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Thermoelastic solutions for annular disks with arbitrary variable thickness

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Abstract. This article presents a unified analytical solution for the analysis of thermal deformations and stresses in elastic annular disks with arbitrary cross-sections of continuously variable thickness. The annular disk is assumed to be under steady heat flow conditions, in which the inner surface of the annular disk is at an initial temperature and the outer surface at zero temperature. The governing second-order differential equation is derived from the basic equations of the thermal annular disks and solved with the aid of some hypergeometric functions. Numerical results for thermal stresses and displacement are given for various annular disks. These disks include annular disks of thickness profiles in the form of general parabolic and exponential functions. Additional annular disks with nonlinearly variable thickness and uniform thickness are also included.

Keywords: annular disks; steady heat flow; variable thickness; different profiles.

1. Introduction

Solid and annular disks are common structure types that can be used in applications involving turbine motors, flywheels, gears and shrink fits. The research on disks is always an important topic and their benefits have been included in the literature (Timoshenko and Goodier 1970). For instance, the subject of stationary disks under external pressure is also studied by Gamer (1983). Güven (1998) has considered an annular disk profile in exponential form and studied the effect of the application of external pressure analytically. Various thickness profiles including hyperbolic, exponential and power function form for annular disks were studied numerically by Eraslan (2002) with radial constrained and free boundary conditions. Zenkour and Allam (2006) have developed analytical solution for the analysis of deformation and stresses in viscoelastic rotating solid and annular disks with arbitrary cross-sections of continuously variable thickness.

For the effects of material inhomogeneity, Shaffer (1967) has obtained the general solutions for a non-homogeneous orthotropic annular disk subjected to uniform pressures at the internal and external surfaces. Horgan and Chan (1999a) have investigated on the response of linearly elastic isotropic hollow circular cylinders or disks under uniform internal or external pressure. In addition,

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Horgan and Chan (1999b) have investigated the effects of material inhomogeneity on the response of linearly elastic isotropic solid circular disks or cylinders, rotating at constant angular velocity about a central axis. In a recent paper, Zenkour (2006) has presented accurate elastic solutions for the rotating variable-thickness and/or uniform-thickness orthotropic circular cylinders containing a uniform-thickness solid core of rigid or homogeneously isotropic material. In Zenkour (2005), the closed form solutions for the rotating exponentially graded annular disk subjected to various boundary conditions are obtained.

When the structures are exposed to a temperature field, the thermal stresses are then induced. The research for thermoelastic problems is of increasing interest in engineering science and many works have been done. Parida and Das (1972) have early studied the transient thermal stresses in a homogeneous orthotropic thin circular disc due to an instantaneous point heat source. Kennedy and Gorman (1977) have determined the natural frequencies of transverse vibration of a variable-thickness disk subjected to combined actions of centrifugal loading and complex radial temperature distribution. Yoshihiro *et al.* (2004) have presented a method of material design for the weight reduction, the high thermal radiation and the relaxation of in-plane thermal stress and centrifugal stress in a solid rotating disk composed of functionally graded material with arbitrary thermal and mechanical non-homogeneities in the radial direction. Callioglu *et al.* (2005) have presented the thermal stress analysis of a curvilinear orthotropic rotating annular disc under internal and external pressures. The temperature distribution is chosen to vary parabolically from the inner surface to the outer surface along the radial section.

The objective of this investigation is to obtain thermoelastic solution for variable-thickness annular disks. The problem of an annular disk subjected to purely radial temperature variation is treated here. A state of plane stress and small deformations are presumed. Closed form solutions are obtained and numerical results are presented. The results include the radial stress, circumferential stress, and radial displacement for combinations of uniform and variable thickness disks. The distributions of displacement and stresses through the radial direction are obtained and comparisons between uniform thickness and variable thickness cases are made at the same temperature field.

2. Basic equations

The problems of composite thin disk are in a state of plane stress with neglecting the axial stress, $\sigma_z = 0$. As the effect of thickness variation of such disks can be taken into account in their equilibrium equation, the theory of the variable thickness disks can give excellent results as that of the uniform thickness disks as long as they meet the assumption of plane stress. Let us consider the axisymmetric problem of an annular disk. The disk is subjected to a temperature distribution, which varies with r only and is independent of θ. The equilibrium equation of disks with variable thickness can be written as

$$\frac{d}{dr}(h\sigma_r) + h\frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1}$$

where r is the radial coordinate, σ_r and σ_θ are the radial and circumferential stresses, and h is the thickness of the annular disk.

The relations between the radial displacement u_r and the strains are irrespective of the thickness of the disk. They can be written as

$$\varepsilon_r = \frac{du_r}{dr}, \quad \varepsilon_\theta = \frac{u_r}{r} \tag{2}$$

where ε_r and ε_θ are the radial and circumferential strains, respectively. For the elastic deformation, the relations between the stresses and strains (constitutive equations) can be described with Hooke's law

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} + \begin{Bmatrix} \alpha T \\ \alpha T \end{Bmatrix} \tag{3}$$

where α is the coefficient of linear thermal expansion. The stresses σ_r and σ_θ can be expressed explicitly in terms of strains by solving the above equation. These are

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}[\varepsilon_r + \nu\varepsilon_\theta - (1+\nu)\alpha T] \\ \sigma_\theta &= \frac{E}{1-\nu^2}[\varepsilon_\theta + \nu\varepsilon_r - (1+\nu)\alpha T] \end{aligned} \tag{4}$$

3. Thermoelastic solution

Substituting for σ_r and σ_θ in Eq. (4) into the radial equilibrium equation, Eq. (1), one obtains

$$r^2\frac{d^2u_r}{dr^2} + r\left[1 + \frac{r}{h}\frac{dh}{dr}\right]\frac{du_r}{dr} - \left[1 - \frac{r\nu}{h}\frac{dh}{dr}\right]u_r - \alpha r^2(1+\nu)\left[\frac{dT}{dr} + \frac{T}{h}\frac{dh}{dr}\right] = 0 \tag{5}$$

