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Buckling analysis of partially embedded pile in elastic soil using differential transform method

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Abstract. The parts of pile, above the soil and embedded in the soil are called the first region and second region, respectively. The forth order differential equations of both region for critical buckling load of partially embedded pile with shear deformation are obtained using the small-displacement theory and Winkler hypothesis. It is assumed that the behavior of material of the pile is linear-elastic and that axial force along the pile length and modulus of subgrade reaction for the second region to be constant. Shear effect is included in the differential equations by considering shear deformation in the second derivative of the elastic curve function. Critical buckling loads of the pile are calculated for by differential transform method (DTM) and analytical method, results are given in tables and variation of critical buckling loads corresponding to relative stiffness of the pile are presented in graphs.

Keywords: static stability; differential transform method; critical buckling load; partially embedded pile; non-trivial solution.

1. Introduction

The piles partially embedded in the soil are used marine, harbor, bridge structure and modeled mostly by equivalent soil spring model. In this model, soil is idealized by Winkler hypothesize (Chen 1997). Elastic soil is idealized by Winkler foundation modulus in this study also and effect of friction through the pile length is neglected.

For designing of these piles require calculation of the buckling load of the piles.

Many researches have studied the behavior of the beams on an elastic foundation and elastic buckling of columns, beams, plates and shells in the past. Hetenyi (1995) has studied beams on Winkler foundations. Reddy and Valsangkar (1970) have obtained buckling loads for fully and partially embedded piles using vibration functions and the Rayleight-Ritz energy method. Smith (1979) has obtained discrete element matrices for stability analysis of slender piles, assuming conservative or non-conservative ground resistance. Pavlovic and Tsikkos (1982) have studied the problem of beam supported on quasi-Winkler foundations. West *et al.* (1997) have investigated stability of end-bearing piles in non-homogeneous elastic foundation. They have neglected shear effect and assumed the coefficient of horizontal subgrade reaction varies linearly with depth. Capron and Williams (1988) have obtained the dynamic stiffness of Timoshenko column embedded in

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elastic medium. Heelis *et al*. (2004) have calculated buckling load of Euler-Bernoulli pile embedded in Winkler foundation using analytical analyses of the pile. Heelis *et al.* (1999) have investigated the stability of uniform-friction piles in homogeneous and non-homogeneous elastic foundation using a power-series solution and neglecting shear effect. West and Mafi (1984) have determined buckling loads, natural frequencies of Euler-Bernoulli beam rested on elastic supports by using an initial-value numerical method. Chen (1998) has calculated displacements, bending moments and shear forces of Euler-Bernoulli beam resting on an elastic foundation using the differential quadrature element method. Doyle and Pavlovic (1987) have solved the partial differential equation for free vibration of beams attached to elastic foundation using variable separating method and neglecting axial force and shear deformation. Valsangkar and Pradhanang (1987) studied the variations of natural circular frequency values of the piles partially embedded in the soil according to relative stiffness of the piles, length of the piles and buckling load ignoring the shear force effect. Budkowska and Szymczak (1996) investigated the initial post-buckling load equilibrium path of the pile partially embedded in soil after flexural buckling. The equilibrium path is determined by utilizing a perturbation approach. Çatal and Alku (1996a) have obtained the second order stiffness matrix of Euler-Bernoulli beam on elastic foundation using analytical method. Chen (1997) determined fixity depths required in equivalent cantilever pile model. Çatal and Alku (1996b) have calculated vertical displacements of Timoshenko beam on elastic foundation using finite difference equations and matrix-displacement method and compared the solutions. Aydoğan (1995) has obtained a stiffness matrix for a Timoshenko beam on elastic foundation using differential-equation. Ergüven and Gedikli (2003) have derived a finite element formulation for Timoshenko beam on elastic foundation by considering second-order effects. Li (2001a) has obtained critical buckling load of multi-step cracked columns with shear deformation by using transfer matrix. Li (2001b) has governed differential equation for buckling of a multi-step non-uniform beam. The shear effect of the beam was neglected in the equation. Banarjee and Williams (1994) have investigated the effects of shear deformation on the critical buckling of columns. Yang and Ye (2002) have studied a dynamic elastic local buckling analysis for a pile subjected to an axial impact load using a perturbation technique. Wang *et al*. (2002) have investigated exact stability criteria and buckling loads of Timoshenko columns under intermediate and end concentrated loads using analytical method. Çatal (2002) has obtained fourth order differential equations for free vibration of partially embedded pile in soil.

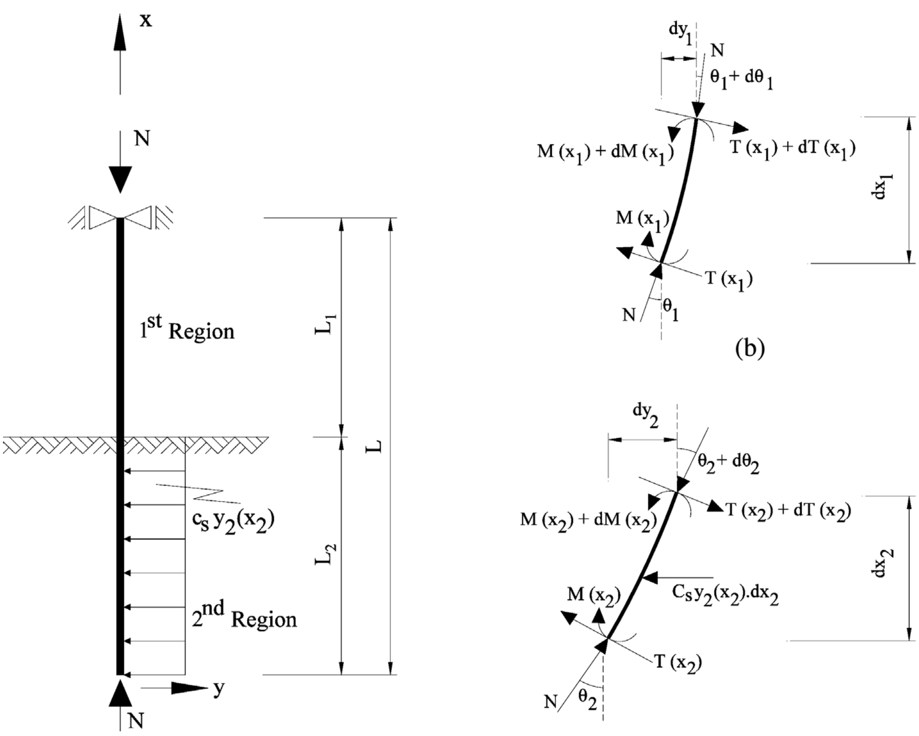
The differential transform method (DTM) which was introduced by Zhou in 1986 for the solution of initial value problems in electric circuit analysis is based on Taylor series expansions. In recent works, DTM is applied to vibration analysis of continuous systems as beams and plates. Jang and Chen (1997), the differential transformation method is Jang and Chen, the differential transformation method is applied to solve a second order non-linear differential equation that describes the under damped and over damped motion of a system subject to external excitations. According to types of conditions at both end of a prismatic Bernoulli-Euler beam, frequency equations and fundamental frequencies of the beam have obtained using DTM by Malik and Dang (1998). Chen and Ho (1996), using differential transform technique proposed a method to solve eigenvalue problems for the free and transverse vibration problems of a rotating twisted Timeshenko beam under axial loading. Özdemir and Kaya (2006), flapwise bending vibration of a rotating tapered cantilever Bernoulli-Euler Beam is considered by using differential transform technique. Ruotolo and Surace (2004) calculated natural frequencies of a bar with many cracks using transfer matrix approach and finite element method. Hosking *et al.* (2004) studied natural flexural vibrations of Bernoulli-Euler beam mounted on discrete elastic supports using transfer matrices. Coupling lateral and torsional vibrations of symmetric rotating shaft modeled by the Timoshenko beam is examined using modified transfer matrix method by Hsieh *et al.* (2006).

The DTM used in this study was proposed by Zhou (1986). DTM is one of the solution methods of ordinary and partial differential equations. DTM has advantage of reducing the ordinary differential equation to the algebraic equation and reducing the partial differential equation to the algebraic equation system. In DTM, the orthogonal polynoms as Taylor series are used for solution of the differential equations and to apply mathematical operations to these polynms are easier.

In this study, forth-order differential equations of elastic curves for critical buckling load of partially embedded pile in elastic soil are developed considering shear effect, these differential equations are solved using differential transform method (DTM) and analytical method, and critical buckling loads for the first three modes of the pile are obtained. Numerical results are presented and the differential transform solutions are compared with the analytical solutions.

