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# Using cable finite elements to analyze parametric vibrations of stay cables in cable-stayed bridges

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**Abstract.** This paper uses the finite element method to simultaneously consider the coupled cable-deck vibrations and the parametric vibrations of stay cables in dynamic analysis of a cable-stayed bridge. The stay cables are represented by some cable finite elements, which can consider the parametric vibration of the cables. In addition to modeling stay cables using multiple link cable elements, a procedure for removing the self-weight term of cable element is proposed. A eigenvalue analysis process using dynamic condensation method for sorting out the natural modes of the girder-tower vibrations and the Rayleigh damping considering element damping for damping matrix are also proposed for dynamic analyses of cable-stayed bridges. The possibilities of using cable elements and of using global and local vibrations to evaluate the parametric vibrations of stay cables in a cable-stayed bridge are confirmed, respectively.

**Keywords:** stay cable; parametric vibrations; finite element method; cable-stayed bridge.

## 1. Introduction

Over the last two decades, significant cable vibrations have occurred in several bridges in different countries. The first report of cable vibrations was a 1985 study of the Far$\phi$ Bridge in Denmark (Langsoe *et al.* 1987). These large-amplitude vibrations in some of the bridge's stay cables are caused by bending vibrations in the girder and/or towers (Yoshimura *et al.* 1989). Since

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multi-cable systems are widely used in cable-stayed bridges (Gimsing 1983), the natural frequencies of the global modes easily approach the natural frequencies of the stay cables. This tends to produce large-amplitude vibrations in the cables. The same phenomenon has been observed in other cable-stayed bridges in Europe (Royer-Carfagni 2003). Large-amplitude vibrations of stay cables have also occurred in some cable-stayed bridges in Japan. During a forced excitation experiment on the Hitsuishijima cable-stayed bridge (420 m, steel, 1992), local vibrations in some stay cables occurred when the bridge was excited at the frequency of the torsional second global mode (Okauchi *et al.* 1992). The same type of vibration appeared during a dynamic experiment on the Yohkura cable-stayed bridge (77 m, timber, 1993) (Fujino *et al.* 1997). During a recent vibration test, large-amplitude vibrations were found in the stay cables of the Tatara cable-stayed bridge in Japan, the center span of which, at 896 m, is the largest in the world (Manabe *et al.* 1999). In the U.S.A., considerable attention is given to local parametric vibrations in stay cables of cable-stayed bridges (Smith *et al.* 2001).

Kovács considered these large-amplitude cable vibrations to be local parametric vibration (i.e., dynamic instability) in the cables. These vibrations are caused by excitations at the extremities of the cables, which are anchored to the girder and tower (Kovács *et al.* 1982). Under certain resonance conditions, energy can flow into the stay cables and produce large-amplitude cable vibrations. Parametric vibrations in cables have been analyzed and verified using analytical models. Applying the harmonic balance method and the eigenvalue method, Takahashi (1991) calculated the boundaries of the instability regions of the parametric resonances in a flat-sag cable. Fujino *et al.* (1993) treated with the linear and nonlinear internal resonance in a stay cable of cable-stayed bridges and used a physical 3-DOF model of a cable-stayed, cantilevered beam to study the influence of parametric vibrations. Laboratory tests were performed to check the validity of this approach (Warnitchai *et al.* 1995). Lee and Perkins (1993) focused their experimental investigation on the parametric responses of lower-mode interaction in sagging cables. Lilien and Pinto Da Costa (1994) examined the amplitudes of vibrations caused by parametric excitation of cable-stayed structures. They developed non-dimensional analytical formulas that can be applied to any stay cable for calculating threshold amplitudes and limit cycle amplitudes produced by parametric excitation, making possible the development of large-span cable-stayed bridges. Pinto Da Costa *et al.* derived the governing differential equations of motion. Parametric resonance curves showed significant responses when the motion of the anchorage was either one or two times the first natural frequency of the cables (Pinto Da Costa *et al.* 1995, 1996).

Focusing on the fact that periodic time-varying displacements were given at the supports of the single cable in the above studies, Wu *et al.* (2003) proposed a single-cable model that considered the parametric vibrations of the stay cable under random excitation. They also discussed the possibility and properties of parametric vibrations in the stay cables of a cable-stayed bridge subjected to environmental and service loadings. Since global vibrations and local cable vibrations are treated separately, the stay cables were modeled as single truss elements in the analysis of global vibrations. This is a very common practice when performing numerical analyses of cable-stayed bridges, but leads to the exclusion of the local cable vibrations in a global girder/tower system.

In the other aspect, the importance of considering the interaction between cable vibrations and girder/tower vibrations in cable-stayed bridges is emphasized. Abdel-Ghaffar *et al.* (1991) discretized the cables using multiple link truss elements and showed that cable vibrations strongly affect the natural vibration characteristics of the deck/towers system. Tuladhar *et al.* (1995) stressed the inadequacy of using single elements to model the cables and the necessity of discretizing the cables into several elements. Caetano *et al.* (2000) configured the interaction between cable vibrations and deck/towers vibrations using a physical model of a cable-stayed bridge and demonstrated the great influence of local cable vibrations on the global vibrations of girder and towers in terms of the response to seismic excitations Although the interaction between simultaneous cable vibrations and girder/tower vibrations was taken into account in their research, the parametric vibrations of stay cables could not be considered because truss elements were used to model the stay cables.

This paper uses the finite element method in dynamic analysis of a cable-stayed bridge to consider these two aspects: the coupled cable-deck vibrations and the parametric vibrations of cables. The stay cables are represented by some cable finite elements. This cable finite element can consider the parametric vibration of the cables. In addition to modeling the stay cables using multiple link cable finite elements, this paper proposes a procedure for removing the self-weight term of the cable elements in the dynamic analysis. The idea that cable finite elements can take into account the non-linearity and the parametric vibration of cables is examined and confirmed.

This paper also describes a dynamic analysis of a cable-stayed bridge that demonstrates the applicability of the cable finite element for modeling the stay cables in cable-stayed bridges.

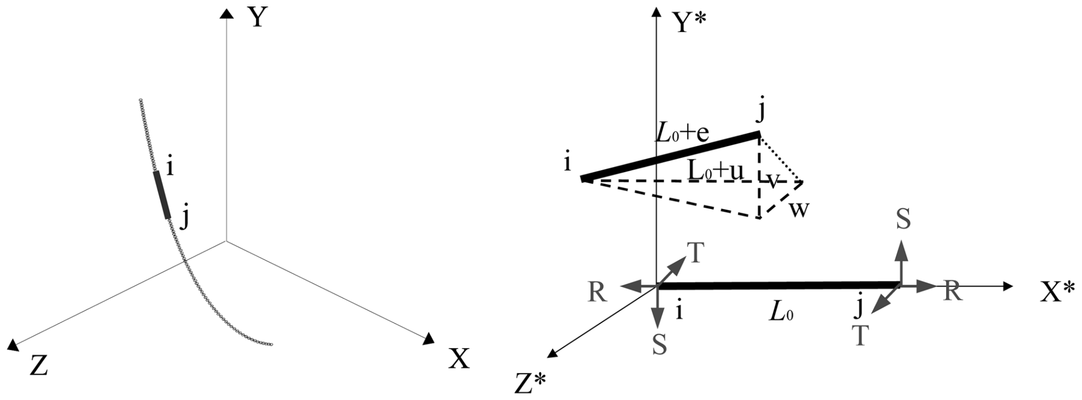
The local vibrations of stay cables are taking into account by using normal eigenvalue analysis of cable-stayed bridges. This analysis extracts the enormous natural modes of the stay cable vibrations and girder/tower vibrations. In order to sort out the natural modes of the girder and tower vibrations, which are necessary for understanding the properties of cable-stayed bridges, a method using the substructure synthesis method is adopted. This paper proposes an eigenvalue analysis process based on the dynamic condensation method.