Let the inner surface ($r = a$) of the annular disk is at temperature T_0 while the outer surface ($r = b$) is temperature free. Assuming steady state conditions, the temperature at any distance r from the center is given by the expression

$$T = c\ln\left(\frac{c_1}{r}\right) \tag{6}$$

where c and c_1 are constants should be determined from the condition

$$T = \begin{cases} T_0 & \text{at} \quad r = a \\ 0 & \text{at} \quad r = b \end{cases} \tag{7}$$

Then, one can easily obtain

$$c_1 = b, \quad c = T_0/\ln\left(\frac{b}{a}\right) \tag{8}$$

Next, it is to be noted that the thermo-elastic model developed herein is not limited to a fixed thickness variation. It can be used for variable thickness with any functional form of thickness variability. The thickness of the disk is assumed sufficiently small compared to its diameter so that the plane stress assumption is justified. Therefore, the thicknesses of the annular disk are given using various distributions through the radial direction as follows:

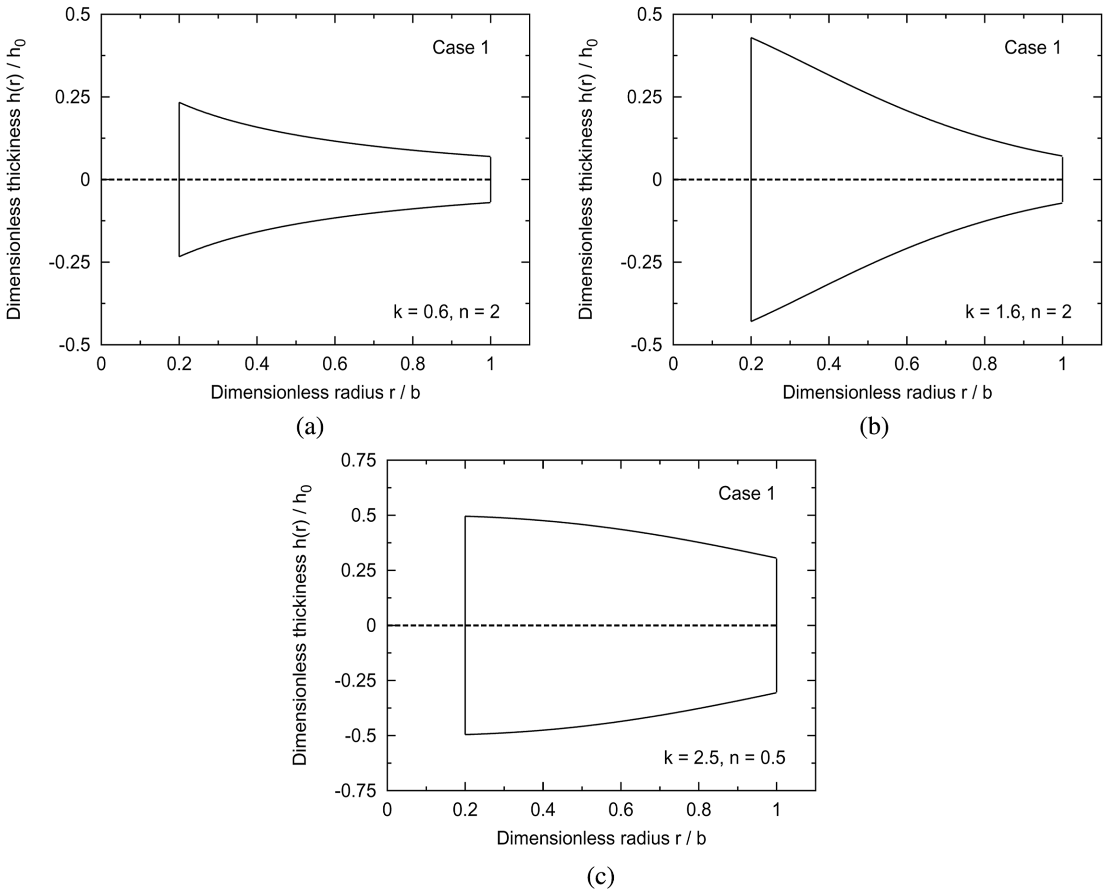


Fig. 1 Disk profiles according to Case 1: (a) $k = 0.6$, $n = 2$, (b) $k = 1.6$, $n = 2$, and (c) $k = 2.5$, $n = 0.5$

Case (1): *Exponentially distribution*
In this case,

$$h(r) = h_0 e^{-n(r/b)^k} \tag{9}$$

where h_0 is the thickness at the middle of the disk, k and n are geometric parameters. The above exponential law assumption reflects a simple rule of mixtures applies only to the radial direction. The geometric parameters k and n may be varied to obtain different distributions of the components materials through the radial direction of the disk. Fig. 1 displays the dimensionless thickness $h(r)/h_0$ as a function of the dimensionless radius r/b, for three different sets of geometric parameters k and n. Figs. 1(a) and (b) correspond to small k and large n ($k = 0.6$ and 1.6, $n = 2$) while the remaining convex profile, Fig. 1(c) corresponds to large k and small n ($k = 2.5$, $n = 0.5$). It is to be noted that the parameter n determines the thickness at the outer edge of the disk relative to h_0 while k determine the shape of the profile.