2. Problem formulation

A pile partially embedded in the soil is presented in Fig. 1(a). The pile parts above the soil and embedded in the soil are called the first region and the second region, respectively. The internal



(a) (b) (c)

Fig. 1(a) Pile partially embedded in the soil, (b) internal forces and deformations of segment in the first region, (c) internal forces and deformations of segment in the second region forces and deformations of segment of the pile having the length of dx_1 and dx_2 at the first and second regions are presented in Fig. 1(b) and Fig. 1(c), respectively.

The buckling loads of the partially embedded pile are calculated under the following assumptions: material behavior of the pile is linear-elastic; soil behavior coincides with Winkler hypothesis; effect of friction along the pile length is neglected.

Using the equilibrium equations of the lateral load and bending moment acting to segment of the pile in the first and the second region and neglecting infinitesimal quantities of second order gives

$$\frac{dM_1(x_1)}{dx_1} = T_1(x_1) \qquad (0 \le x_1 \le L_1) \tag{1}$$

$$\frac{dT_1(x_1)}{dx_1} = N \cdot \frac{d\theta_1}{dx_1} \qquad (0 \le x_1 \le L_1) \tag{2}$$

$$\frac{dM_2(x_2)}{dx_2} = T_2(x_2) \qquad (0 \le x_2 \le L_2) \tag{3}$$

$$\frac{dT_2(x_2)}{dx_2} = N\frac{d\theta_2}{dx} + C_s y_2(x_2) \qquad (0 \le x_2 \le L_2) \tag{4}$$

where $M_1(x_1)$, $M_2(x_2)$ and $T_1(x_1)$, $T_2(x_2)$ are bending moment and shear force functions for the first and the second regions, respectively; N is the constant axial compressive force; θ_1 and θ_2 are slope of elastic curve in the first and second region, respectively, $C_s = C_0 \cdot b$ in which C_0 is the modulus of subgrade reaction and b is width of the pile.

Substituting Eqs. (2) and (4) into the second order derivate with respect to x of elastic curve equations gives

$$\left[1 - \frac{N}{\bar{k}AG}\right]\frac{d^2 y_1(x_1)}{dx_1^2} + \frac{M_1(x_1)}{EI} = 0 \qquad (0 \le x_1 \le L_1) \tag{5}$$

$$\left[1 - \frac{N}{\bar{k}AG}\right]\frac{d^2 y_2(x_2)}{dx_2^2} - \frac{C_s}{\bar{k}AG}y_2(x_2) + \frac{M_2(x_2)}{EI} = 0 \qquad (0 \le x_2 \le L_2) \tag{6}$$

Where, $y_1(x_1)$ and $y_2(x_2)$ are elastic curve functions for the first and second regions, respectively.

Substituting Eqs. (1) and (3) into the first order derivate with respect to x of Eqs. (5) and (6) gives

$$\left[1 - \frac{N}{\bar{k}AG}\right]\frac{d^3 y_1(x_1)}{dx_1^3} + \frac{T_1(x_1)}{EI} = 0 \qquad (0 \le x_1 \le L_1) \tag{7}$$

$$\left[1 - \frac{N}{\bar{k}AG}\right]\frac{d^3 y_2(x_2)}{dx_2^3} - \frac{C_s}{\bar{k}AG}\frac{dy_2(x_2)}{dx_2} + \frac{T_2(x_2)}{EI} = 0 \qquad (0 \le x_2 \le L_2) \tag{8}$$

Differentiating Eqs. (7) and (8) with respect to x gives

$$\left[1 - \frac{N}{\bar{k}AG}\right]\frac{d^4 y_1(x_1)}{dx_1^4} + \frac{dT_1(x_1)}{dx_1}\frac{1}{EI} = 0 \qquad (0 \le x_1 \le L_1) \tag{9}$$

$$\left[1-\frac{N}{\bar{k}AG}\right]\frac{d^4y_2(x_2)}{dx_2^4}-\frac{C_s}{\bar{k}AG}\frac{d^2y_2(x_2)}{dx_2^2}+\frac{dT_2(x_2)}{dx_2}\frac{1}{EI} = 0 \qquad (0 \le x_2 \le L_2) \tag{10}$$

Substituting Eqs. (2) and (4) into Eqs. (9) and (10) respectively, given

$$\frac{d^4y_1(x_1)}{dx_1^4}+\frac{\bar{k}AGN}{(\bar{k}AG-N)EI}\frac{d^2y_1(x_1)}{dx_1^2} = 0 \qquad (0 \le x_1 \le L_1) \tag{11}$$

$$\frac{d^4y_2(x_2)}{dx_2^4}+\frac{\bar{k}AGN-EIC_S}{(\bar{k}AG-N)EI}\frac{dy_2^2(x_2)}{dx_2^2}-\left[\frac{\bar{k}AG\cdot C_S}{EI(N-\bar{k}AG)}\right]y_2(x_2) = 0 \tag{12}$$

where $\bar{k}$ is the shape factor due to cross-section geometry of the pile, I, A, E, G are moment of inertia, cross-section area, modulus of elasticity, shear modulus, respectively, of the pile.

Writing the dimensionless parameters z_1, z_2 instead of the position parameters x_1, x_2 in Eqs. (11) and (12) gives the elastic curve function of the pile at the first and the second region as

$$\frac{d^4y_1(z_1)}{dz_1^4}+D_1\frac{d^2y_1(z_1)}{dz_1^2} = 0 \qquad \left(0 \le z_1 \le \frac{L_1}{L}\right) \tag{13}$$

$$\frac{d^4y_2(z_2)}{dz_2^4}+\beta_1\frac{d^2y_2(z_2)}{dz_2^2}+\beta_2y_2(z_2) = 0 \qquad \left(0 \le z_2 \le \frac{L_2}{L}\right) \tag{14}$$

Where $\beta_1 = \dfrac{L^2(\bar{k}AGN-EIC_S)}{EI(\bar{k}AG-N)}$; $\beta_2 = \dfrac{L^4\cdot\bar{k}AG\cdot C_S}{EI(N-\bar{k}AG)}$; $D_1 = \dfrac{NL^2}{EI}\left(\dfrac{\bar{k}AG}{\bar{k}AG-N}\right)$; $\alpha = \dfrac{C_SL^4}{EI}$

α being the relative stiffness value; L_1 = pile length above the soil; L_2 = pile length embedded in the soil; L = total length of the pile; $z_1 = x_1/L$; $z_2 = x_2/L$

3. Differential transformation

The differential transformation technique, which was first proposed by Zhou (1986), is one of the numerical methods for ordinary and partial differential equations that use the form of polynomials as the approximation to the exact solutions that are sufficiently differentiable. The function that will be solved and the calculation of following derivatives necessary in the solution become more difficult when the order increases. This is in contrast with the traditional high-order Taylor series method. Instead, the differential transform technique provides an iterative procedure to obtain higher-order series; therefore, it can be applied to the case high order.

The differential transformation of the function $y(z)$ is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^ky(z)}{dz^k}\right]_{z=z_0} \tag{15}$$

Where $y(z)$ is the original function and $Y(k)$ is transformed function which is called the T-function (it is also called the spectrum of the $y(z)$ at $z = z_0$, in the K domain). The differential inverse transformation of $Y(k)$ is defined as:

Table 1 Some basic mathematical operations of DTM

Original function $y(z)$	Transformed function $Y(k)$
$Ay(z)$	$AY(k)$
$y_1(z) \pm y_2(z)$	$Y_1(k) \pm Y_2(k)$
$dy(z)/dz$	$(k+1)\ Y(k+1)$
$d^2y(z)/dz^2$	$(k+1)(k+2)\ Y(k+2)$
$d^3y(z)/dz^3$	$(k+1)(k+2)(k+3)\ Y(k+3)$
$d^4y(z)/dz^4$	$(k+1)(k+2)(k+3)(k+4)\ Y(k+4)$

$$y(z) = \sum_{k=0}^{\infty} (z-z_0)^k Y(k) \tag{16}$$

from Eq. (13) and Eq. (14) we get

$$y(k) = \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!}\left[\frac{d^k y(z)}{dz^k}\right]_{z=z_0} \tag{17}$$

Eq. (16) implies that the concept of the differential transformation is derived from Taylor's series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivative are calculated by iterative procedure that are described by the transformed equations of the original functions.

The basic operations of transformed functions which are given Table 1 can easily be proofed using Eqs. (15) and (16).

The function is expressed by finite series and Eq. (16) can be written as $y(z) = \sum_{k=0}^{n} (z-z_0)^k Y(k)$. Eq. (16) implies that $y(z) = \sum_{k=n+1}^{\infty} (z-z_0)^k Y(k)$ is negligibly small. In fact, n is decided by the convergence of natural frequency in this paper.