The damping matrix should be correctly evaluated in dynamic analysis of cable-stayed bridges, because the stay cables have very small structural damping in the range of 0.001 to 0.005, while the girder and towers have relatively greater damping in the range of 0.01 to 0.03. The Rayleigh damping considering element damping is proposed as a way to consider the different damping constants of the girder, the tower and the stay cables. This paper also examined possibility that cable finite element can be used to model the parametric-induced vibrations of stay cables in cable-stayed bridges.

Finally, this paper examines the concept of using global and local vibrations to evaluate parametric vibrations of stay cables in cable-stayed bridges.

## 2. Cable modeling using cable finite elements

Three main approaches are used for examining the nonlinear behavior of stay cables in cable-stayed bridges. In the first approach, each cable is represented by a single truss element or single spring element with the equivalent modulus approach (Ernst 1965). This approach is often used for common dynamic analysis of cable-stayed bridges (Karoumi 1999). Abdel-Ghaffar pointes out the inadequacy of the first approach and proposes dividing each cable into several straight truss elements for cable modeling. This led to the development of the second approach (Abdel-Ghaffar *et al.* 1991, Tuladhar *et al.* 1995, Caetano *et al.* 2000), but because the approach uses truss elements, it cannot consider the transverse vibrations and out-of-plane vibrations of cables (Au *et al.* 2001). The




(a) Global coordinate system (b) Local coordinate system

Fig. 1 Cable element in global and local coordinate system

third approach uses isoparametric cable elements (Ali *et al.* 1995). However, these elements are stiffer and required numerical integration in order to formulate the element stiffness matrix (Leonard 1988).

This paper considers the parametric vibrations of cables in cable structures, and models each cable by multiple link cable elements based on the second approach. The cable finite element proposed by Broughton and Ndumbaro (1994) is used in this paper. This cable finite element can account for the in-plane (longitudinal and transverse) and out-of-plane responses of cables.

### *2.1 Stiffness matrix of cable element*

In the local coordinate system of the cable element, shown in Fig. 1, the original length of the element is $L_0$, the initial basic force is $P_0$, and the displacements in three directions ($x^*$, $y^*$, $z^*$) are ($u_i$, $v_i$, $w_i$) in node-*i* and ($u_j$, $v_j$, $w_j$) in the node-*j*. The equilibrium equation of one cable element is given below.

$$P\left[-\frac{L_0+u}{L_0+e} \quad -\frac{v}{L_0+e} \quad -\frac{w}{L_0+e} \quad \frac{L_0+u}{L_0+e} \quad \frac{v}{L_0+e} \quad \frac{w}{L_0+e}\right]^T = \{F\}^e \tag{1}$$

where $\{F\}^e = \{-R \ -S \ -T \ R \ S \ T\}$ is the load vector as applicable to the ends of the element $(i-j)$, $P = P_0 + E_c A_c / L_0 \times e$ is the updated element basic force, $e = \sqrt{(L_0+u)^2+v^2+w^2} - L_0$ is the element extension, $u = u_j - u_i$, $v = v_j - v_i$, $w = w_j - w_i$, and $Ec$ and $Ac$ are the Young's modulus and cross-sectional area of the cable element.

By transforming the partial basic forces into partial intermediate forces and partial intermediate displacements, the following equation can be obtained.

$$[K]^e\{\delta X\}^e = \{\delta F\}^e \tag{2}$$

$$[K]^e = \frac{E_c A_c}{L_0 (L_0+e)^2}\begin{bmatrix} (L_0+u)^2 & (L_0+u)v & (L_0+u)w & -(L_0+u)^2 & -(L_0+u)v & (L_0+u)w \\ & v^2 & vw & -(L_0+u)v & -v^2 & -vw \\ & & w^2 & -(L_0+u)w & -vw & -w^2 \\ & & & (L_0+u)^2 & (L_0+u)v & (L_0+u)w \\ & sym. & & & v^2 & vw \\ & & & & & w^2 \end{bmatrix}$$

$$+\frac{P}{(L_0+e)^3}\begin{bmatrix} v^2+w^2 & -(L_0+u)v & -(L_0+u)w & -v^2-w^2 & (L_0+u)v & (L_0+u)w \\ & (L_0+u)^2+w^2 & -vw & (L_0+u)v & -(L_0+u)^2-w^2 & vw \\ & & (L_0+u)^2+v^2 & (L_0+u)w & vw & -(L_0+u)^2-v^2 \\ & & & v^2+w^2 & -(L_0+u)v & -(L_0+u)w \\ & sym. & & & (L_0+u)^2+w^2 & -vw \\ & & & & & (L_0+u)^2+v^2 \end{bmatrix} \tag{3}$$

where $[K]^e$ is the element incremental stiffness matrix and $\{\delta X\}^e = \{\delta u_i \;\; \delta v_i \;\; \delta w_i \;\; \delta u_j \;\; \delta v_j \;\; \delta w_j\}^T$.

### *2.2 Relationships of stiffness matrix among cable element, truss element, and chord element*

In the truss element, only the longitudinal displacement $u$ in the local coordinate is considered. By letting $v = w = P = 0$ in $[K]^e$ of Eq. (3), the stiffness matrix of the truss element can be obtained (Ross 1991).

$$[K]^e_{Truss} = \frac{E_c A_c}{L_0}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & sym. & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \tag{4}$$

The $v$ and $w$ displacements are zero. Therefore, the truss element cannot take into account the transverse and out-of-plane actions of the cables.

The chord element is called in this paper when considering the initial axial force $P_0$ in the truss element. The stiffness matrix of the chord element is shown below (ARK 2003):

$$[K]^e_{Chord} = \frac{E_c A_c}{L_0}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & sym. & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} + \frac{P_0}{L_0}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & -1 & 0 \\ & & 1 & 0 & 0 & -1 \\ & & & 0 & 0 & 0 \\ & sym. & & & 1 & 0 \\ & & & & & 1 \end{bmatrix} \tag{5}$$

This stiffness matrix of the chord element can be obtained from the stiffness matrix of the cable element by assuming $v = w = 0$, $P_0 = P$ and $L_0 + u = L_0$ in $[K]^e$ of Eq. (3).

The stiffness matrix of the chord element doesn't change when $u$, $v$ and $w$ change, so the chord element cannot be used to evaluate the parametric vibrations of cables. Moreover, it is also confirmed in the following section that the chord element cannot simulate the parametric vibrations of cables.

### 2.3 Mass matrix of cable element

The lumped mass matrix is used for the cable element, as shown below.

$$[M]^e = \frac{\rho_c A_c L_0}{2}\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \tag{6}$$

where $\rho_c$ is the mass density of a cable per unit volume, $A_c$ is the section area, and $L_0$ is the length of cable element.

