement with the results of Sun *et al.* (1991). The results given in this figure agree well, even though the time steps used in the proposed method are much larger than those for Sun's post-correction integration solution.

It is known that the range of parameters arise in comparing the numerical efficiency of various techniques, including number of degrees of freedom, band width of matrices, number of time steps, and required accuracy. For this purpose, let consider the numerical solution of the problem given by example 3. The proposed method was compared with the finite difference, Wilson-θ and Newmark's method. The solution times for proposed and Newmark's method can be given by

$$t_{HDQ} = t_{HDQ} + tN_i \quad \text{and} \quad t_{Newmark} = t_{Newmark} + tN_j$$

where t_{HDQ} and $t_{Newmark}$ are the times to generate the stiffness and mass matrices, t is the time per time step, and N_i and N_j are the number of time steps in HDQ and Newmark's methods respectively. The ratio of time steps for comparable accuracy and ratio of computer times for

Table 1 Time-step ratios for comparable accuracy

$\Delta t_{Newmark}/T$	$t_{HDQ}/t_{Newmark}$
0.01	12.45
0.02	8.36
0.03	6.90
0.04	5.86
0.05	5.02
0.06	5.87
0.07	4.83
0.08	4.11
0.09	3.97
0.1	3.65
0.125	2.49

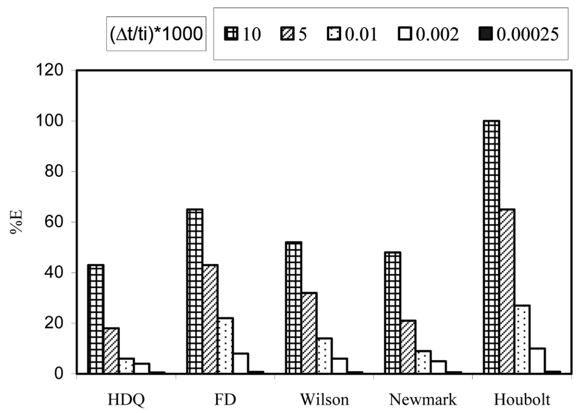


Fig. 14 % Error for different [($\Delta t/t_i$)*1000] ratios

proposed and Newmark's methods are listed in Table 1. It is proposed that $\Delta t_{Newmark}/T \leq 1/10$ value is suitable for MDOF system (Wood 1990), where T is the highest frequency which is considered is important. It is shown that for similar accuracy, a time step 3.65 times greater than Newmark's method for $\Delta t_{Newmark}/T = 1/10$.

In order to evaluate these methods, the error of the solution can be given as the mean value of the relative discrepancies between the numerical methods and exact solution. This error term is given by

$$E = \frac{1}{20}\sum_{i=1}^{2}\sum_{j=2}^{12}\left|\frac{u_{ij}-\bar{u}_{ij}}{\bar{u}_{ij}}\right|$$

The first time step is excluded from the analysis since the small response at this step causes large relative errors (Senjanovij 1984). Maximum value of E is obtained for Houbolt followed by Wilson-θ, Central difference and Newmark. In order to verify the stability of the solutions obtained by considered methods, which are all unconditionally stable except the finite differences, the same example (example 3) is solved for different $\Delta t/t_i$ and the error results are given in Fig. 14. Where t_i is the CPU times for each method.

It is shown that the result was determined more rapidly using the HDQ method than using Newmark's or other numerical schemes. In addition, as less time steps are required for similar accuracy, the time-stepping procedure requires less computational effort than others.

5. Conclusions

A harmonic type DQ method was introduced to study the linear and nonlinear dynamic response of SDOF and MDOF systems. The method of HDQ that was using the paper proposes a very simple algebraic formula to determine the connections weighting coefficients required by DQ approximation without restricting the choice of mesh grids. The known initial conditions are easily incorporated in the HDQ as well as the other type DQ. The discretizing and programming procedures are straightforward and easy. It is also concluded that the results obtained with non-equally sampling grid points are more accurate than the values calculated by equally sampling grid points. The simple examples presented here demonstrate that the HDQ method gives accurately acceptable results for a variety of linear problems and nonlinear hysteric systems. Furthermore, results for examples are given demonstrating good agreement with solutions generated from other numerical approaches.

Acknowledgements

The financial support of the Scientific Research Projects Unit of Akdeniz University is gratefully acknowledged.