4. Solution of motion equations by differential transformation

The boundary conditions of the pile whose both ends simply supported shown in Fig. 2 are given in Eqs. (18)-(35).

$$y_1(z_1 = L_1/L) = 0 \tag{18}$$

$$y_2(z_2 = 0) = 0 \tag{19}$$

$$\left.\frac{d^2 y_2(z_2)}{dz_2^2}\right|_{z_2=0} = -\beta_1 y_2(z_2 = 0) \tag{20}$$

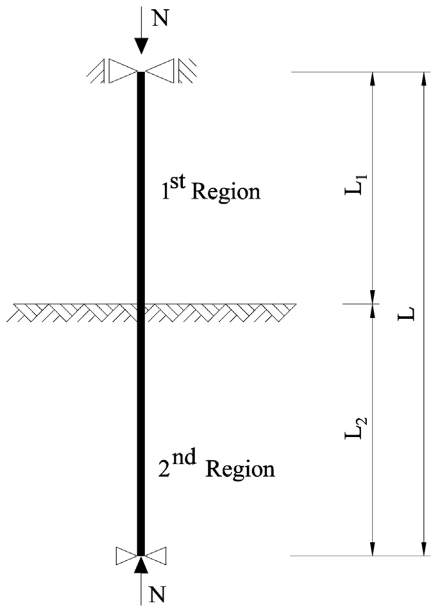


Fig. 2 Pile whose both ends are simply supported

$$\left.\frac{d^2y_1(z_1)}{dz_1^2}\right|_{z_1=\frac{L_1}{L}} = -D_1y_1(z_1 = L_1/L) \tag{21}$$

$$y_1(z_1 = 0) = y_2(z_2 = L_2/L) \tag{22}$$

$$\left.\frac{dy_1(z_1)}{dz_1}\right|_{z_1=0} = \left.\frac{dy_2(z_2)}{dz_2}\right|_{z_2=\frac{L_2}{L}} \tag{23}$$

$$\left.\frac{d^3y_2(z_2)}{dz_2^3}\right|_{z_2=\frac{L_2}{L}} + \beta_1\left.\frac{dy_2(z_2)}{dz_2}\right|_{z_2=\frac{L_2}{L}} = \left.\frac{d^3y_1(z_1)}{dz_1^3}\right|_{z_1=0} + D_1\left.\frac{dy_1(z_1)}{dz_1}\right|_{z_1=0} \tag{24}$$

$$\left.\frac{d^2y_2(z_2)}{dz_2^2}\right|_{z_2=\frac{L_2}{L}} + \beta_1y_2(z_2 = L_2/L) = \left.\frac{d^2y_1(z_1)}{dz_1^2}\right|_{z_1=0} + D_1y_1(z_1 = 0) \tag{25}$$

By applying the DTM to Eqs. (13), (14), (18), (20) and using the relationship in Table 1 following equations are obtained.

$$Y_2(k+4) = -\beta_1\frac{Y_2(k+2)}{(k+3)(k+4)} - \beta_2\frac{Y_2(k)}{(k+1)(k+2)(k+3)(k+4)} \tag{26}$$

$$Y_1(k+4) = -D_1\frac{Y_1(k+2)}{(k+3)(k+4)} \tag{27}$$

$$Y_2(0) = 0 \tag{28}$$

$$Y_2(2) = 0 \tag{29}$$

The recurrence relations of the first region for $k = 0(1)n$ are obtained from Eq. (26) using Eqs. (28) and (29) as follows:

$$\left.\begin{aligned}
Y_2(2k) &= 0\\
Y_2(5) &= \frac{1}{5!}\{-\beta_1 3!\, Y_2(3) - \beta_2 Y_2(1)\}\\
Y_2(7) &= \frac{1}{7!}\{(\beta_1^2 - \beta_2)3!\, Y_2(3) - \beta_1\beta_2 Y_2(1)\}\\
Y_2(9) &= \frac{1}{9!}\{(-\beta_1^3 + 2\beta_1\beta_2)3!\, Y_1(3) + (-\beta_1^2\beta_2 + \beta_2^2)Y_2(1)\}\\
Y_2(11) &= \frac{1}{11!}\{(\beta_1^4 - 3\beta_1^2\beta_2 + \beta_2^2)3!\, Y_1(3) + (\beta_1^3 - 2\beta_1\beta_2^2)Y_1(1)\}\\
Y_2(13) &= \frac{1}{13!}\{(-\beta_1^5 + 4\beta_1^3\beta_2 - 3\beta_1\beta_2^2)3!\, Y_2(3) + (-\beta_1^4\beta_2 + 3\beta_1^2\beta_2^2 - \beta_2^3)Y_2(1)\}\\
&\vdots
\end{aligned}\right\} \tag{30}$$

The recurrence relations of the second region for $k = 0(1)n$ are obtained from Eq. (27) as:

$$\left.\begin{aligned}
Y_1(4) &= \frac{1}{4!}\{-D_1 2!\, Y_1(2)\}\\
Y_1(5) &= \frac{1}{5!}\{-D_1 3!\, Y_1(3)\}\\
Y_1(6) &= \frac{1}{6!}\{(D_1^2)2!\, Y_1(2)\}\\
Y_1(7) &= \frac{1}{7!}\{(D_1^2)3!\, Y_1(3)\}\\
Y_1(8) &= \frac{1}{8!}\{(-D_1^3)2!\, Y_1(2)\}\\
Y_1(9) &= \frac{1}{9!}\{(-D_1^3)3!\, Y_1(3)\}\\
Y_1(10) &= \frac{1}{10!}\{(D_1^4)2!\, Y_1(2)\}\\
Y_1(11) &= \frac{1}{11!}\{(D_1^4)3!\, Y_1(3)\}\\
Y_1(12) &= \frac{1}{12!}\{(-D_1^5)2!\, Y_1(2)\}\\
Y_1(13) &= \frac{1}{13!}\{(-D_1^5)3!\, Y_1(3)\}\\
&\vdots
\end{aligned}\right\} \tag{31}$$

By applying the DTM to Eqs. (19), (21), (22), (23), (24), (25) and using the recurrence relations (30), (31) following equations are obtained

$$b_{11}Y_1(0) + b_{12}Y_1(1) + b_{13}2!Y_1(2) + b_{14}3!Y_1(3) = 0 \tag{32}$$

$$b_{21}Y_1(0) + b_{22}Y_1(1) + b_{23}2!Y_1(2) + b_{24}3!Y_1(3) = 0 \tag{33}$$

$$b_{35}Y_2(1) + b_{36}3!Y_2(3) = Y_1(0) \tag{34}$$

$$b_{45}Y_2(1) + b_{46}3!Y_2(3) = Y_1(1) \tag{35}$$

$$b_{55}Y_2(1) + b_{56}3!Y_2(3) = 3!Y_1(3) + D_1Y_1(1) \tag{36}$$

$$b_{65}Y_2(1) + b_{66}3!Y_2(3) = 2!Y_1(2) + D_1Y_1(0) \tag{37}$$

where

$$b_{11} = 1;\ b_{12} = \frac{L_1}{L};\ b_{13} = 0;\ b_{14} = 0;\ b_{21} = D_1;\ b_{22} = \left(\frac{L_1}{L}\right)D_1;\ b_{23} = 1;\ b_{24} = \frac{L_1}{L}$$

$$b_{35} = \frac{L_2}{L} + \sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k \ge 2m}\binom{k-m-1}{m-1}\beta_1^{k-2m}\beta_2^{m}(-1)^m\right\}$$

$$b_{36} = \sum_{k=1}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k \ge 2m-1}\binom{k-m}{m-1}\beta_1^{k-2m+1}\beta_2^{m-1}(-1)^m\right\}$$

$$b_{45} = 1 + \sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k \ge 2m}\binom{k-m-1}{m-1}\beta_1^{k-2m}\beta_2^{m}(-1)^m\right\}$$

$$b_{46} = \sum_{k=1}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k \ge 2m-1}\binom{k-m}{m-1}\beta_1^{k-2m+1}\beta_2^{m-1}(-1)^m\right\}$$

$$b_{55} = \beta_1 + \left(\frac{L_2}{L}\right)^2\frac{-\beta_2}{2!} + \sum_{k=3}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k \ge 2m+1}\binom{k-m-2}{m-1}\beta_1^{k-2m-1}\beta_2^{m+1}(-1)^m\right\}$$

$$b_{56} = 1 + \sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k \ge 2m}\binom{k-m-1}{m-1}\beta_1^{k-2m}\beta_2^{m}(-1)^m\right\}$$

$$b_{65} = \left(\frac{L_2}{L}\right)\beta_1 + \left(\frac{L_2}{L}\right)^3\frac{-\beta_2}{3!} + \sum_{k=3}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k \ge 2m+1}\binom{k-m-2}{m-1}\beta_1^{k-2m-1}\beta_2^{m+1}(-1)^m\right\}$$

$$b_{66} = \left(\frac{L_2}{L}\right) + \sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k \ge 2m}\binom{k-m-1}{m-1}\beta_1^{k-2m}\beta_2^{m}(-1)^m\right\}$$