### 2.4 Procedure of nonlinear dynamic analysis

The equation of motion of a cable subjected to dynamic excitation can be expressed as follows:

$$[M]\{\ddot{X}\}_t + [C]\{\dot{X}\}_t + \{R\}_t = \{F\}_t + \{F\}_g \tag{7}$$

where $\{F\}_t$ is the dynamic load vector, $\{F\}_g$ is the self-weight load vector, and $\{R\}_t$ is the restoring load that is a combination of $P\left[-\frac{L_0+u}{L_0+e} \quad -\frac{v}{L_0+e} \quad -\frac{w}{L_0+e} \quad \frac{L_0+u}{L_0+e} \quad \frac{v}{L_0+e} \quad \frac{w}{L_0+e}\right]^T$ in Eq. (1).

Not only the dynamic term but also the self-weight term is included in the restoring load $\{R\}_t$. Generally, the self-weight term should be removed when performing the dynamic analysis of cables (Irvine 1981). Therefore, in this paper, the self-weight term in Eq. (7) is removed.

The equilibrium equation of a cable under the self-weight load is as follows:

$$\{R\}_g = \{F\}_g \tag{8}$$

Substituting Eq. (8) for Eq. (7), the following equation is obtained as

$$[M]\{\ddot{X}\}_t + [C]\{\dot{X}\}_t + (\{R\}_t - \{R\}_g) = \{F\}_t \tag{9}$$

$(\{R\}_t - \{R\}_g)$ is the restoring force of removing the self-weight term in the global coordinate of the cable. It is obtained from the restoring force of the cable element in the local coordinate $(\{R\}_t - \{R\}_g)^e$.

In one cable element in the local coordinate, Eq. (1) becomes the following equation if the initial tension in one cable element is $P_0$ under the initial self-weight load $\{F\}_g^e$

$$\{R\}_g^e = P_0[-1 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0]^T \tag{10}$$

The restoring force in one cable element under the excitation of $\{F\}_g^e + \{F\}_t^e$ is as follows:

$$\{R\}_t^e = P\left[-\frac{L_0+u}{L_0+e} \quad -\frac{v}{L_0+e} \quad -\frac{w}{L_0+e} \quad \frac{L_0+u}{L_0+e} \quad \frac{v}{L_0+e} \quad \frac{w}{L_0+e}\right]^T \tag{11}$$

Therefore, the restoring force in the local coordinate that removes the self-weight term of the cable element can be obtained as follows:

$$(\{R\}_t - \{R\}_g)^e = \left[-P\frac{L_0+u}{L_0+e}+P_0 \quad -P\frac{v}{L_0+e} \quad -P\frac{w}{L_0+e} \quad P\frac{L_0+u}{L_0+e}-P_0 \quad P\frac{v}{L_0+e} \quad P\frac{w}{L_0+e}\right]^T \tag{12}$$

The restoring force of the cable element is a function of nodal displacements and element forces, and therefore, as the structure deforms, it needs to be reformulated by using the Newton-Raphson method. The stiffness matrix $[K]^e$ in Eq. (3) should be used during the iterative procedure.

By using the proposed procedure of removing the self-weight term of the cable finite element, a time-history dynamic analysis of direct integration can be applied using the direct integration method, such as Newmark $\beta$ method (Bhatt 2002).

## 3. Numerical verification

In order to examine the accuracy of the cable finite element and the proposed procedure, a cable is formulated as a continuum and analyzed, and the results are compared.

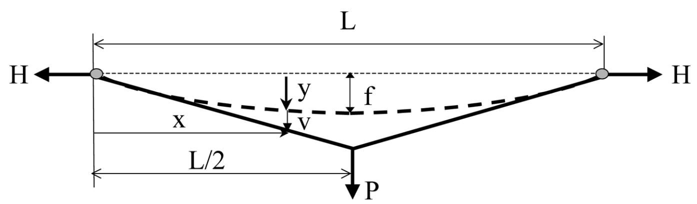
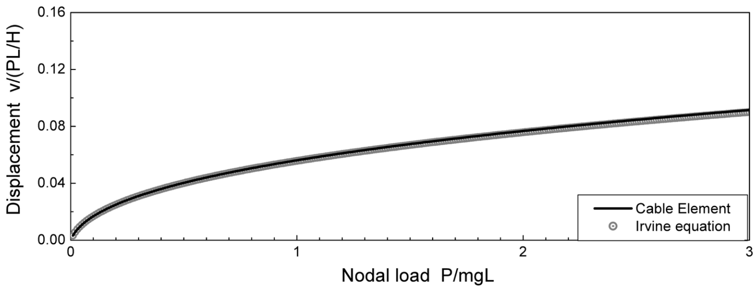
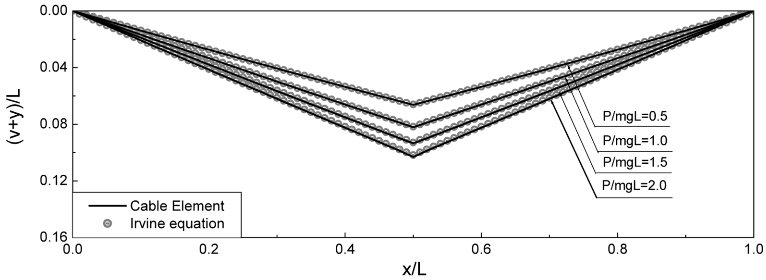



Fig. 2 Geometry of a horizontal cable under nodal load




(a) Load-displacement curve




(b) Space shape

Fig. 3 Displacement of a cable with $\gamma = 0.026$ and $k^2 = 900$

### *3.1 Static analysis of a cable*

A static analysis is performed using a horizontal cable with a uniform cross section $A$ and uniform weight per unit length $m = \rho_c A_c$ and hanging between two points (Fig. 2). The span is $L$, the sag is $f$, and the sag-to-span ratio $\gamma = f/L$ is 0.01. The initial horizontal tension is $H$ and the ratio of the elongation stiffness to the longitudinal tension $k^2 = EcAc/H$ is 900 (Wu *et al*. 2004). A nodal load $P$ is applied at the center of the cable. Fig. 3(a) shows the vertical displacements $v$ at the center point of the cable. Fig. 3(b) shows the space shapes when $P/mgL = 0.5$, 1.0, 1.5 and 2.0. The deformations derived from cable finite elements agree well with those derived from the Irvine equation (Irvine 1981).