References

Bathe, K.J. (1982), *Finite Element Procedures in Engineering Analysis*. Englewood Cliffs. NJ: Prentice-Hall.
Bathe, K.J. and Wilson, E.L. (1973), "Stability and accuracy analysis of direct integration methods", *Earthq. Eng. Struct. Dyn.*, **1**, 283-291.
Bert, C.W., Jang, S.K. and Striz, A.G. (1987), "Two new approximate methods for analyzing free vibration of structural components", *AIAA J.*, **26**(5), 612-618.
Bert, C.W. and Malik, M. (1996), "Differential quadrature method in computational mechanics: A review", *Appl. Mech. Rev.*, **49**(1), 1-28.
Bert, C.W. and Stricklin, J.D. (1988), "Comparative evaluation of six different numerical methods for nonlinear dynamic systems", *J. Sound Vib.*, **127**(2), 221-229.
Chopra, A.K. (1995), *Dynamics of Structures, Theory and Applications to Earthquake Engineering.* Prentice-Hall, New Jersy.
Civalek, Ö. (2004), "Application of differential quadrature (DQ) and harmonic differential quadrature (HDQ) for buckling analysis of thin isotropic plates and elastic columns", *Eng. Struct., An Int. J.*, **26**(2), 171-186.
Civalek, Ö. (2002), *Differential Quadrature (DQ) for Static and Dynamic Analysis of Structures* (in Turkish), Fırat University.
Civalek, Ö. and Ülker, M. (2004a), "Harmonic differential quadrature (HDQ) for axisymmetric bending analysis of thin isotropic circular plates", *Struct. Eng. Mech.*, **17**(1), 1-14.
Civalek, Ö. and Ülker, M. (2004b), "Free vibration analysis of elastic beams using harmonic differential quadrature (HDQ)", *Math. and Comput. Appl.*, **9**(2), 257-264.

Civalek, Ö. (2004b), "Geometrically non-linear static and dynamic analysis of plates and shells resting on elastic foundation by the method of polynomial differential quadrature (PDQ)", PhD. Thesis, Fırat University, (in Turkish), Elazığ.

Civalek, Ö. (2003), "Linear and nonlinear dynamic response of multi-degree-of freedom-systems by the method of harmonic differential quadrature (HDQ)", PhD. Thesis, Dokuz Eylül University, İzmir.

Clough, R.W. and Penzien, J. (1975), *Dynamics and Structures*, McGraw-Hill, New York.

Du, H., Liew, K.M. and Lim, M.K. (1996), "Generalized differential quadrature method for buckling analysis", *J. Eng. Mech.,* ASCE, **22**(2), 95-100.

Dokainish, M.A. and Subbaraj, K. (1989), "A survey of direct time-integration methods in computational structural dynamics-I, explicit methods", *Comput. Struct.*, **32**(6), 1371-1386.

Dokainish, M.A. and Subbaraj, K. (1989), "A survey of direct time-integration methods in computational structural dynamics-II, implicit methods", *Comput. Struct.*, **32**(6), 1387-1401.

Fung, T.C. (1996), "Unconditionally stable higher-order accurate Hermittian time finite elements", *Int. J. Numer. Meth. Eng*., **39**, 3475-3495.

Fung, T.C. (1998), "Higher order time step integration methods with complex time step", *J. Sound Vib.*, **210**(1), 69-89.

Fung, T.C. (2001), "Solving initial value problems by differential quadrature method-Part 1: First-order equations", *Int. J. Numer. Meth. Eng*., **50**, 1411-1427.

Fung, T.C. (2001a), "Solving initial value problems by differential quadrature method-Part 2: Second-and higher-order-order equations", *Int. J. Numer. Meth. Eng*., **50**, 1429-1454.

Fung, T.C. (2002), "Stability and accuracy of differential quadrature method in solving dynamic problems", *Comput. Meth. Appl. Mech. Eng.*, **191**, 1311-1331.

Fung, T.C. (2002a), "On the equivalence of the time domain differential quadrature method and dissipative Runge-Kutta collocation method", *Int. J. Numer. Meth. Eng.*, **53**, 409-431.

Fung, T.C. (2003), "Imposition of boundary conditions by modifying the weighting coefficient matrices in the differential quadrature method", *Int. J. Numer. Meth. Eng.*, **56**, 405-432.

Fung, T.C. (2003a), "Generalized Lagrange functions and weighting coefficient formulae for the harmonic differential quadrature method", *Int. J. Numer. Meth. Eng.*, **57**, 415-440.

Houbolt, J.C. (1950), "A recurrence matrix solution for the dynamic response of elastic aircraft", *J. of Aeronautical Sciences*, **17**, 540-550.