Substituting Eqs. (34) and (35) into Eqs. (36) and (37), respectively, gives:

$$3!Y_1(3) = (b_{55} - D_1 b_{45})Y_2(1) + (b_{56} - D_1 b_{46})3!Y_2(3) \tag{38}$$

$$2!Y_1(2) = (b_{65} - D_1 b_{35})Y_2(1) + (b_{66} - D_1 b_{36})3!Y_2(3) \tag{39}$$

Substituting Eqs. (34), (35), (38) and (39) into Eqs. (32) and (33), respectively, gives:

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{Bmatrix} Y_2(1) \\ 3!Y_2(3) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{40}$$

where

$$B_{11} = b_{11}b_{35} + b_{12}b_{45} + b_{13}(b_{65} - D_1 b_{35}) + b_{14}(b_{55} - D_1 b_{45})$$

$$B_{12} = b_{11}b_{36} + b_{12}b_{46} + b_{13}(b_{66} - D_1 b_{36}) + b_{14}(b_{56} - D_1 b_{46})$$

$$B_{21} = b_{21}b_{35} + b_{22}b_{45} + b_{23}(b_{65} - D_1 b_{35}) + b_{24}(b_{55} - D_1 b_{45})$$

$$B_{22} = b_{21}b_{36} + b_{22}b_{46} + b_{23}(b_{66} - D_1 b_{36}) + b_{24}(b_{56} - D_1 b_{46})$$

Thus, the frequency equation of the beam resting on elastic foundation is obtained using Eq. (28) as:

$$f^{(n)} = B_{11}B_{22} - B_{12}B_{21} = 0 \tag{41}$$

Solving (41) we get $N = N_i^{(n)}$, $i = 1, 2, 3, \ldots$ where $N_i^{(n)}$ is the nth estimated N axial compressive load circular frequency corresponding to n, and n is indicated by

$$\left| N_i^{(n)} - N_i^{(n-1)} \right| \le \varepsilon$$

where $N_i^{(n-1)}$ is the ith estimated axial compressive load corresponding to $n - 1$ and ε is a positive and small value.

5. Analytical solution of differential equations

The solution of differential equation of the elastic curve for the first region of the pile, Eq. (13), is obtained as (Ross 1984):

$$y_1(z_1) = C_1 + C_2 z_1 + \cos(D_2 z_1)C_3 + \sin(D_2 z_1)C_4 \qquad \left(0 \le z_1 \le \frac{L_1}{L_2}\right) \tag{42}$$

Where $D_2 = \left[\frac{NL^2}{EI}\left[\frac{\bar{k}AG}{\bar{k}AG - N}\right]\right]^{0.5}$; C_1, ..., C_4 = constant of integration.

The solution of Eq. (16) is obtained due to the sign of γ; four possible conditions exist due to the signs of Δ_1 and Δ_2 when γ is positive.

Where $\Delta_1 = -\frac{\beta_1}{2} - (\beta_2)^{0.5}$; $\Delta_2 = -\frac{\beta_1}{2} + (\beta_2)^{0.5}$; $D_3 = (\Delta_1)^{0.5}$; $D_4 = (\Delta_2)^{0.5}$; $\gamma = \left(\frac{\beta_1}{2}\right)^2 + \beta_2$

I. $\gamma > 0$, $\Delta_1 > 0$ and $\Delta_2 > 0$

$$y_2(z_2) = [C_5\cosh(D_3z_2) + C_6\sinh(D_3z_2) + C_7\cosh(D_4z_2) + C_8\sinh(D_4z_2)] \quad \left(0 \le z_2 \le \frac{L_2}{L}\right) \quad (43)$$

II. $\gamma > 0$, $\Delta_1 > 0$ and $\Delta_2 < 0$

$$y_2(z_2) = [C_5\cosh(D_3z_2) + C_6\sinh(D_3z_2) + C_7\cos(D_4z_2) + C_8\sin(D_4z_2)] \quad \left(0 \le z_2 \le \frac{L_2}{L}\right) \quad (44)$$

III. $\gamma > 0$, $\Delta_1 < 0$ and $\Delta_2 > 0$

$$y_2(z_2) = [C_5\cos(D_3z_2) + C_6\sin(D_3z_2) + C_7\cosh(D_4z_2) + C_8\sinh(D_4z_2)] \quad \left(0 \le z_2 \le \frac{L_2}{L}\right) \quad (45)$$

VI. $\gamma > 0$, $\Delta_1 < 0$ and $\Delta_2 < 0$

$$y_2(z_2) = [C_5\cos(D_3z_2) + C_6\sin(D_3z_2) + C_7\cos(D_4z_2) + C_8(D_4z_2)] \quad \left(0 \le z_2 \le \frac{L_2}{L}\right) \quad (46)$$

V. $\gamma < 0$

$$y_2(z_2) = \{C_5[\cosh(r\alpha_1z_2)\cos(r\alpha_2z_2)] + C_6[\sinh(r\alpha_1z_1)\cos(r\alpha_2z_2)] + C_7[\cosh(r\alpha_1z_2)\sin(r\alpha_2z_2)] + C_8[\sinh(r\alpha_1z_2)\sin(r\alpha_2z_2)]\} \quad \left(0 \le z_2 \le \frac{L_2}{L}\right) \quad (47)$$

Where $\lambda = \mathrm{Arctg}\left[\frac{-2\sqrt{-\left(\frac{\beta_1}{2}\right)^2 - \beta_2}}{\beta_1}\right]$; $\alpha_1 = \sin(\lambda/2)$; $\alpha_2 = \cos(\lambda/2)$; $r = \sqrt[4]{-\beta_2}$

Bending moment functions with respect to z for the first and the second regions of pile are obtained from Eqs. (5) and (6), respectively, as:

$$M_1(z_1) = -N \cdot C_3 - NC_4 \quad \left(0 \le z_1 \le \frac{L_1}{L}\right) \quad (48)$$

$$M_2(z_2) = -\frac{EI}{L^2}\left[\frac{\bar{k}AG - N}{\bar{k}AG}\right]\frac{d^2y_2(z_2)}{dz^2} + \left[\frac{EIC_S}{\bar{k}AG} - N\right]y_2(z_2) \quad \left(0 \le z_2 \le \frac{L_2}{L}\right) \quad (49)$$

The summation of horizontal components $V_1(z_1)$ and $V_2(z_2)$ of axial (N) and shear forces ($T_1(z_1)$, $T_2(z_2)$) at initial ends of differential parts at the first and the second regions of pile are written, respectively, as:

$$V_1(z_1) = T_1(z_1) - \frac{N}{L}\frac{dy_1(z_1)}{dz_1} = \frac{1}{L}\left[\frac{dM_1(z_1)}{dz_1} - N\frac{dy_1(z_1)}{dz_1}\right] \quad \left(0 \le z_1 \le \frac{L_1}{L}\right) \quad (50)$$

$$V_2(z_2) = T_2(z_2) - \frac{N}{L}\frac{dy_2(z_2)}{dz_2} = \frac{1}{L}\left[\frac{dM_2(z_2)}{dz_2} - N\frac{dy_2(z_2)}{dz_2}\right] \quad \left(0 \le z_2 \le \frac{L_2}{L}\right) \quad (51)$$

substituting Eqs. (48) and (49) into Eqs. (50) and (51), respectively, gives

$$V_1(z_1) = -\frac{EI}{L^3}\left[\frac{\bar{k}AG - N}{\bar{k}AG}\right]\frac{d^3y_1(z_1)}{dz_1^3} - \frac{N}{L}\frac{dy_1(z_1)}{dz_1} \qquad \left(0 \le z_1 \le \frac{L_1}{L}\right) \tag{52}$$

$$V_2(z_2) = -\frac{EI}{L^3}\left[\frac{\bar{k}AG - N}{\bar{k}AG}\right]\frac{d^3y_1(z_1)}{dz_1^3} + \frac{1}{L}\left[\frac{EIC_S}{\bar{k}AG} - N\right]\frac{dy_1(z_1)}{dz_1} \qquad \left(0 \le z_2 \le \frac{L_2}{L}\right) \tag{53}$$

Constants of integration C_1, ..., C_8 of elastic curve functions for the first and the second regions must be obtained by using boundary conditions due to the support type of both ends in order to the calculate the buckling load of the pile partially embedded in the soil.