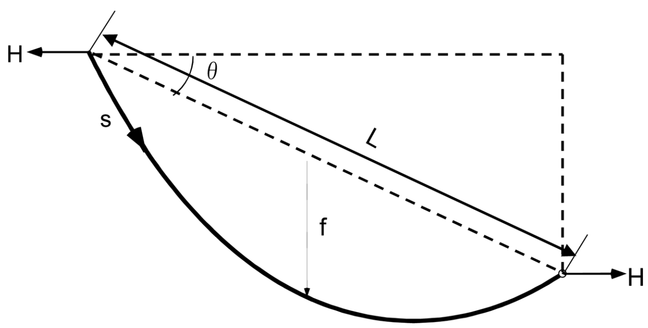
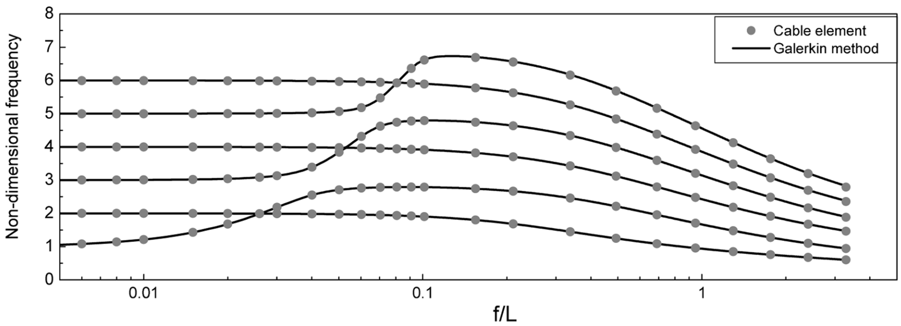
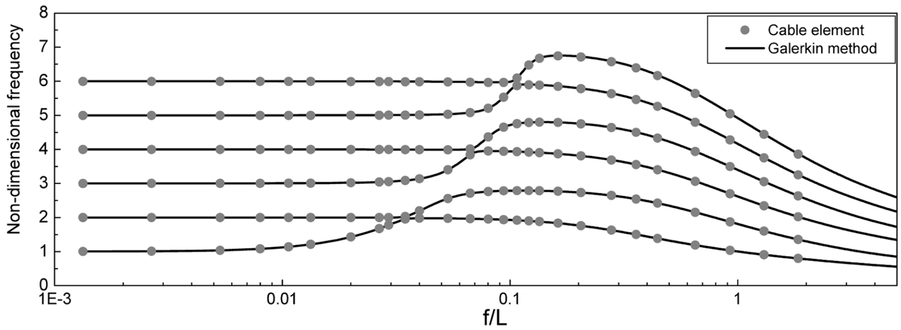


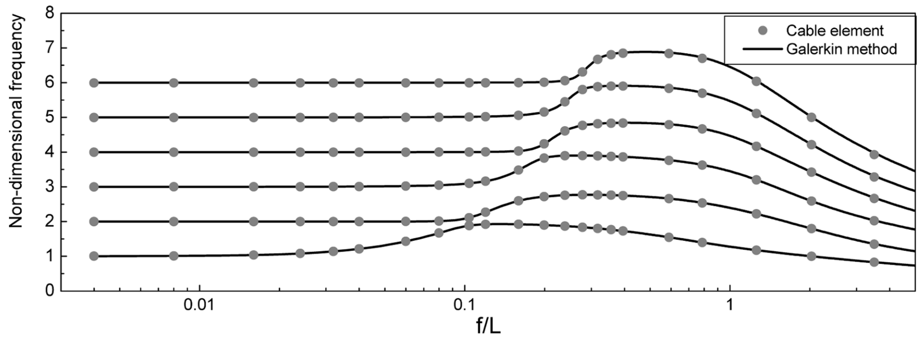

Fig. 4 Geometry of an inclined cable




(a) $\theta = 0°$




(b) $\theta = 30°$




(c) $\theta = 60°$

Fig. 5 Natural frequencies of in-plane modes of cables with $k^2 = 900$

### *3.2 Natural vibrations of cables*

An inclined cable with an inclination angle $\theta$, a uniform cross section $A$, and uniform weigh per unit length $m$, and hanging between two points is analyzed, as shown in Fig. 4. The horizontal tension of the cable is $H$ and the ratio of the elongation stiffness to the longitudinal tension $k^2 = EcAc/H$ is assumed to 900. The initial tension $P_0$ for each cable element in the inclined cables is obtained from $P_0 = H \cdot dx/ds$. A Galerkin method is used to verify the accuracy of the results from the cable finite element (Yamaguchi 1997).

Fig. 5(a) shows the natural frequencies of horizontal cables ($\theta = 0^{\circ}$). Figs. 5(b) and 5(c) show the natural frequencies of inclined cables with inclination angles of $\theta = 30^{\circ}$ and $\theta = 60^{\circ}$. The ordinate is the sag-to-span ratio $\gamma = f/L$ and the abscissa is the non-dimensional frequency $\overline{\omega} = \omega_{cable}/\omega_0$. $\omega_{cable}$ is the circular frequency of the cable. $\omega_0 = \sqrt{H\sec\theta/m}(\pi/L)$ is the first natural circular frequency of the inclined taut string that has a span $L$ and no sag.

For both horizontal cables and inclined cables, the results obtained using the cable finite element overlap those obtained using the Galerkin method. In other word, these two methods provide the same results.

Therefore, the cable element method can be used to evaluate the properties of not only horizontal cables but also inclined cables without a small-sag limitation.

### *3.3 Parametric vibrations of a cable subjected to periodic support excitation*

In the finial application in this section, a dynamic analysis is performed on a cable subjected to periodic excitation at one support. Wu *et al.* (2004) proves that parametric vibrations can be generated under either the excitation displacement or the excitation axial force. Therefore, this paper uses periodic support displacement as the excitation force.

Fig. 6 shows the geometry of a horizontal cable subjected to horizontal displacement at one support. The sag-to-span ratio of cable $\gamma = f/L$ is 0.026 and the ratio of elongation stiffness to longitudinal tension $k^2 = EcAc/H$ is 900. The periodic displacement is assumed to be a sine wave $A^*\sin\Omega t$. The non-dimensional amplitude $A^*$ is set to $A^* = A/L = 1.12\times 10^{-4}$. The frequency $\Omega$ of the excitation displacement is assumed to be equal to the first natural frequency $\omega_{cable}$ of the in-plane cable modes. In the other order, the excitation corresponds to the parametric excitation in the second unstable region. The finite difference method is used as a method of comparison (Wu *et al.* 2004).

Fig. 7 shows the nonlinear time-history responses of the cable. The responses obtained from the cable finite element method agree well with those obtained from the finite difference method.

The chord element is used to model the cable in certain types of numerical analyses. Fig. 8 lists the time-domain displacement at the center point of the cable modeled by chord finite element using TDAP software (ARK 2003). Comparing the response in Fig. 7(a) with that in Fig. 8 confirms that the chord element cannot satisfactorily simulate parametric vibrations in cables.

The results support the use of the cable finite element and the proposed procedure for evaluating parametric vibrations of cables in cable structures.