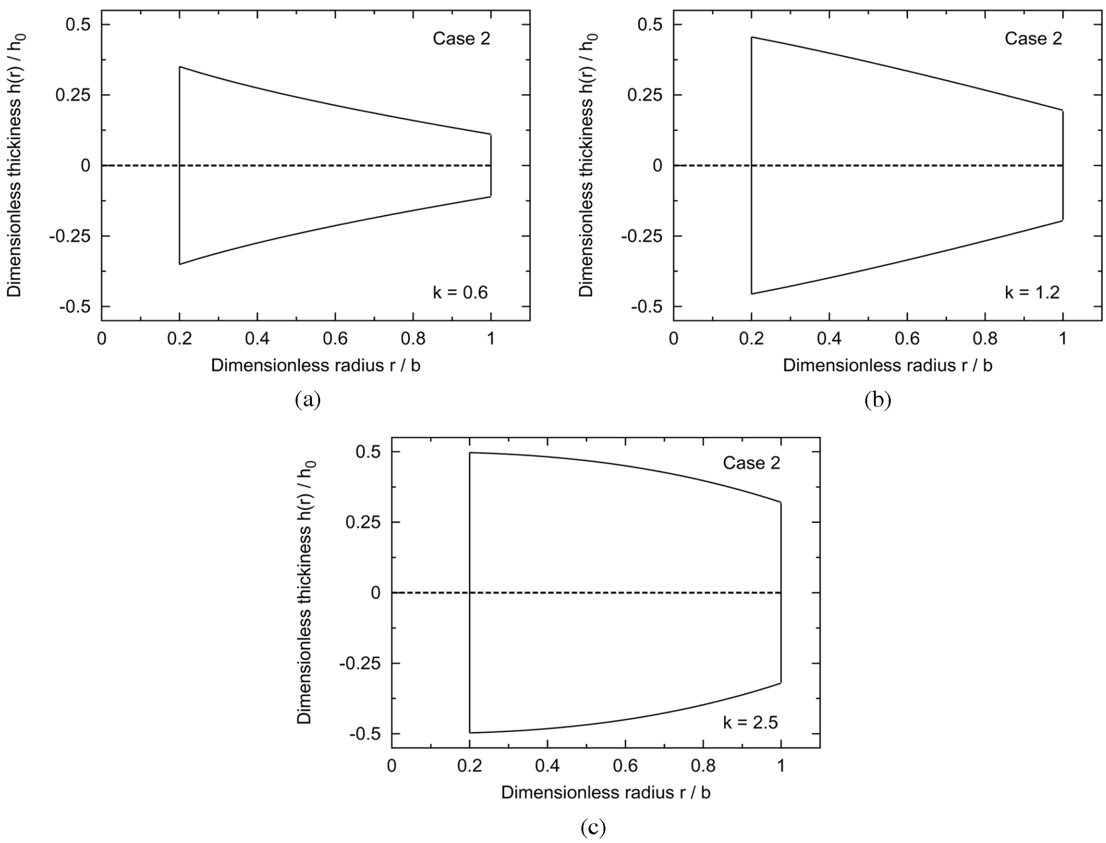


Fig. 2 Disk profiles according to Case 2 (n/b = 0.5): (a) k = 0.6, (b) k = 1.2, and (c) k = 2.5

Case (2): *General parabolic distribution*

Here, we consider an annular thin disk whose thickness vary continuously in the form of a general parabolic function $h(r)$,

$$h(r) = h_0\left(1 - \left(\frac{r}{b+n}\right)^k\right), \qquad n > 0, k \geq 0 \tag{10}$$

With this form of the profile function, a linearly decreasing disk thickness is obtained by setting $k = 1$. Fig. 2 displays $h(r)/h_0$ versus r/b for different values of k at a fixed value $n/b = 0.5$. Note that, for $k < 1$ the profile is concave (see Fig. 2(a)) and for $k > 1$ it is convex as shown in Figs. 2(b) and (c). In addition, the shape of the profile is smoothed as n increasing.