Boundary conditions of the pile whose both ends simply supported (Fig. 2) are given in relations (54).

$$\left.\begin{aligned}
y_1\left(z_1 = \frac{L_1}{L}\right) &= 0 \\
M_1\left(z_1 = \frac{L_1}{L}\right) &= 0 \\
y_1(z_1 = 0) &= y_2\left(z_2 = \frac{L_2}{L}\right) \\
\left.\frac{dy_1(z_1)}{dz_1}\right|_{z_1 = 0} &= \left.\frac{dy_2(z_2)}{dz_2}\right|_{z_2 = \frac{L_2}{L}} \\
M_1(z_1 = 0) &= M_2\left(z_2 = \frac{L_2}{L}\right) \\
V_1(z_1 = 0) &= V_2\left(z_2 = \frac{L_2}{L}\right) \\
y_2(z_2 = 0) &= 0 \\
M_2(z_2 = 0) &= 0
\end{aligned}\right\} \tag{54}$$

Elastic curve function for the second region of the pile, $y_2(z_2)$ used in Eqs. (54), must be obtained from Eqs. (43)-(47) due to the values of γ, Δ_1 and Δ_2. A set of eight linear homogeneous equations is obtained from Eqs. (54) due to boundary conditions of the pile partially embedded in the soil. This equation set is written in matrix form as:

$$[S]\{C\} = \{0\} \tag{55}$$

Where $\{C\}$ and $[S]$ indicate the unknown coefficients vector and coefficient matrix, respectively. Hence, the non-trivial solution of this problem is given by

$$|S| = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} \\ S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} \\ S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} \end{bmatrix} \tag{56}$$

Where, $U_1 = \dfrac{EIC_S}{\bar{k}AG}$, $U_2 = \dfrac{EI}{L^2}\left(\dfrac{\bar{k}AG - N}{\bar{k}AG}\right)$, $U_3 = \dfrac{EI}{L^3}\left(\dfrac{\bar{k}AG - N}{\bar{k}AG}\right)$, $A_1 = U_1 - D_3^2 U_2$

$$A_2 = U_1 - D_4^2 U_2,\ A_3 = U_1 + D_4^2 U_2,\ A_4 = U_1 + D_3^2 U_2,\ A_5 = U_1 - (D_3^2 - D_4^2) U_2$$

$$A_6 = 2D_3 D_4 U_2,\ B_1 = D_3\left[\left(U_1 - \frac{N}{L}\right) - D_3^2 U_3\right],\ B_2 = D_4\left[\left(U_1 - \frac{N}{L}\right) - D_4^2 U_3\right]$$

$$B_3 = D_4\left[\left(-U_1 + \frac{N}{L}\right) - D_4^2 U_3\right],\ B_4 = D_3\left[\left(-U_1 + \frac{N}{L}\right) - D_3^2 U_3\right],\ B_5 = U_1 - \frac{N}{L} - (D_3^2 - D_4^2) U_3$$

$$S_{11} = 1,\ S_{12} = \frac{L_1}{L},\ S_{13} = \cos\left(D_2 \frac{L_1}{L}\right),\ S_{14} = \sin\left(D_2 \frac{L_1}{L}\right),\ S_{15} = 0,\ S_{16} = 0,\ S_{17} = 0,\ S_{18} = 0,\ S_{21} = 0$$

$$S_{22} = 0,\ S_{23} = -N,\ S_{24} = -N,\ S_{25} = 0,\ S_{26} = 0,\ S_{27} = 0,\ S_{28} = 0,\ S_{31} = 1,\ S_{32} = 0$$

$$S_{33} = 1,\ S_{34} = 0,\ S_{41} = 0,\ S_{42} = 1,\ S_{43} = 0,\ S_{44} = -1,\ S_{51} = 0,\ S_{52} = 0,\ S_{53} = -N,\ S_{54} = -N$$

$$S_{61} = 0,\ S_{62} = -\frac{N}{L},\ S_{63} = 0,\ S_{64} = 0$$

for $\gamma > 0$, $\Delta_1 > 0$ and $\Delta_2 > 0$

$$S_{35} = -\cosh\left(D_3 \frac{L_2}{L}\right),\ S_{36} = -\sinh\left(D_3 \frac{L_2}{L}\right),\ S_{37} = -\cosh\left(D_4 \frac{L_2}{L}\right),\ S_{38} = -\sinh\left(D_4 \frac{L_2}{L}\right)$$

$$S_{45} = -\frac{D_3}{L}\sinh\left(D_3 \frac{L_2}{L}\right),\ S_{46} = -\frac{D_3}{L}\cosh\left(D_3 \frac{L_2}{L}\right),\ S_{47} = -\frac{D_4}{L}\sinh\left(D_4 \frac{L_2}{L}\right)$$

$$S_{48} = -\frac{D_4}{L}\cosh\left(D_4 \frac{L_2}{L}\right),\ S_{55} = -A_1\cosh\left(D_3 \frac{L_2}{L}\right),\ S_{56} = -A_1\sinh\left(D_3 \frac{L_2}{L}\right)$$

$$S_{57} = -A_2\cosh\left(D_4 \frac{L_2}{L}\right),\ S_{58} = -A_2\sinh\left(D_4 \frac{L_2}{L}\right),\ S_{65} = -B_1\sinh\left(D_3 \frac{L_2}{L}\right)$$

$$S_{66} = -B_1\cosh\left(D_3\frac{L_2}{L}\right),\; S_{67} = -B_2\sinh\left(D_4\frac{L_2}{L}\right),\; S_{68} = -B_2\cosh\left(D_4\frac{L_2}{L}\right),\; S_{75} = 1,\; S_{76} = 0$$

$$S_{77} = 1,\; S_{78} = 0,\; S_{85} = A_1,\; S_{86} = 0,\; S_{87} = A_2,\; S_{88} = 0$$

for $\gamma > 0$, $\Delta_1 > 0$ and $\Delta_2 < 0$

$$S_{35} = -\cosh\left(D_3\frac{L_2}{L}\right),\; S_{36} = -\sinh\left(D_3\frac{L_2}{L}\right),\; S_{37} = -\cos\left(D_4\frac{L_2}{L}\right),\; S_{38} = -\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{45} = -\frac{D_3}{L}\sinh\left(D_3\frac{L_2}{L}\right),\; S_{46} = -\frac{D_3}{L}\cosh\left(D_3\frac{L_2}{L}\right),\; S_{47} = \frac{D_4}{L}\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{48} = -\frac{D_4}{L}\cos\left(D_4\frac{L_2}{L}\right),\; S_{55} = -A_1\cosh\left(D_3\frac{L_2}{L}\right),\; S_{56} = -A_1\sinh\left(D_3\frac{L_2}{L}\right)$$

$$S_{57} = -A_3\cos\left(D_4\frac{L_2}{L}\right),\; S_{58} = -A_3\sin\left(D_4\frac{L_2}{L}\right),\; S_{65} = -B_1\sinh\left(D_3\frac{L_2}{L}\right),\; S_{66} = -B_1\cosh\left(D_3\frac{L_2}{L}\right)$$

$$S_{67} = -B_2\sin\left(D_4\frac{L_2}{L}\right),\; S_{68} = B_2\cos\left(D_4\frac{L_2}{L}\right),\; S_{75} = 1,\; S_{76} = 0$$

$$S_{77} = 1,\; S_{78} = 0,\; S_{85} = A_1,\; S_{86} = 0,\; S_{87} = -A_3,\; S_{88} = 0$$

for $\gamma > 0$, $\Delta_1 < 0$ and $\Delta_2 > 0$

$$S_{35} = -\cos\left(D_3\frac{L_2}{L}\right),\; S_{36} = -\sin\left(D_3\frac{L_2}{L}\right),\; S_{37} = -\cosh\left(D_4\frac{L_2}{L}\right),\; S_{38} = -\sinh\left(D_4\frac{L_2}{L}\right)$$

$$S_{45} = \frac{D_3}{L}\sin\left(D_3\frac{L_2}{L}\right),\; S_{46} = -\frac{D_3}{L}\cos\left(D_3\frac{L_2}{L}\right),\; S_{47} = -\frac{D_4}{L}\sinh\left(D_4\frac{L_2}{L}\right)$$

$$S_{48} = -\frac{D_4}{L}\cosh\left(D_4\frac{L_2}{L}\right),\; S_{55} = -A_4\cos\left(D_3\frac{L_2}{L}\right),\; S_{56} = -A_4\sinh\left(D_3\frac{L_2}{L}\right)$$