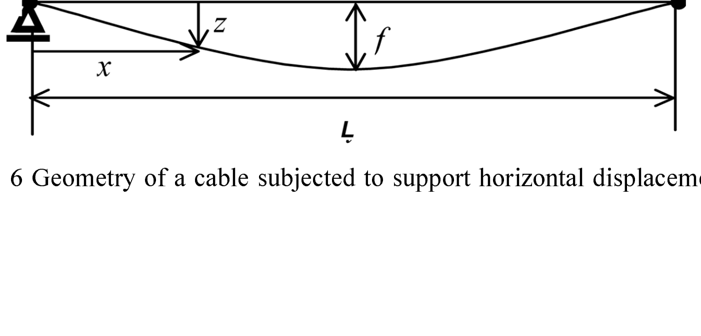
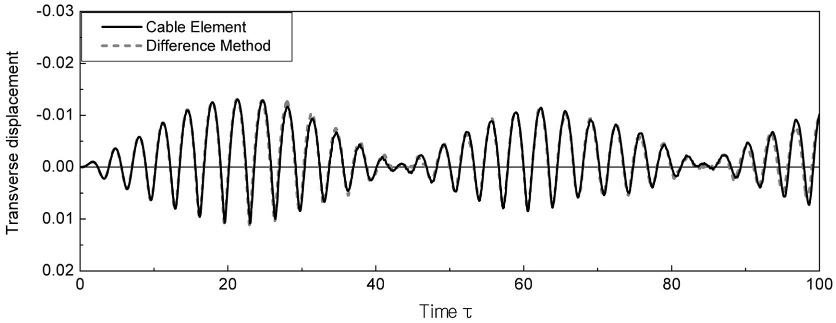
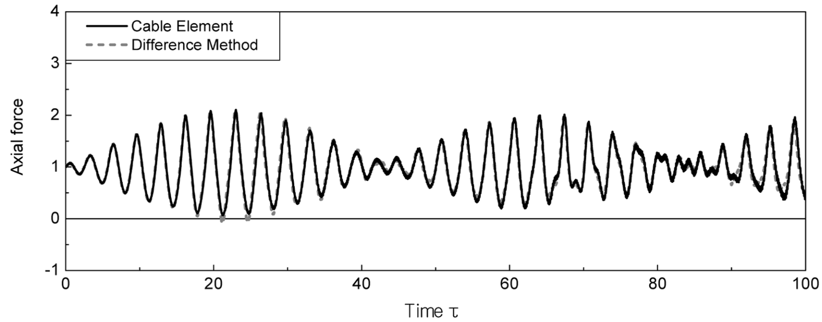


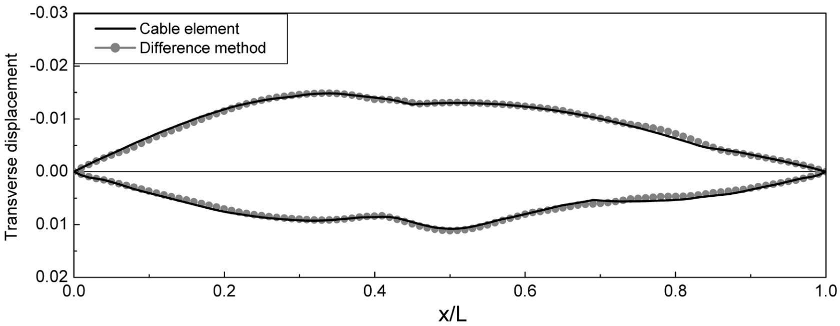

Fig. 6 Geometry of a cable subjected to support horizontal displacement




(a) Transverse displacement




(b) Axial force




(b) Maximum transverse displacement

Fig. 7 Response in the center point of the cable with $\gamma = 0.026$ and $k^2 = 900$ using cable element

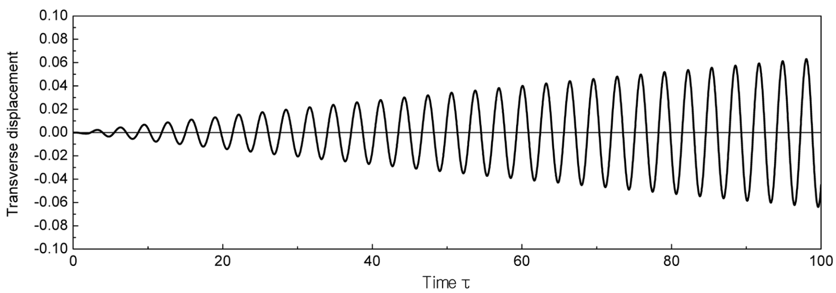



Fig. 8 Response in the center point of the cable with $\gamma = 0.026$ and $k^2 = 900$ using chord element

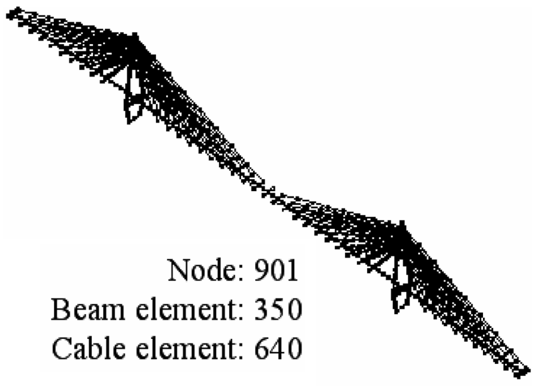



Fig. 9 MECS Cable-stayed bridge models

## 4. Numerical evaluations of a cable-stayed bridge

The following numerical example demonstrates the use of the cable element to model stay cables in cable-stayed bridges. The cable-stayed bridge analyzed in this paper has three spans. The main span is 350 m and the side spans are 160 m. The towers are A-shaped, and the cables are a two-plane, multiple system. The cables are numbered sequentially from the side span to the main span.

A three-dimensional finite element model of the bridge is shown in Fig. 9. The MECS (Multi-Element-Cable-System) models are used in this analysis (Abdel-Ghaffar *et al*. 1991). The girder and towers are modeled using three-dimensional linear beam elements. Each cable is represented by eight cable finite elements with the original modulus. Since the initial shapes of cables with sag are taken into account, there is no need to consider the equivalent modulus that allows for sagging (Au *et al*. 2001).

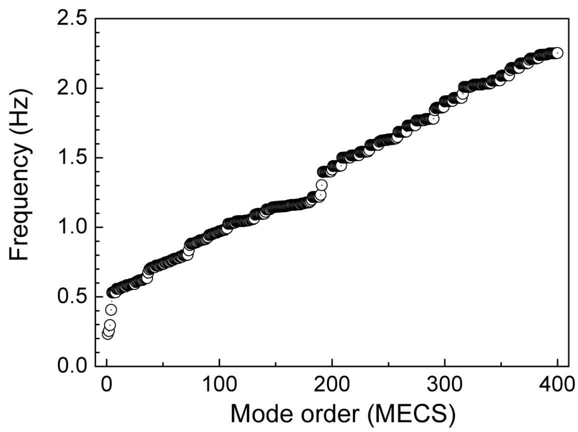



Fig. 10 Natural frequencies of MECS model

### *4.1 Eigenvalue analysis*

A subspace iteration algorithm (Bhatt 2002) is used to extract the first 400 modes of the MECS model. Fig. 10 shows the natural frequencies of those modes. These 400 modes range from about 0.0 Hz to 2.5 Hz. As with the common natural vibration analysis of cable-stayed bridges, the modes of the girder/tower vibrations are very important for understanding the properties of the bridge. If single elements are used to model the cables of the cable-stayed bridge, which is an OECS (One-Element-Cable-System) model (Abdel-Ghaffar *et al.* 1991), the modes of the girder/tower vibrations can be easily obtained. An eigenvalue analysis of the OECS model is performed to determine the number of natural modes of the girder/tower vibrations. 23 natural modes in the range of 0.0-2.5 Hz are obtained in the OECS model.

How these 23 modes of the girder/tower vibrations are exacted from the 400 modes in the MECS model is important. One method is to observe the participation factors and modal shapes of all modes (Abdel-Ghaffar *et al.* 1991, Caetano *et al.* 2000). This is very troublesome, however, since the MECS model produces enormous natural modes for the stay cable vibrations besides the girder/tower vibrations. The more the stay cables are divided, the more difficult this method becomes. Therefore, the present paper proposes, using the substructure synthesis method to calculate the natural frequencies of the girder/tower modes in the MECS cable-stayed bridge model.