Case (3): *Nonlinearly distribution*

In this case,

$$h(r) = h_0\left(\frac{r}{b}\right)^{-2k} \tag{11}$$

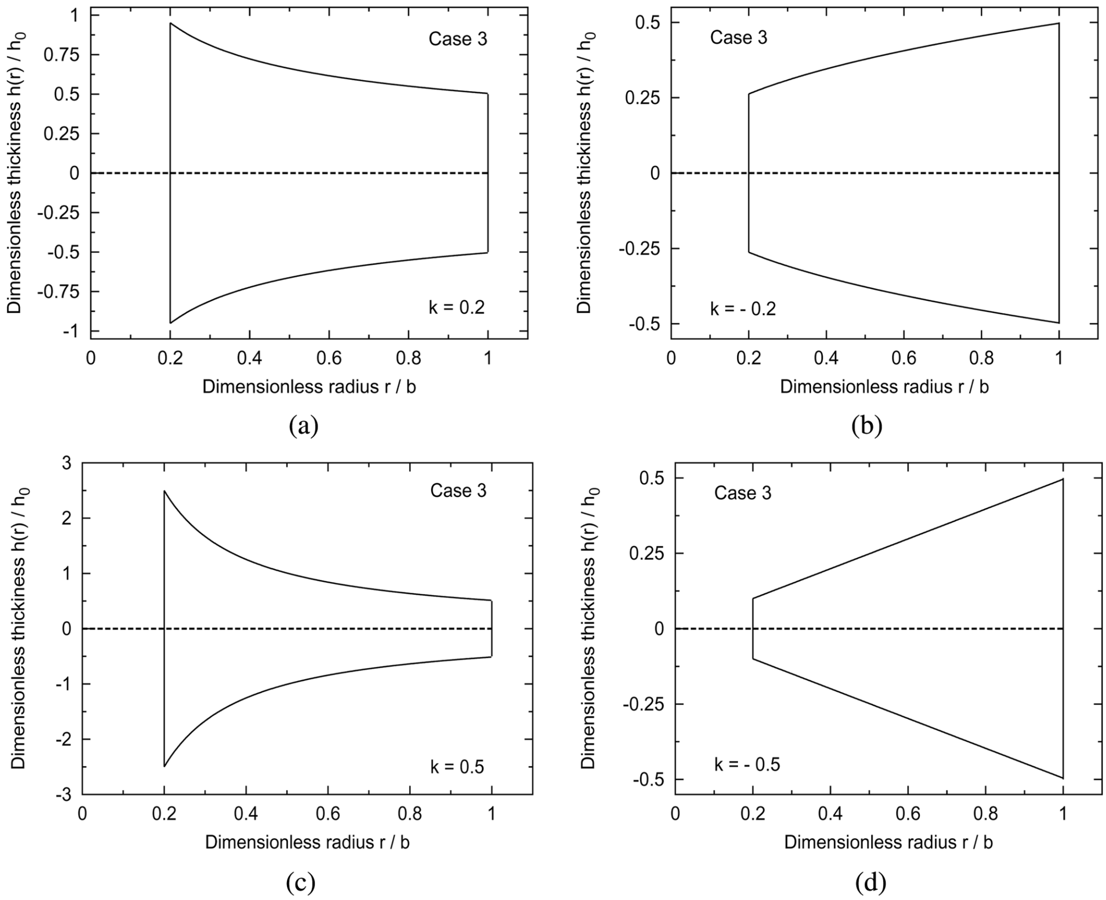


Fig. 3 Disk profiles according to Case 3: (a) k = 0.2, (b) k = −0.2, (c) k = 0.5, and (d) k = −0.5

Fig. 3 displays the dimensionless thickness as a function of the dimensionless radius for $k = \pm 0.2$ and ±0.5. For positive values of k, the profile is concave whereas it may be convex for negative values of k.

Case (4): *Uniform thickness and density*

Putting $k = 0$ in the above case, one obtains the case of uniform thickness annular disk. In addition, putting the value n equals to zero in Case (1) represents a uniform disk. However, a disk of uniform thickness is obtaining from Case (2) by setting $n \to \infty$.

Therefore, the general solution of Eq. (5) can be written for different cases as

$$u_r = AF_i(r) + BG_i(r) + U_i(r) \tag{12}$$

where A and B are arbitrary constants and the index "i" represents the number of the case studied. So, for the first two cases, one gets

$$F_1(r) = rH\left([\alpha_1], [\beta_1], n\left(\frac{r}{b}\right)^k\right) \tag{13a}$$

$$G_1(r) = \frac{1}{r}H\left([\alpha_1 - \beta_1 + 1], [2 - \beta_1], n\left(\frac{r}{b}\right)^k\right) \tag{13b}$$

$$F_2(r) = rH\left([\alpha_2, \beta_2], [\gamma], \left(\frac{r}{b+n}\right)^k\right) \tag{14a}$$

$$G_2(r) = \frac{1}{r}H\left([\beta_2 - \gamma + 1, \alpha_2 - \gamma + 1], [2 - \gamma], \left(\frac{r}{b+n}\right)^k\right) \tag{14b}$$

in which

$$\alpha_1 = \frac{1+\nu}{k}, \quad \beta_1 = 1 + \frac{2}{k} \tag{15a}$$

$$\alpha_2 = \frac{1}{2} + \frac{1}{k} - \frac{\sqrt{k^2 + 4(1 - k\nu)}}{2k}, \quad \beta_2 = \frac{1}{2} + \frac{1}{k} + \frac{\sqrt{k^2 + 4(1 - k\nu)}}{2k}, \quad \gamma = 1 + \frac{2}{k} \tag{15b}$$

The functions $H([\alpha_1], [\beta_1], z)$ and $H([\alpha_2, \beta_2], [\gamma], z)$ are the generalized hyper-geometric functions,

$$H([\alpha_1], [\beta_1], z) = \sum_{q=0}^{\infty} \frac{(\alpha_1)_q}{(\beta_1)_q q!} z^q, \quad \beta_1 \neq 0, \quad |z| < 1 \tag{16a}$$

$$H([\alpha_2, \beta_2], [\gamma], z) = \sum_{q=0}^{\infty} \frac{(\alpha_2)_q (\beta_2)_q}{(\gamma)_q q!} z^q, \quad \gamma \neq 0, \quad |z| < 1 \tag{16b}$$

where $(\eta)_q$ is the Pochhammer symbol given by

$$(\eta)_q = \eta(\eta+1)(\eta+2)\dots(\eta+q-1) = \frac{\Gamma(\eta+q)}{\Gamma(\eta)} \tag{16c}$$

in which Γ represents Gamma function.

For Cases 3 and 4, one can get easily

$$F_3(r) = r^{k + \sqrt{k^2 + 2k\nu + 1}}, \quad G_3(r) = r^{k - \sqrt{k^2 + 2k\nu + 1}} \tag{17}$$

and

$$F_4(r) = r, \quad G_4(r) = \frac{1}{r} \tag{18}$$

For all cases studied, the particular solution $U_i(r)$ for Eq. (12) is obtained using variation of parameters as

$$U_i(r) = F_i(r)P_i(r) + G_i(r)Q_i(r) \tag{19}$$

where

$$P_i(r) = -\int_0^r \frac{G_i(\xi) f_i(\xi)}{\Delta(\xi)} d\xi, \quad Q_i(r) = \int_0^r \frac{F_i(\xi) f_i(\xi)}{\Delta(\xi)} d\xi \tag{20}$$

in which

$$f_1(r) = -\frac{(1+\nu)\alpha c}{r}\left[1 + kn\left(\frac{r}{b}\right)^k \ln\left(\frac{b}{r}\right)\right] \tag{21a}$$

$$f_2(r) = -\frac{(1+\nu)\alpha c}{r}\left\{1 + k\left(\frac{r}{b+n}\right)^k \left[1 - \left(\frac{r}{b+n}\right)^k\right]^{-1} \ln\left(\frac{b}{r}\right)\right\} \tag{21b}$$