$$S_{57} = -A_2\cosh\left(D_4\frac{L_2}{L}\right),\; S_{58} = -A_2\sinh\left(D_4\frac{L_2}{L}\right),\; S_{65} = -B_4\sin\left(D_3\frac{L_2}{L}\right)$$

$$S_{66} = -B_4\cos\left(D_3\frac{L_2}{L}\right),\; S_{67} = -B_2\sinh\left(D_4\frac{L_2}{L}\right),\; S_{68} = -B_2\cosh\left(D_4\frac{L_2}{L}\right),\; S_{75} = 1,\; S_{76} = 0$$

$$S_{77} = 1,\; S_{78} = 0,\; S_{85} = A_4,\; S_{86} = 0,\; S_{87} = A_5,\; S_{88} = 0$$

for $\gamma > 0$, $\Delta_1 < 0$ and $\Delta_2 < 0$

$$S_{35} = -\cos\left(D_3\frac{L_2}{L}\right),\; S_{36} = -\sin\left(D_3\frac{L_2}{L}\right),\; S_{37} = -\cos\left(D_4\frac{L_2}{L}\right),\; S_{38} = -\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{45} = \frac{D_3}{L}\sin\left(D_3\frac{L_2}{L}\right),\ S_{46} = -\frac{D_3}{L}\cos\left(D_3\frac{L_2}{L}\right),\ S_{47} = -\frac{D_4}{L}\sin\left(D_4\frac{L_2}{L}\right),\ S_{48} = -\frac{D_4}{L}\cos\left(D_4\frac{L_2}{L}\right)$$

$$S_{55} = -A_4\cos\left(D_3\frac{L_2}{L}\right),\ S_{56} = -A_4\sin\left(D_3\frac{L_2}{L}\right),\ S_{57} = -A_3\cos\left(D_4\frac{L_2}{L}\right),\ S_{58} = -A_3\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{65} = -B_4\sin\left(D_3\frac{L_2}{L}\right),\ S_{66} = -B_4\cos\left(D_3\frac{L_2}{L}\right),\ S_{67} = -B_3\sin\left(D_4\frac{L_2}{L}\right),\ S_{68} = -B_3\cos\left(D_4\frac{L_2}{L}\right)$$

$$S_{75} = 1,\ S_{76} = 0,\ S_{77} = 1,\ S_{78} = 0,\ S_{85} = A_4,\ S_{86} = 0,\ S_{87} = A_5,\ S_{88} = 0$$

for $\gamma < 0$

$$S_{35} = \cosh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right),\ S_{36} = \sinh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right),\ S_{37} = \cosh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{38} = \sinh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{45} = -\frac{D_3}{L}\sinh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right) + \frac{D_4}{L}\cosh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{46} = -\frac{D_3}{L}\cosh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right) + \frac{D_4}{L}\sinh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{47} = -\frac{D_3}{L}\sinh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right) - \frac{D_4}{L}\cosh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right)$$

$$S_{48} = -\frac{D_3}{L}\cosh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right) - \frac{D_4}{L}\sinh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right)$$

$$S_{55} = -A_5\cosh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right) - A_6\sinh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{56} = -A_5\sinh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right) - A_6\cosh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{57} = -A_5\cosh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right) + A_6\sinh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right)$$

$$S_{58} = -A_5\sinh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right) + A_6\cosh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right)$$

$$S_{65} = -(B_5D_3 + 2B_6D_3D_4^2)\sinh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right) + (B_5D_3 - 2B_6D_3^2D_4)\cosh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{66} = -(B_5D_3 + 2B_6D_3D_4^2)\cosh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right) + (B_5D_3 - 2B_6D_3^2D_4)\sinh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{67} = -(B_5D_3 + 2B_6D_3D_4^2)\sinh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right) - (B_5D_3 - 2B_6D_3^2D_4)\cosh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right)$$

$$S_{68} = -(B_5D_3 + 2B_6D_3D_4^2)\cosh\left(D_3\frac{L_2}{L}\right)\sin\left(D_4\frac{L_2}{L}\right) - (B_5D_3 - 2B_6D_3^2D_4)\sinh\left(D_3\frac{L_2}{L}\right)\cos\left(D_4\frac{L_2}{L}\right)$$

$$S_{75} = 1,\ S_{76} = 0,\ S_{77} = 1,\ S_{78} = 0,\ S_{85} = A_5,\ S_{86} = 0,\ S_{87} = 0,\ S_{88} = -A_6$$

6. Numerical analysis

A pile model is considered for numerical analysis. The piles made up using I 600 steel profile. The buckling loads of the piles partially embedded in soil having modulus of subgrade reaction of 15.000 kN/m^2 are calculated for support conditions given in Fig. 2 by a computer program having an iteration algorithm and prepared by the writers.

The characteristics of the steel pile used numerical analysis are presented in the following:

$$I = 139*10^{-5}\ \text{m}^4;\ A = 254*10^{-4}\ \text{m}^2;\ EI = 291900\ \text{kN/m}^2\ ;\ AG = 2053790.5\ \text{kN};\ \bar{k} = 0.4347$$

Buckling loads and relative stiffness values (α) of the steel pile are calculated by taking pile lengths of the first and the second regions, L_1 and L_2, from Table 1 and by using DTM and analytical method for L_2/L = 0.25, L_2/L = 0.50, L_2/L = 0.75. Euler critical buckling load of piles are calculated using $N_E = \pi^2\ EI/(L_b)^2$ by neglecting the effects of modulus of subgrade reaction, shear deformation and rotation restraining stiffness and by taking $L_b = L$ for both ends simply supported pile.

$N_r = N/N_E$ values are calculated according to α, L_2/L and series size (n) values using DTM and according to N_r and L_2/L values by using analytical method; and the values obtained are presented Tables 2(a),(b),(c).

Table 1 Values of L with respect to α, values of L_1 and L_2 with respect to L_2/L

L (m.)	$\alpha = C_s\ L^4/EI$	L_2/L = 0.25		L_2/L = 0.50		L_2/L = 0.75	
		L_1 (m)	L_2 (m)	L_1 (m)	L_2 (m)	L_1 (m)	L_2 (m)
2.10	1	1.575	0.525	1.05	1.05	0.525	1.575
3.73	10	2.797	0.933	1.865	1.865	0.933	2.797
6.64	100	4.980	1.660	3.320	3.320	1.660	4.980
11.81	1000	8.857	2.953	5.905	5.905	2.953	8.857
21.00	10000	15.750	5.250	10.50	10.50	5.250	15.750
37.35	100000	28.012	9.338	18.675	18.675	9.338	32.012
66.42	1000000	49.815	16.605	33.21	33.21	16.605	49.815

Table 2(a) N_r values for the first, second and third modes of the pile

α	Method	n	L_1/L = 0.25			L_1/L = 0.50			L_1/L = 0.75		
			1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
1.0	D T M	4	0.57869297	0.96824322	1.20676493	0.58263466	1.02121335	1.15173288	0.58707537	0.97075671	1.20579137
		6	0.57843581	1.01717216	1.21864203	0.58262394	1.02005304	1.18550891	0.58683045	1.01854984	1.21757510
		8	0.57843581	1.01938257	1.19178040	0.58262394	1.02004998	1.18722029	0.58683045	1.02066534	1.19180795
		10	0.57843581	1.01939788	1.18707640	0.58262394	1.02004998	1.18723254	0.58683045	1.02068065	1.18752032
		12	0.57843581	1.01939788	1.18701058	0.58262394	1.02004998	1.18723254	0.58683045	1.02068065	1.18745909
		14	0.57843581	1.01939788	1.18700915	0.58262394	1.02004998	1.18723254	0.58683045	1.02068065	1.18745909
		16	0.57843581	1.01939788	1.18700915	0.58262394	1.02004998	1.18723254	0.58683045	1.02068065	1.18745909
	Analytic method		0.57843581	1.01939788	1.18700915	0.58262394	1.02004998	1.18723254	0.58683045	1.02068065	1.18745909
10.0	D T M	4	0.82152121	1.87783313	3.04269313	0.86244949	2.09306519	3.01674532	0.90491349	1.90283922	3.03300073
		6	0.82098515	2.07214468	3.11892357	0.86242534	2.08828902	2.91072779	0.90451749	2.08584056	3.10827980
		8	0.82098515	2.08148454	2.94843976	0.86244949	2.08827453	2.92101419	0.90451749	2.09424837	2.94873918
		10	0.82098515	2.08155215	2.91929496	0.86244949	2.08827453	2.92109146	0.90451749	2.09430632	2.92371377
		12	0.82098515	2.08155698	2.91889413	0.86244949	2.08827453	2.92109146	0.90451749	2.09430632	2.92335640
		14	0.82098515	2.08155698	2.91888930	0.86244949	2.08827453	2.92109146	0.90451749	2.09430632	2.92335640
		16	0.82098515	2.08155698	2.91888930	0.86244949	2.08827453	2.92109146	0.90451749	2.09430632	2.92335640
	Analytic method		0.82098515	2.08155698	2.91888930	0.86244949	2.08827453	2.92109146	0.90451749	2.09430632	2.92335640
100.0	D T M	4	1.02242338	2.69830826	5.95093931	1.41807527	3.25726813	4.82216908	1.85855230	2.95056255	5.85408088
		6	1.02141332	3.13580103	6.24140745	1.41793754	3.24790214	5.44800703	1.85888899	3.27500536	6.13510652
		8	1.02141332	3.15975164	5.58161013	1.41793754	3.24787154	5.48334375	1.85888899	3.28958998	5.58392102
		10	1.02141332	3.15993529	5.46377005	1.41793754	3.24787154	5.48360392	1.85888899	3.28969711	5.50827381
		12	1.02141332	3.15993529	5.46219375	1.41793754	3.24787154	5.48360392	1.85888899	3.28969711	5.50720254
		14	1.02141332	3.15993529	5.46216314	1.41793754	3.24787154	5.48360392	1.85888899	3.28969711	5.50718724
		16	1.02141332	3.15993529	5.46216314	1.41793754	3.24787154	5.48360392	1.85888899	3.28969711	5.50718724
	Analytic method		1.02141332	3.15993529	5.46216314	1.41793754	3.24787154	5.48360392	1.85888899	3.28969711	5.50718724