Kutta, W. (1901), "Beitrag Zur Näherungsweisen Integration Totaler Differentialgleichungen", *Zeitschrift für Mathematik un Physik*, **46**, 435-453.

Liew, K.M. and Teo, T.M. (1999), "Three dimensional vibration analysis of rectangular plates based on differential quadrature method", *J. Sound Vib*., **220**(4), 577-599.

Liew, K.M., Teo, T.M. and Han, J.B. (1999a), "Comparative accuracy of DQ and HDQ methods for three-dimensional vibration analysis of rectangular plates", *Int. J. Numer. Meth. Eng.*, **45**, 1831-1848.

Liew, K.M., Teo, T.M. and Han, J.-B. (2001), "Three-dimensional static solutions of rectangular plates by variant differential quadrature method", *Int. J. Mech. Sci.,* **43**, 1611-1628.

Liew, K.M., Huang, Y.Q. and Reddy, J.N. (2002), "A hybrid moving least squares and differential quadrature (MLSDQ) meshfree method", *Int. J. Comput. Eng. Sic.*, **3**(1), 1-12.

Liew, K.M. and Liu, F.-L. (1997), "Differential cubature method: A solution technique for Kirchhoff plates of arbitrary shape", *Comput. Meth. Appl. Mech. Eng.*, **145**, 1-10.

Liew, K.M. and Han, J.-B. (1997a), "A four-node differential quadrature method for straight-sided quadrilateral Reissner/Mindlin plates", *Comm. Numer. Meth. Eng.,* **13**(2), 73-81.

Liew, K.M., Han, J.-B, Xiao, Z.M. and Du, H. (1996), "Differentiel quadrature method for mindlin plates on winkler foundations", *Int. J. Mech. Sci.*, **38**(4), 405-421.

Liew, K.M. and Teo, T.M. (1998), "Modeling via differential quadrature method: Three-dimensional solutions for rectangular plates", *Comp. Meth. in Appl. Mech. Eng*., **159**, 369-381.

Newmark, N.M. (1959), "A method of computation for structural dynamics", *J. Eng. Mech. Div.,* ASCE, **85**, 67-94.

Runge, C. (1895), "Über die numerische Auflösung von Differentialgleichungen", *Mathematische Annalen*, **46**, 167-178.

Senjanovic, I. (1984), "Harmonic accelaration method for dynamic structural analysis", *Comput. Struct.*, **18**(1), 71-80.

Shu, C. and Xue, H. (1997), "Explicit computations of weighting coefficients in the harmonic differential quadrature", *J. Sound Vib.,* **204**(3), 549-555.

Shu, C., Yao, Q. and Yeo, K.S. (2002), "Block-marching in time with DQ discretization: An efficient method for time-dependent problems", *Comput. Meth. Appl. Mech. Eng*., **191**, 4587-4597.

Striz, A.G., Wang, X. and Bert, C.W. (1995), "Harmonic differential quadrature method and applications to analysis of structural components", *Acta Mechanica*, **111**, 85-94.

Sun, K., Pires, J.A. and Tao, J.R. (1991), "A post-correction integration algorithm for non-linear dynamic analysis of structures", *Earth. Eng. Struct. Dyn.,* **20**, 1083-1097.

Tanaka, M. and Chen, W. (2001), "Dual reciprocity BEM applied to transient elastodynamics problems with differenential quadrature method in time", *Comput. Meth. Appl. Mech. Eng*., **190**, 2331-2347.

Tanaka, M. and Chen, W. (2001a), "Coupling dual reciprocity BEM and differential qudrature method for time-dependent diffusion problems", *Appl. Mathematical Modeling*, **25**(3), 257-268.

Wang, X. and Bert, C.W. (1993), "A new approach in applying differential quadrature to static and free vibrational analyses of beams and plates", *J. Sound Vib.,* **162**, 566-572.

Wood, W.L. (1990), *Practical Time-Stepping Schemes*, Clarendon Press, Oxford, U.K.

Wu, T.Y. and Liu, G.R. (2000), "The generalized differenential quadrature rule for initial-value differential equations", *J. Sound Vib*., **233**, 195-213.

Zienkiewicz, O.C. (1977), *The Finite Element Method in Engineering Science*, 3rd edt. McGraw- Hill, London.

Zienkiewicz, O.C. and Lewis, R.W. (1973), "An analysis of various time-stepping schemes for initial value problems", *Earth. Eng. Struct. Dyn*., **1**, 407-408.