$$f_3(r) = -\frac{(1+\nu)\alpha c}{r}\left[1 + 2k\ln\left(\frac{b}{r}\right)\right] \tag{21c}$$

$$f_4(r) = -\frac{(1+\nu)\alpha c}{r} \tag{21d}$$

and $\Delta(r)$ is the Wronskian given by

$$\Delta(r) = F_i(r)\frac{dG_i(r)}{dr} - G_i(r)\frac{dF_i(r)}{dr} \tag{22}$$

The particular solution for Cases (1) and (2) may be given by

$$U_i(r) = -F_i(r)\int_0^r \frac{G_i(\xi) f_i(\xi)}{H_i(\xi)} d\xi + G_i(r)\int_0^r \frac{F_i(\xi) f_i(\xi)}{H_i(\xi)} d\xi, \quad i = 1, 2 \tag{23}$$

where

$$\begin{aligned} H_1(r) = &\left\{\frac{kR_1(\alpha_1 - \beta_1 + 1)}{r(2-\beta_1)} H([\alpha_1 - \beta_1 + 2], [3 - \beta_1], R_1)\right. \\ &\left. -\frac{2}{r} H([\alpha_1 - \beta_1 + 1], [2 - \beta_1], R_1)\right\} H([\alpha_1], [\beta_1], R_1) \\ &-\frac{kR_1\alpha_1}{r\beta_1} H([\alpha_1 + 1], [\beta_1 + 1], R_1) H([\alpha_1 - \beta_1 + 1], [2 - \beta_1], R_1) \end{aligned} \tag{24a}$$

$$\begin{aligned} H_2(r) = &-\frac{1}{r}\left\{2H([\alpha_2 - \gamma + 1, \beta_2 - \gamma + 1], [2 - \gamma], R_2) - \frac{kR_2}{r(2-\gamma)}[\alpha_2\beta_2 + (1-\gamma)\right. \\ &\left.(\alpha_2 + \beta_2 + 1 - \gamma)] H([\alpha_2 - \gamma + 2, \beta_2 - \gamma + 2], [3 - \gamma], R_2)\right\} H([\alpha_2, \beta_2], [\gamma], R_2) \\ &+ \frac{kR_2\alpha_2\beta_2}{r\gamma} H([\alpha_2 - \gamma + 1, \beta_2 - \gamma + 1], [2 - \gamma], R_2) H([\alpha_2 + 1, \beta_2 + 1], [\gamma + 1], R_2) \end{aligned} \tag{24b}$$

in which

$$R_1 = n\left(\frac{r}{b}\right)^k, \quad R_2 = \left(\frac{r}{n+b}\right)^k \tag{24c}$$

The particular solution for Cases (3) and (4) may be given, respectively, by

$$U_3(r) = \frac{cr\alpha}{1+\nu}\left[1-(1+\nu)\ln\left(\frac{r}{b}\right)-\frac{1-\nu}{2k}\right] \tag{25a}$$

and

$$U_4(r) = \frac{1}{4}cr\alpha(1+\nu)[1-2\ln(r)] \tag{25b}$$

The radial and circumferential stresses are obtained with the help of Eqs. (2) and (12) from Eq. (4) as

$$\sigma_r = \frac{E}{1-\nu^2}\left[A\left(\frac{\nu F_i}{r}+\frac{dF_i}{dr}\right)+B\left(\frac{\nu G_i}{r}+\frac{dG_i}{dr}\right)+\frac{\nu U_i}{r}+\frac{dU_i}{dr}-(1+\nu)c\alpha\ln\left(\frac{b}{r}\right)\right] \tag{26a}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[A\left(\frac{F_i}{r}+\nu\frac{dF_i}{dr}\right)+B\left(\frac{G_i}{r}+\nu\frac{dG_i}{dr}\right)+\frac{U_i}{r}+\nu\frac{dU_i}{dr}-(1+\nu)c\alpha\ln\left(\frac{b}{r}\right)\right] \tag{26b}$$

The elastic solution for the annular disk is completed by the application of the boundary conditions. The radial stress should be vanished at the inner and outer radii, i.e., $\sigma_r = 0$ at $r = a$ and $r = b$. Hence

$$A = \frac{\bar{A}}{\bar{\Delta}}, \quad B = \frac{\bar{B}}{\bar{\Delta}} \tag{27}$$

where

$$\begin{aligned}\bar{A} &= b[ac\alpha(1+\nu)\ln(\bar{a})+\nu U_i(a)+aU_i'(a)]G_i'(b)-a[\nu U_i(b)+bU_i'(b)]G_i'(a)\\&+\nu[ac\alpha(1+\nu)\ln(\bar{a})+\nu U_i(a)+aU_i'(a)]G_i(b)-\nu[\nu U_i(b)+bU_i'(b)]G_i(a)\end{aligned}$$

$$\begin{aligned}\bar{B} &= a[\nu U_i(b)+bU_i'(b)]F_i'(a)-b[ac\alpha(1+\nu)\ln(\bar{a})+\nu U_i(a)+aU_i'(a)]F_i'(b)\\&+\nu[\nu U_i(b)+bU_i'(b)]F_i(a)-\nu[ac\alpha(1+\nu)\ln(\bar{a})+\nu U_i(a)+aU_i'(a)]F_i(b)\end{aligned}$$

$$\begin{aligned}\bar{\Delta} &= b[\nu G_i(a)+aG_i'(a)]F_i'(b)-a[\nu G_i(b)+bG_i'(b)]F_i'(a)\\&+\nu[\nu G_i(a)+aG_i'(a)]F_i(b)-\nu[\nu G_i(b)+bG_i'(b)]F_i(a), \quad \bar{a} = a/b\end{aligned} \tag{28}$$

4. Numerical examples and discussion

The thermoelastic solutions for all cases studied are considered. The results are presented in terms of the following dimensionless and normalized variables:

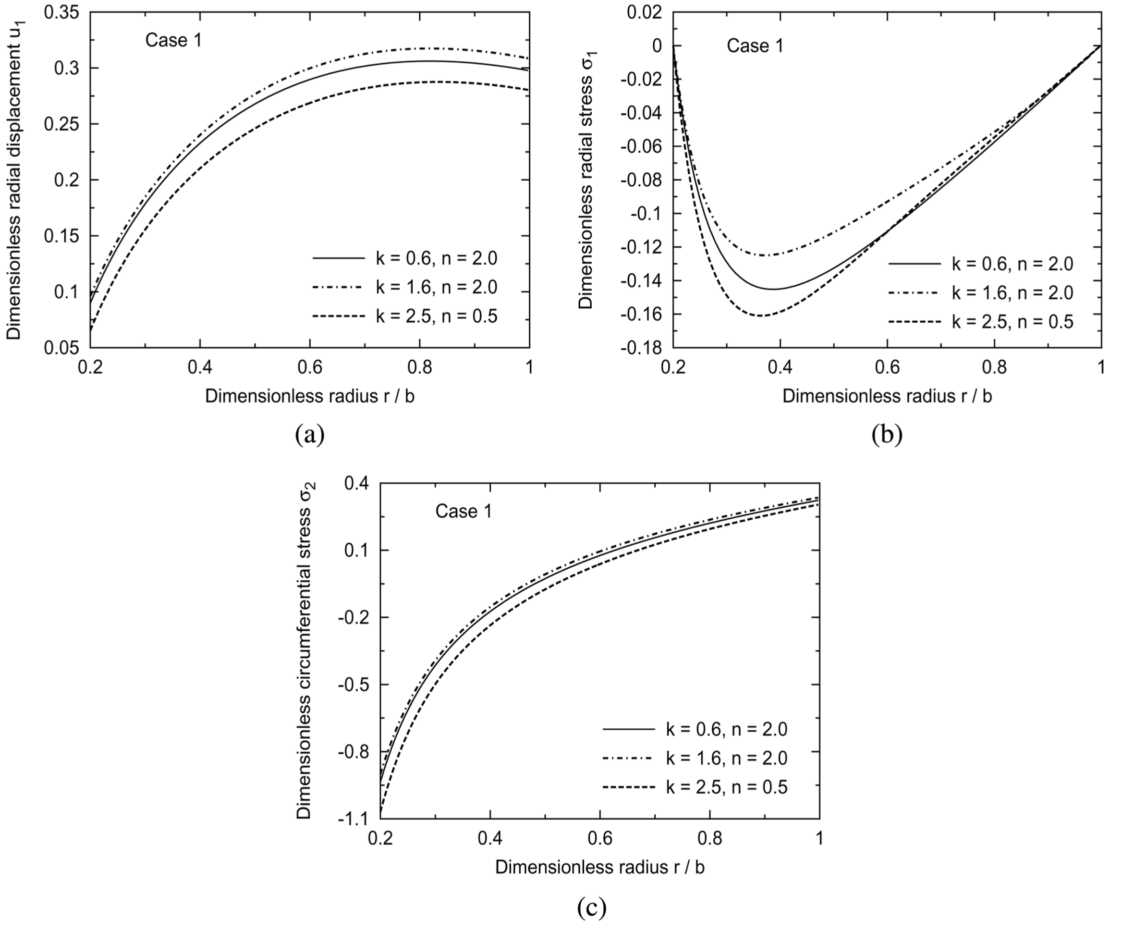


Fig. 4 Dimensionless stresses and displacement for an annular disk (Case 1): (a) radial displacement u_1, (b) radial stress σ_1, and (c) circumferential stress σ_2

displacement: $u_1 = \dfrac{u_r}{\alpha b T_0}$

radial stress: $\sigma_1 = \dfrac{\sigma_r}{\alpha E T_0}$, and

circumferential stress: $\sigma_2 = \dfrac{\sigma_\theta}{\alpha E T_0}$

Taking the value of Poisson's ratio is fixed $\nu = 0.3$, Young's modulus $E = 70$ GPa and $\overline{a} = 0.2$. The results of the present investigations are displayed in Figs. 4-8. Radial displacement u_1, radial stresses σ_1, and circumferential stress σ_2 are plotted in Fig. 4 for Case 1. Similar results for Cases 2 and 3, respectively, are plotted in Figs. 5 and 6. The circumferential stress σ_2 is monotone increasing in $\overline{r}\,(\equiv r/b)$ with its maximum occurring at the outer surface $\overline{r} = 1$. The radial