Table 2(b) N_r values for the first, second and third modes of the pile

α	Method	n	$L_1/L = 0.25$			$L_1/L = 0.50$			$L_1/L = 0.75$		
			1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
1000.0	D T M	4	1.65811207	3.41585901	9.38136368	3.24215157	6.62436324	7.77156823	5.57534072	8.06703548	10.65541190
		6	1.65307707	4.31102361	9.94295964	3.23919835	7.53632730	8.15689084	5.33782436	8.52255797	10.72299707
		8	1.65302866	4.41041641	9.46797534	3.23914994	7.56416503	8.20530428	5.33792119	8.26833859	10.70663332
		10	1.65302866	4.41162675	7.90828780	3.23914994	7.56435869	8.20574000	5.33796960	8.23812860	10.50949377
		12	1.65302866	4.41162675	7.90102578	3.23914994	7.56435869	8.20574000	5.33796960	8.23764446	10.49758407
		14	1.65302866	4.41162675	7.90097737	3.23914994	7.56435869	8.20574000	5.33796960	8.23764446	10.49743883
		16	1.65302866	4.41162675	7.90097737	3.23914994	7.56435869	8.20574000	5.33796960	8.23764446	10.49743883
	Analytic method		1.65302866	4.41162675	7.90097737	3.23914994	7.56435869	8.20574000	5.33796960	8.23764446	10.49743883
10000.0	D T M	4	2.49175761	5.46524278	25.83480251	4.98810747	9.66470698	21.04798888	10.28358989	26.02369726	32.38198058
		6	2.54472162	6.79929288	25.03303485	4.67384415	11.87128554	20.65382032	9.81257759	26.14141207	34.19530904
		8	2.54533392	7.21228969	26.10528633	4.68302866	12.51665046	20.55217841	11.09687826	18.78967090	25.60656743
		10	2.54533392	7.22453570	26.15396423	4.68287559	12.53364180	19.91462033	11.16530286	25.59952597	37.47356687
		12	2.54533392	7.22453570	13.59185045	4.68287559	12.53379488	19.89472055	11.16132291	20.30848274	25.59952597
		14	2.54533392	7.22453570	13.56215387	4.68287559	12.53379488	19.89456748	11.16132291	20.29975745	24.22705400
		16	2.54533392	7.22453570	13.56184772	4.68287559	12.53379488	19.89456748	11.16132291	20.29975745	24.48467951
	Analytic method		2.54533392	7.22453570	13.56184772	4.68287559	12.53379488	19.89456748	11.16132291	20.29975745	24.48467951
100000.0	D T M	4	2.76395712	10.74447170	63.20154993	4.41322796	9.30583880	143.15535291	9.89223545	29.69026466	163.56307133
		6	2.91406691	12.23443247	65.07017475	5.87558789	13.72487746	116.28812083	11.26549796	40.48267465	146.60061480
		8	2.91745469	8.670532542	66.70249771	5.84701861	16.69656717	36.96913701	11.84124165	43.76087887	95.46256587
		10	2.91745649	8.49524591	67.37072841	5.96032729	17.04472504	61.20121586	11.75553380	45.16658446	77.11721209
		12	2.91745649	8.49282479	16.52563569	5.95596926	17.04085124	32.39320941	15.25406046	15.25406046	67.66126349
		14	2.91745649	8.49282479	16.53870977	5.95596926	17.04085124	32.30798578	16.70818857	16.70818857	62.69795578
		16	2.91745649	8.49282479	16.53967822	5.95596926	17.04085124	32.30653311	17.57110777	17.57107777	60.81964645
	Analytic method		2.91745649	8.49282479	16.53967822	5.95596926	17.04085124	32.30653311	17.57110777	17.57107777	60.81964645

Table 2(c) N_r values for the first, second and third modes of the pile

α	Method	n	L_1/L = 0.25			L_1/L = 0.50			L_1/L = 0.75		
			1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode	1st. Mode	2nd. Mode	3rd. Mode
1000000	D T M	4	2.69817294	8.11442588	376.56356930	4.31064519	9.31956896	452.73503841	10.65793624	31.48378866	490.42011971
		6	3.30151013	45.62852386	316.98172431	4.58168753	13.72209291	450.00991926	12.09583884	44.15080713	479.92717505
		8	3.21881925	32.95231751	314.12276354	4.50818452	15.11558744	304.06050794	13.05597189	48.38794931	309.39100737
		10	3.22494450	9.49566992	15.44788229	1.13623401	13.52761620	245.39591925	13.58886870	50.09995688	249.09097674
		12	3.22494450	9.48341941	18.22874612	6.11146890	17.97607953	35.61833289	13.62562021	51.24844139	111.47650032
		14	3.22494450	9.48341941	18.22874612	6.92918987	20.13216778	38.14346749	13.62102627	52.18713606	112.99862512
		16	3.22494450	9.48341941	18.73867324	6.82965454	19.97597388	39.23529343	14.11717158	52.87163283	114.38446310
	Analytic method		3.22494450	9.48341941	18.75398637	6.82965454	19.97597388	39.23529343	14.11717158	52.87163283	114.38446310

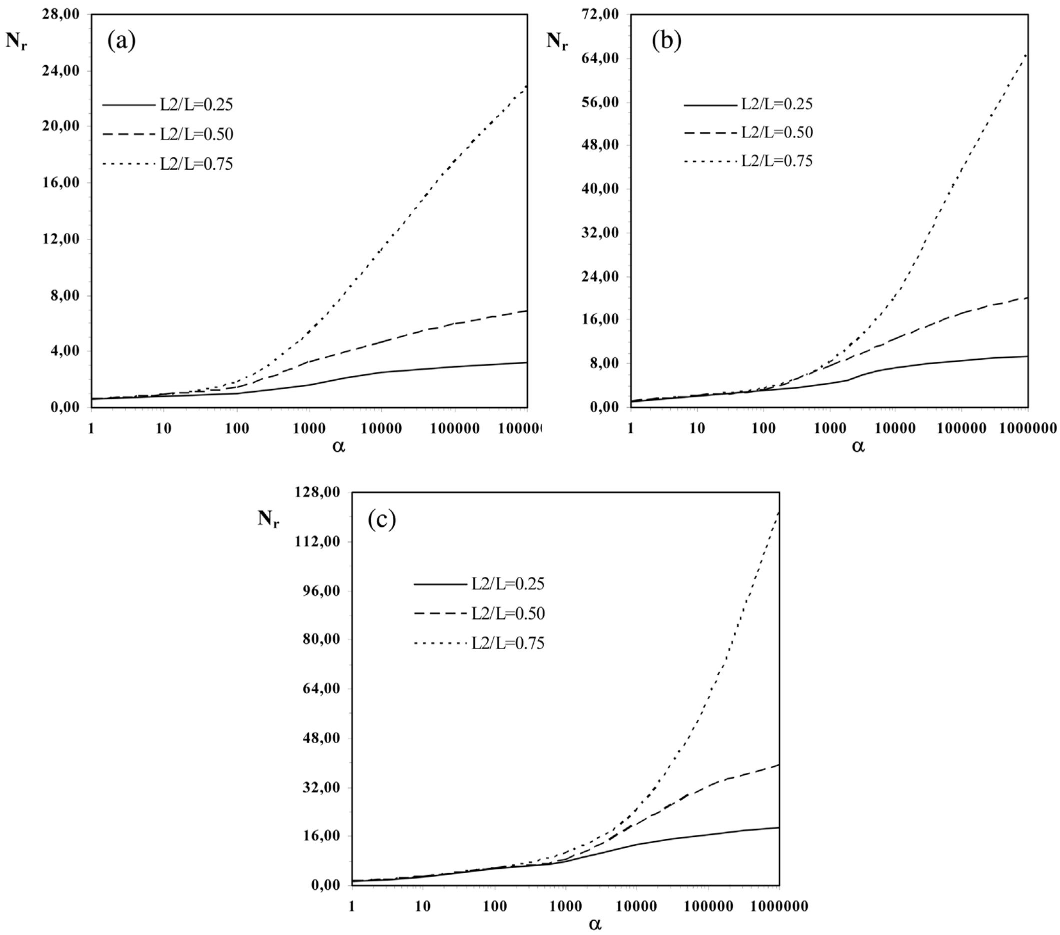


Fig. 3 Variation of N_r value with relative stiffness for the pile (a) the first mode, (b) the second mode, (c) the third mode