Static condensation and dynamic condensation are two methods of mode analysis that use the substructure synthesis method. The static condensation method (Guyan 1965, Irons 1965) correctly represents the stiffness of the substructure but is only accurate in a dynamic context if there is no mass within the substructure. In order to give virtually exact eigenvalues for all modes considered in the reduced eigenproblem, the dynamic condensation method was proposed by Paz (1990). However, dynamic condensation requires an iterative process, which slows down the calculation.

The process of dynamic condensation can be explained by considering the generalized eigenvalue problem expressed in the partitioned matrices below.

$$\begin{bmatrix} K_{ss} - \omega_i^2 M_{ss} & K_{sp} - \omega_i^2 M_{sp} \\ K_{ps} - \omega_i^2 M_{ps} & K_{pp} - \omega_i^2 M_{pp} \end{bmatrix} \begin{Bmatrix} Y_s \\ Y_p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{13}$$

where $K_{ij}$ and $M_{ij}$ are the stiffness and mass matrices and $ij$ corresponds to the $\boldsymbol{p}$ degrees of freedom to be remained or the $\boldsymbol{s}$ degrees of freedom to be reduced, $Y_s$ is the displacement vector corresponding to the $\boldsymbol{s}$ degrees of freedom to be reduced, $Y_p$ is the displacement vector corresponding to the $\boldsymbol{p}$ degrees of freedom to be remained and $\omega_i$ is the eigenvalue that is approximated at each step of the calculation. In the MECS cable-stayed bridge model, the remaining $\boldsymbol{p}$ degrees of freedom are assigned to the girder and towers so that the reduced $\boldsymbol{s}$ degrees of freedom are given to the stay cables.

In order to speed up the calculation, the eigenvalue analysis process using dynamic condensation is shown in Fig. 11. This figure shows that the static condensation is the first step of the dynamic condensation.

The proposed process modifies the process of the dynamic condensation in the following two ways:

1. Iterative error $\varepsilon$ is added so that if $\omega_i^{new} - \omega_i < \varepsilon$, it is unnecessary to perform calculations at the maximum convergent times *indexmax*;
2. The number of modes needed by the subspace method using the dynamic condensation is $\boldsymbol{i}$, not

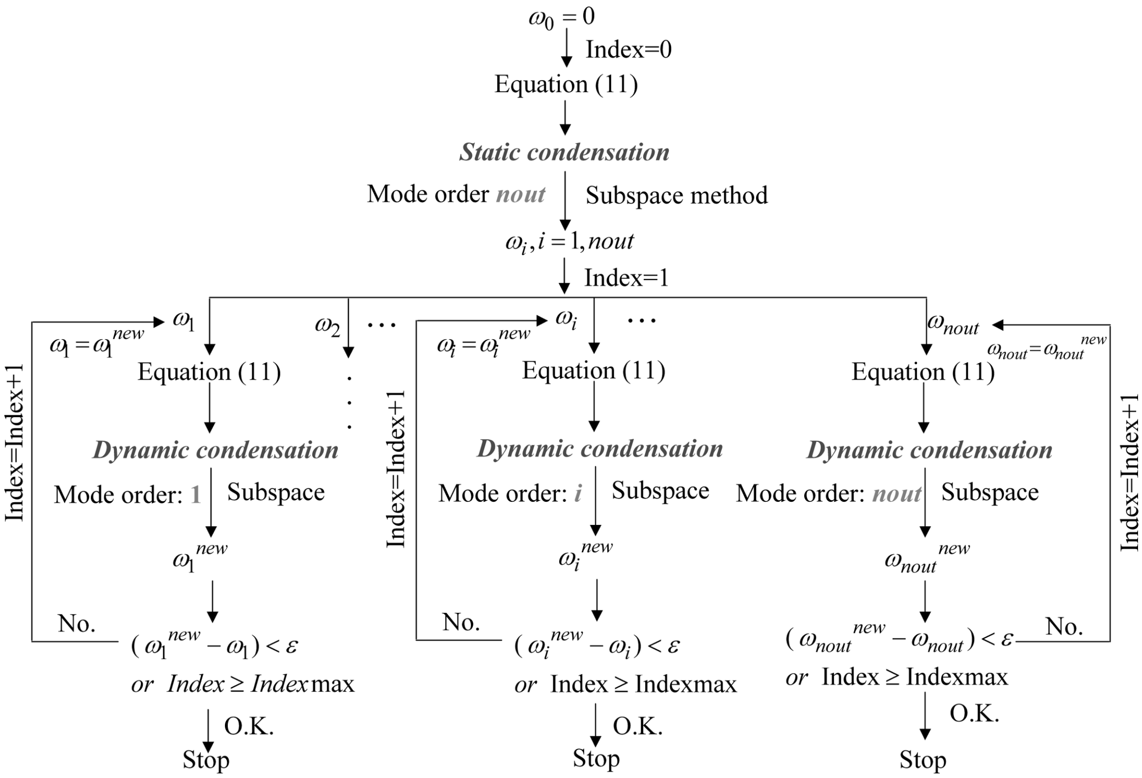



Fig. 11 Eigenvalue analysis process using dynamic condensation

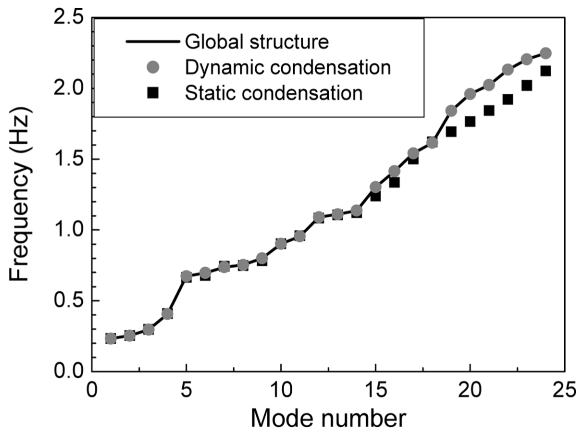



Fig. 12 Natural frequencies of girder/tower modes

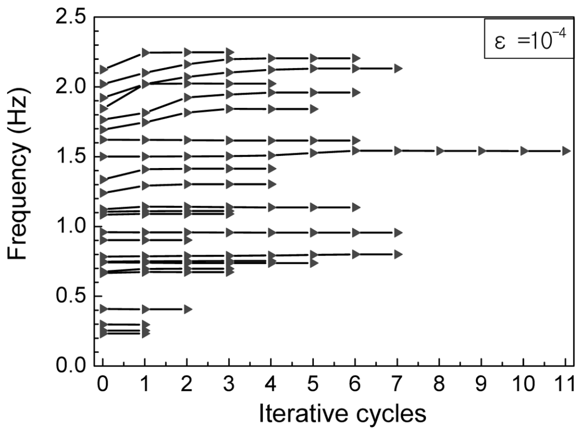



Fig. 13 Iterative cycles in dynamic condensation

***nout***. For example, when the 11th natural mode is the determined eigenvalue in 100 modes, the number of modes needed by the subspace method using dynamic condensation is 11, not 100.