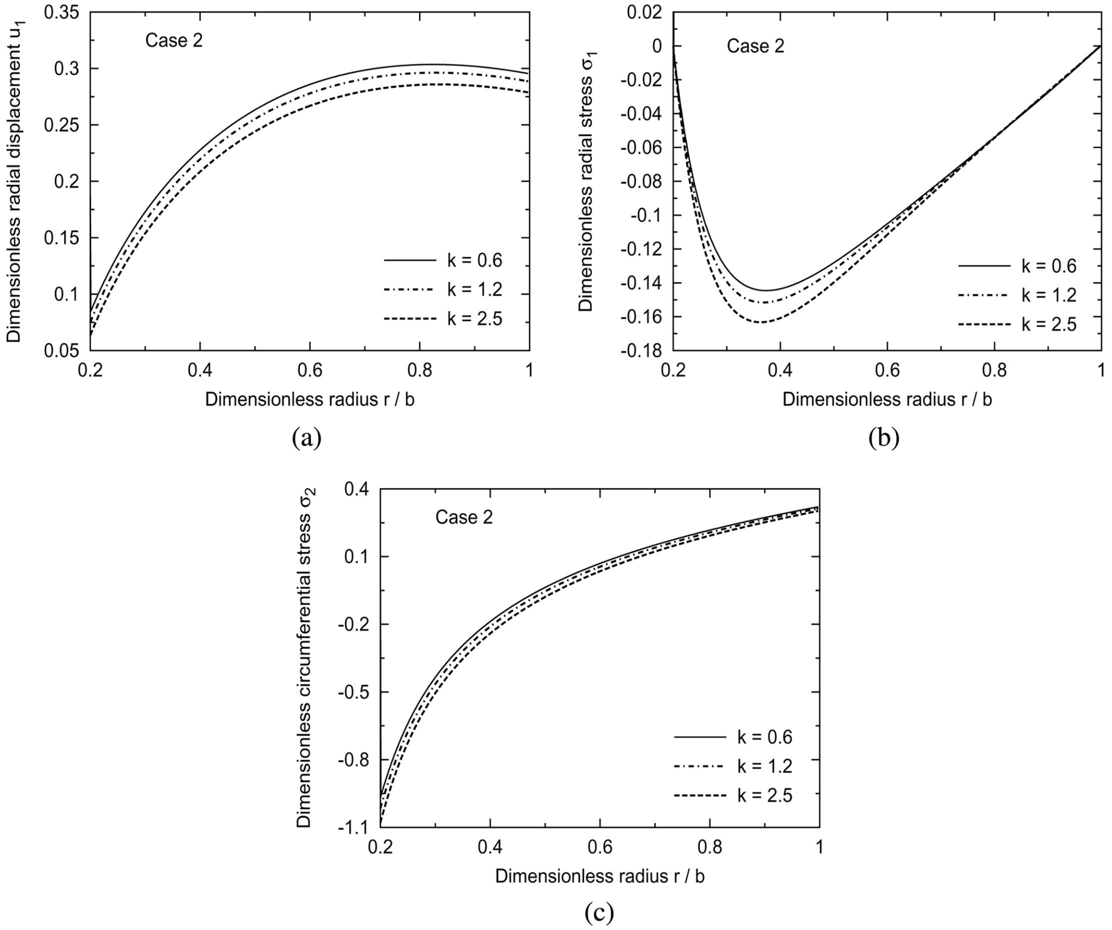
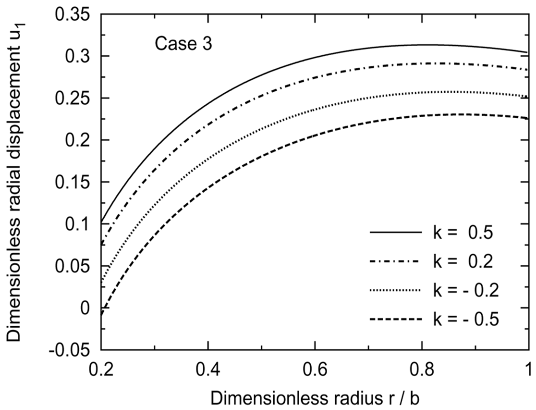


Fig. 5 Dimensionless stresses and displacement for an annular disk (Case 2): (a) radial displacement u_1, (b) radial stress σ_1, and (c) circumferential stress σ_2

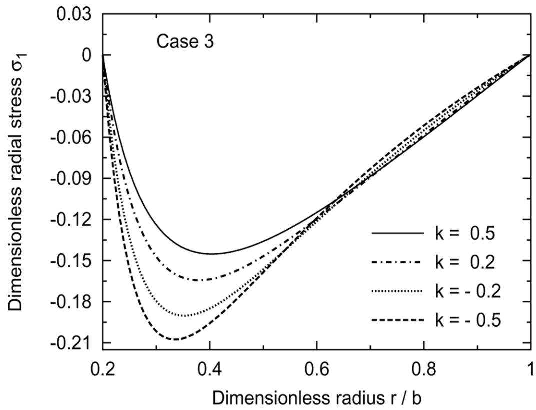
displacement u_1 is no longer monotonic increasing in $\overline{r}$ and has a single maximum at the vicinity of the outer surface. The radial stress σ_1 is no longer monotonic decreasing in $\overline{r}$ and has a single minimum at the vicinity of the inner surface. For all cases, the radial stress σ_1 is compressive at all points, whereas the circumferential stress σ_2 is compressive at the inner surface and tensile at the outer surface.

Fig. 7 shows plots of u_1 and σ_1 for Case 1 (k = 1.5, n = 0.5), Case 2 (k = 1.5, n/b = 0.5), Case 3 (k = 0.1), and Case 4. It is to be noted that, results for the uniform thickness disk (Case 4) are the smallest ones.

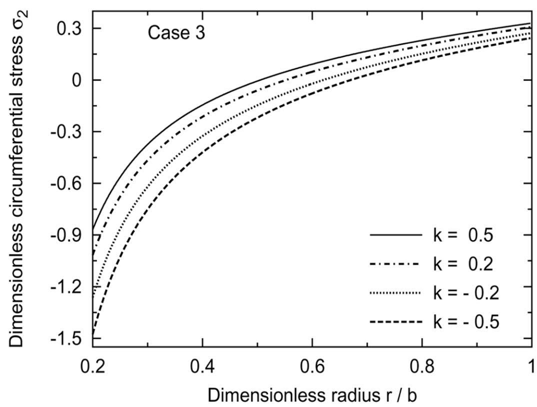
Finally, Fig. 8 shows a comparison of the results of Case 1 and Case 4 for annular disks subjected to a through the radial temperature field. For the considered values of $\overline{r}$ radial displacement u_r/b increases with increase in $\overline{r}$ and temperature. For $\overline{r} > 0.5$, σ_θ also increases with increase in $\overline{r}$ and temperature. However, σ_r decreases with increase in temperature and increases with increase in $\overline{r}$.