Variation of $N_r = N/N_E$ and α according to L_2/L = 0.25, L_2/L = 0.50, L_2/L = 0.75 and series size n = 16 are shown in Figs. 3(a),(b),(c) for the pile both ends simply supported.

Figs. 3 that give the variation between relative stiffness and N_r values of the pile partially embedded in the soil indicates that N_r values of the pile having relative stiffness between 100 and 1.000.000 increases as L_2/L values increase for all modes. N_r values of the pile having relative stiffness between 1 and 100 are same for L_2/L = 0.25, L_2/L = 0.50, L_2/L = 0.75.

7. Conclusions

In this paper, the buckling loads for the first three modes of the both ends simply supported pile are calculated by using DTM and analytical method according modulus of subgrade reactions and variation of L_2/L values.

In the analytical method, the boundary conditions of the pile are used for obtaining closed-form solution function of the buckling load and the calculation of following derivates necessary in these boundary conditions become more difficult when the order of derivates increases. However calculation of high-order derivates necessary in the analytical method are calculated easier while the DTM is being applied for buckling load of the pile, because Taylor series is used as solution function.

Buckling loads of pile values obtained for the first mode and relative stiffness between 1 and 100.000 using DTM for series size n = 4 and n > 4 are same. DTM results indicate that frequency factor values of the first mode are very fast converging for L_2/L value, and that converging speed decrease as the number of modes increase.

It is seen from Table 2(a),(b),(c) that all buckling loads obtained by using analytical method and DTM for n = 16 overlap.

The results of DTM and analytical method in Table 2(a),(b),(c) indicate that the DTM can be applied for buckling problem of partially embedded piles.

References

Aydoğan, M. (1995), "Stiffness-matrix formulation of beams with shear effect on elastic foundation", *J. Struct. Eng.,* ASCE, **121**, 1265-1270.

Banarje, J.R. and Williams, F.W. (1994), "The effect of shear deformation on the critical buckling of columns", *J. Sound Vib.,* **174**, 607-616.

Budkowska, B.B. and Sozymczak, C. (1996), "Initial post-buckling behavior of piles partially embedded in soil", *Comput. Struct.,* **62**, 831-835.

Capron, M.D. and Williams, F.W. (1988), "Exact dynamic stiffness for an axially loaded uniform Timoshenko member embedded in an elastic medium", *J. Sound Vib.*, **124**, 453-466.

Çatal, H.H. (2002), "Free vibration of partially supported piles with the effects of bending moment, axial and shear force", *Eng. Struct.,* **24**, 1615-1622.

Çatal, H.H. and Alku, S. (1996a), "Calculation of the second order stiffness matrix of the beam on elastic foundation", *Turkish J. Eng. Envir. Sci. TUBITAK.* **20**, 145-201.

Çatal, H.H. and Alku, S. (1996b), "Comparison solutions of the continuous footing subjected to bending moment, shear and axial force using finite difference equations and matrix-displacement methods", *Proc. of 2nd. National Computational Mechanic Congress,* 121-129, Trabzon, Turkey, September.

Chen, C.K. and Ho, S.H. (1996), "Application of differential transformation to eigenvalue problem", *J. Appl. Math. Comput.*, **79**, 173-178.

Chen, C.N. (1998), "Solution of beam on elastic foundation by DQEM", *J. Eng. Mech.,* **124**, 1381-1384.

Chen, Y. (1997), "Assessment on pile effective lengths and their effects on design I and II", *Comput. Struct.,* **62**, 265-286.

Doyle, P. and Pavlovic, M. (1982), "Vibrations of beam on partial elastic foundations", *Earthq. Eng. Struct. Dyn.,* **10**, 663-674.

Ergüven, M.E. and Gedikli, A. (2003), "A mixed finite element formulation for Timoshenko beam on Winkler foundation", *Comput. Mech.,* **31**, 229-237.

Heelis, M.E., Pavlovic, M.N. and West, R.P. (1999), "The stability of uniform-friction piles in homogeneous and non-homogeneous elastic foundations", *Int. J. Solids Struct.,* **36**, 3277-3292.

Heelis, M.E., Pavlovic, M.N. and West, R.P. (2004), "The analytical prediction of the buckling loads of fully and partially embedded piles", *Geotechnique,* **54**, 363-373.

Hetenyi, M. (1995), *Beams on Elastic Foundations.* The University of Michigan Press, Michigan.

Hosking, R.J., Husain, S.A. and Milinazzo, F. (2004), "Natural flexural vibrations of a continuous beam on discrete elastic supports", *J. Sound Vib.*, **272**, 169-185.

Hsieh, S.C., Chen, J.H. and Lee, A.C. (2006), "A modified transfer matrix method for coupling lateral and torsional vibrations of symmetric rotor bearing systems", *J. Sound Vib.,* **209**, 294-333.

Jang, M.J. and Chen, C.L. (1997), "Analysis of response of strongly non-linear damped system using a differential transformation technique", *Appl. Math. Comput.,* **88**, 137-151.

Li, Q.S. (2001a), "Buckling of multi-step cracked columns with shear deformation", *Eng. Struct.*, **23**, 356-364.

Li, Q.S. (2001b), "Buckling of multi-step non-uniform beams with elastically restrained boundary conditions", *J. Construct. Steel Res.*, **57**, 753-777.

Malik, M. and Dang, H.H. (1998), "Vibration analysis of continuous systems by differential transformation", *Appl. Math. Comput.*, **97**, 17-26.

Özdemir, Ö. and Kaya, M.O. (2006), "Flapwise bending vibration analysis of a rotating tapered cantilever Bernoulli-Euler beam by differential transform method", *J. Sound Vib.*, **289**, 413-420.

Pavlovic, M.N. and Tsikkos, S. (1982), "Beams on quasi-Winkler foundations", *Eng. Struct.*, **4**, 113-118.

Reddy, A.S. and Vasangkar, A.J. (1970), "Buckling of fully and partially embedded piles", *J. Soil Mech. Foundations Div.,* **96**, 1951-1965.

Ross, S.L. (1984), *Differential Equations*. Third Edition, John Wiley & Sons, New York.

Ruotolu, R. and Surace, C. (2004), "Natural frequencies of a bar with multiple cracks", *J. Sound Vib.*, **272**, 301-316.

Smith, I.M. (1979), "Discrete element analysis of pile instability", *Int. J. Numer. Analytical Methods in Geomechanics,* **3**, 205-211.

Valsangkar, A.J. and Pradhanang, R.B. (1987), "Vibration of partially supported piles", *J. Eng. Mech.,* **113**, 1244-1247.

Wang, C.M., Ng, K.H. and Kitipornchai, S. (2002), "Stability criteria for Timoshenko columns with intermediate and end concentrated axial loads", *J. Construct. Steel Res.,* **58**, 1177-1193.

West, H.H. and Mafi, M. (1984), "Eigenvalues for beam-columns on elastic supports", *J. Struct. Eng.*, ASCE, **110**, 1305-1319.

West, R.P., Heelis, M.E., Pavlovic, M.N. and Wylie, G.B. (1997), "Stability of end-bearing piles in a non-homogeneous elastic foundation", *Int. J. Numer. Analytical Methods in Geomechanics*, **21**, 845-861.

Yang, J. and Ye, J.Q. (2002), "Dynamic elastic local buckling of piles under impact loads", *Struct. Eng. Mech.*, **13**, 543-556.

Zhou, J.K. (1986), *Differential Transformation and Its Applications for Electrical Circuits,* Wuhan China, Huazhong University Press.