(a)

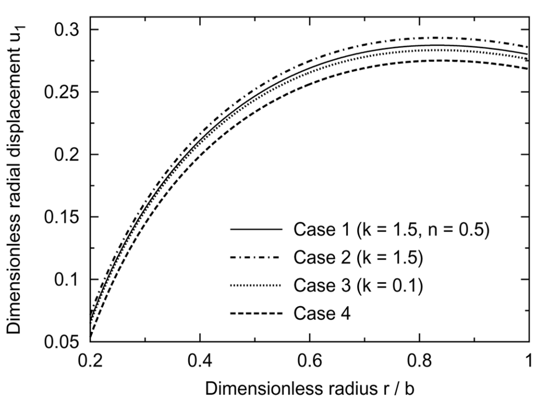


(b)

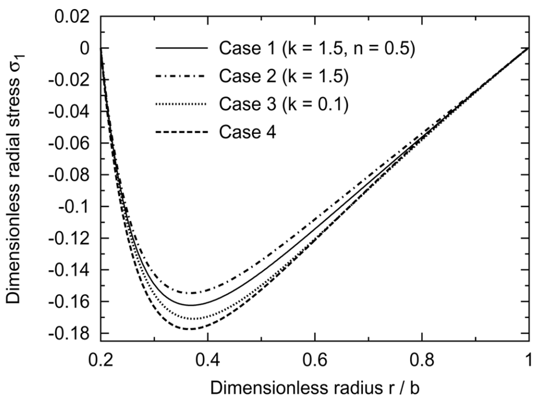


(c)

Fig. 6 Dimensionless stresses and displacement for an annular disk (Case 3): (a) radial displacement u_1, (b) radial stress σ_1, and (c) circumferential stress σ_2



(a)



(b)

Fig. 7 Dimensionless (a) radial displacement and (b) radial stress of an annular disk according to all cases

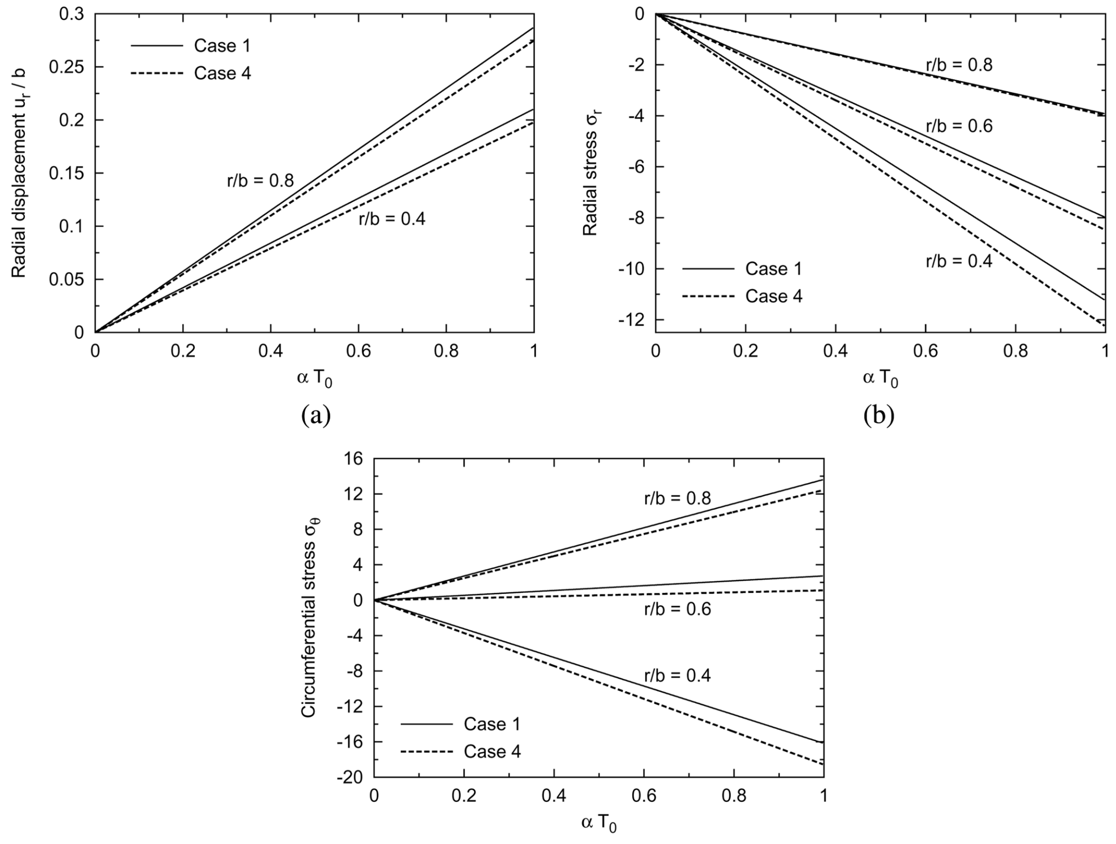


Fig. 8 Stresses and displacement for an annular disk versus αT_0 according to Case 1 and Case 4: (a) radial displacement u_r/b, (b) radial stress σ_r, and (c) circumferential stress σ_θ

5. Conclusions

A thin disk subjected to a temperature distribution which varies with its radius and independent of the circumferential angle, is considered. It is assumed further that it is vary over the thickness and, consequently, it is taken that the stresses and displacement also vary over the thickness. The governing equation is derived from the basic equations of the disks and solved with the help of the generalized hyper-geometric functions. These disks include annular disks with uniform thickness, nonlinearly variable thickness, and disks of thickness profile in the form of general parabolic and exponential functions. If the temperature is constant, then all stresses are zero and the radial displacement is given in terms of the constant temperature. If the temperature at the inner surface is positive, the radial stress is compressive at all points, whereas the circumferential stress is compressive at the inner surface and tensile at the outer surface.

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