Using the previous process, the computation becomes faster. By the way, an efficient algorithm, for example the Cuthill-McKee algorithm or the Multiple Roots Reverse Cuthill-McKee method (Cuthill *et al*. 1969, Lin 1989), can be used into the two substructures: girder/tower and stay cables

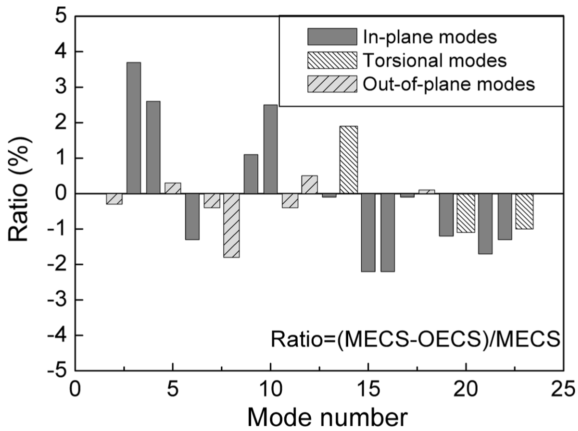



Fig. 14 Difference between MECS model and OECS model

to reduce the bandwidths of these substructures. Therefore, the calculation using dynamic condensation can be speed up.

The natural frequencies of the girder/tower modes using dynamic condensation are shown in Fig. 12. The frequencies obtained by dynamic condensation are identical to those obtained using global structure (correct values).

Fig. 13 shows the iterative cycles *indexmax* using dynamic condensation when $\varepsilon = 10^{-4}$. The iterative cycles change depending on the modes. The maximum number of iterative cycles is 11 in this analysis, not 1 or 2 (Paz 1990). The reason for this is that the three-dimensional MECS cable-stayed bridge model has many degrees of freedom and has too many coupled deck-cable modes.

The natural frequencies of the girder/tower modes using static condensation are also shown in Fig. 12. The frequencies in the region of 1.2-2.5 Hz that were obtained using static condensation differ greatly from those obtained using global structure, which confirms that static condensation doesn't provide great precision in the eigenvalue analysis of a MECS model.

Therefore, by using the proposed process using dynamic condensation method, the natural modes of the girder/tower vibrations can be calculated accurately and efficiently.

The natural frequencies of the in-plane modes, torsional modes and out-of-plane modes for the MECS and OECS models are shown in Fig. 14. The maximum differences in the in-plane, torsional and out-of-plane modes of the MECS model and the OCES model are about 4%, 2% and 2%, respectively. Therefore, not only the in-plane and torsional modes but also the out-of-plane modes of the coupled cable-deck vibrations can be considered.

### 4.2 Dynamic analysis of cable-stayed bridge

The equation of motion for a three-dimensional bridge with MDOF subjected to excitation can be expressed as follows (Clough *et al.* 1975):

$$[M]\{\ddot{Y}\} + [C]\{\dot{Y}\} + [K]\{Y\} = \{F\} \tag{14}$$

where $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices, respectively, $\{F\}$ is the load vector, and $\{Y\}$, $\{\dot{Y}\}$ and $\{\ddot{Y}\}$ are the displacement, velocity and acceleration vector representing

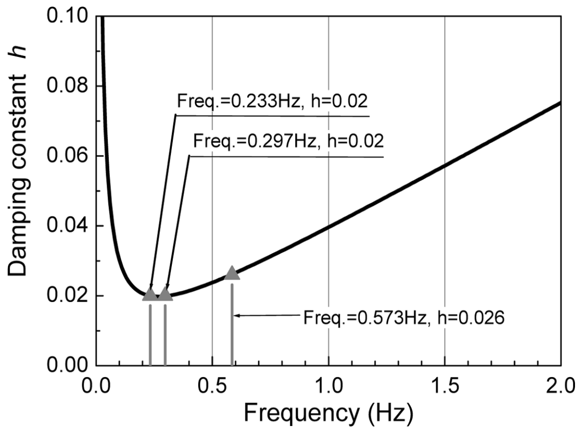



Fig. 15 Rayleigh damping

the displaced shape of the bridge.

The equations of motion in Eq. (14) are directly integrated using the Newmark $\beta$ method ($\beta = 0.25$). The Newton-Raphson method is used for the iterative procedure.

Rayleigh damping is usually adopted for general dynamic analysis of direct integration. The damping matrix $[C]$ using Rayleigh damping has the following formula (Caughey 1960).

$$[C] = \alpha[M] + \beta[K] \tag{15}$$

in which $\alpha$ and $\beta$ are arbitrary proportional factors. These factors are given in terms of two different frequencies ($f_i$, $f_j$) and the corresponding critical damping ratios ($h_i$, $h_j$) ($i < j$) and expressed as follow:

$$\alpha = \frac{4\pi \cdot f_i f_j (h_i f_j - h_j f_i)}{f_j^2 - f_i^2}, \quad \beta = \frac{h_j f_j - h_i f_i}{\pi(f_j^2 - f_i^2)} \tag{16}$$

Generally, the first two natural frequencies of the modes with relatively greater participation factors are used to determine the Rayleigh damping matrix in dynamic analysis of cable-stayed bridges.

For example, the Rayleigh damping curve of the cable-stayed bridge described here is shown in Fig. 15. The two frequencies are assumed to be $f_1 = 0.233$ Hz (corresponding to the frequency of the floating mode) and $f_2 = 0.297$ Hz (corresponding to the frequency of the first vertical mode). Those two modes are the main vibrations of the girder/tower vibrations. The two damping constants are assumed to be $h_1 = h_2 = 0.02$, which is the damping ratio of the steel girder/tower system (Japan Road Association 2002). This curve provides the damping ratio of the corresponding frequency. The first frequency of the in-plane modes in cable C02 is 0.573 Hz, and the corresponding damping constant of cable C02 is 0.026. The actual damping constants of the stay cables are in the range of 0.001-0.005 (Wu *et al.* 2003). Therefore, the damping of the stay cables is overestimated when normal Rayleigh damping is used.

In order to correctly evaluate the damping of the stay cables in cable-stayed bridges, this paper proposes an extended Rayleigh damping method that takes into account element damping.

Assuming that Eq. (15) can be applied to every element of the structure, the Rayleigh damping matrix that takes into account element damping can be extended as follows:

$$[C] = \sum_{k=1}^{N} (\alpha_k [M]_k + \beta_k [K]_k) \tag{17}$$

where $[M]_k$ and $[K]_k$ are the mass and stiffness matrix of element-***k***, $\alpha_k$ and $\beta_k$ are the arbitrary proportional factors of element-***k*** and $N$ is the number of elements. The $\alpha_k$ and $\beta_k$ of element-***k*** can be obtained as follows:

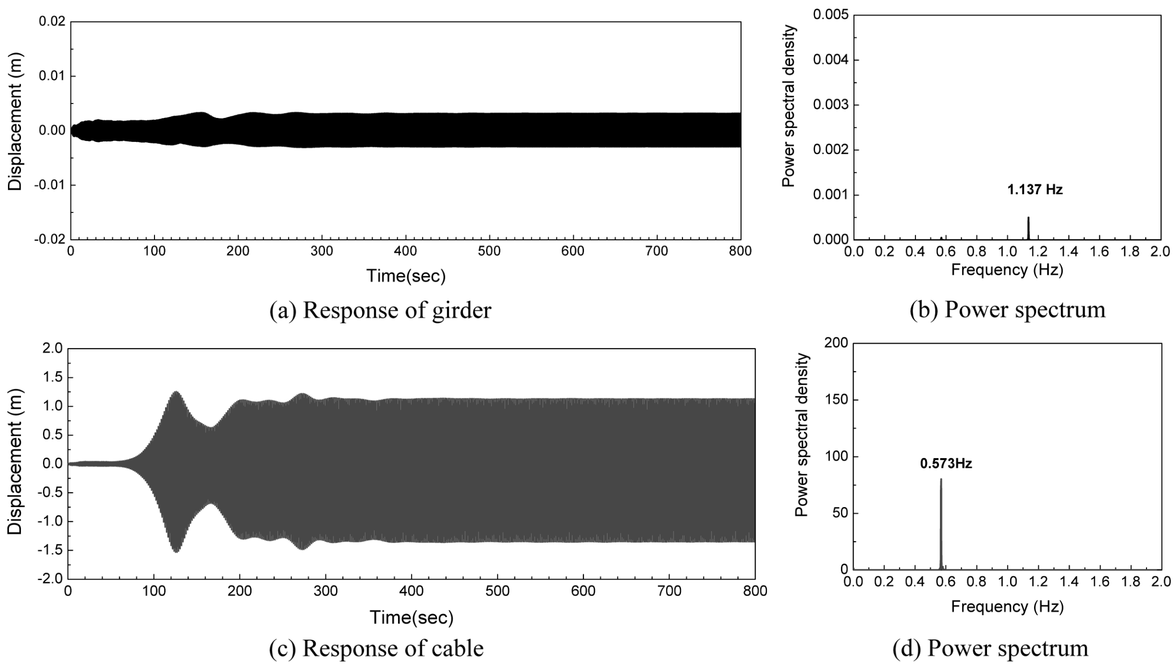
$$\alpha_k = \frac{4\pi \cdot f_i^k f_j^k (h_i^k f_j^k - h_j^k f_i^k)}{(f_j^k)^2 - (f_i^k)^2}, \quad \beta_k = \frac{h_j^k f_j^k - h_i^k f_i^k}{\pi((f_j^k)^2 - (f_i^k)^2)} \tag{18}$$

where $f_i^k$, $f_j^k$ and $h_i$, $h_j$ ($i < j$) are two different frequencies and corresponding damping constants of element-***k***.

By using extended Rayleigh damping, different damping constants can be established for stay cables with small structural damping and for girder/tower with relatively greater damping. Table 1 shows the parameters of the damping matrix for the cable-stayed bridge presented here.

Table 1 Damping parameters

| | $f_1$ (Hz) | $f_2$ (Hz) | $h_1$ | $h_2$ |
|---|---|---|---|---|
| Girder tower | 0.233 | 0.297 | 0.02 | 0.02 |
| Stay cables | First frequency of every cable | Second frequency of every cable | 0.001 | 0.001 |




(a) Response of girder (b) Power spectrum

(c) Response of cable (d) Power spectrum

Fig. 16 Time-histories and frequency-domain responses of Cable C02 by cable element (Rayleigh damping considering element damping)

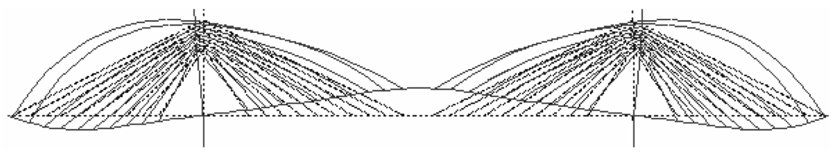
A dynamic analysis is performed on this cable-stayed bridge, which is subjected to periodic sinusoidal loading $P = A\sin\omega t$. The excitation point is at the center of the main span and the freedom of the point is the rotation along the longitudinal direction. The amplitude of the excitation is assumed to be 500 kN·m and the frequency of the excitation is 1.137 Hz, which corresponds to the first torsional frequency of the girder vibrations.

Fig. 16 shows the time-history and frequency-domain responses of cable C02. Parametric vibration in the principal unstable region occurs in cable C02 because the predominate frequency (0.573 Hz) of cable C02 is close to half the frequency of the excitation (1.137 Hz). A large-amplitude vibration is induced in this cable, while the vibrations in the girder are very small.

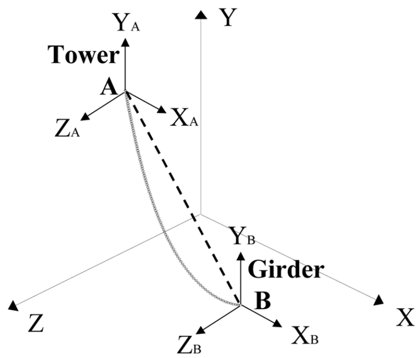
Therefore, the cable finite element can take into account the parametric vibrations of stay cables in cable-stayed bridges.

### *4.3 Global and local vibrations analysis of cable-stayed bridges*

The other concept that considers the vibration of stay cables is the separate treatment of the global vibration and local vibration (Gimsing 1983). Fig. 17 shows the concept of the global vibration of the girder/tower and local vibration of stay cables. Wu *et al.* (2003) used this concept to discuss the characteristics of local cable vibrations in one cable-stayed bridge. The following are the differences between this paper and Wu *et al*'s paper:

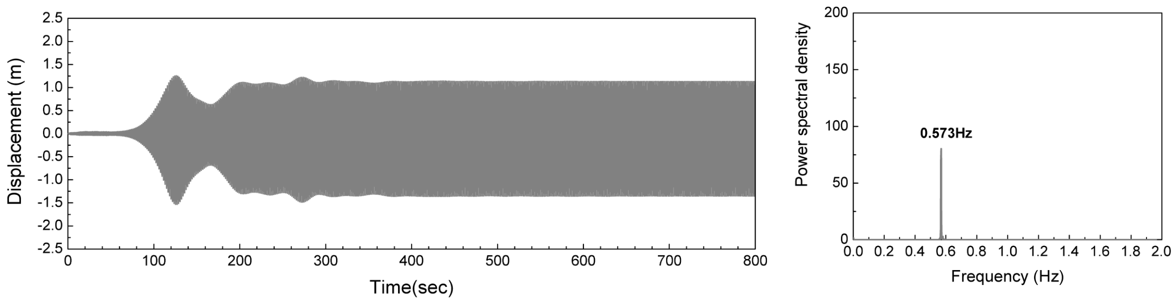


(a) Global vibration




(b) Local vibration

Fig. 17 Global vibration and local vibration




(a) Displacement of cable (b) Power spectrum

Fig. 18 Time-histories and frequency-domain response of Cable C02 from global and local vibration analysis

1. Three-direction excitation displacements at two supports (($X_A$, $Y_A$, $Z_A$) and ($X_B$, $Y_B$, $Z_B$) in Fig. 17(b)) can be considered by using cable finite elements.
2. The loading on the stay cables (for example, earthquake) can be taken into account in this paper.
3. The MECS model is used in the global vibration analysis. In other words, the excitation displacements that needed in local cable vibrations are the responses of girder and tower obtained from the MECS model.

Fig. 18 shows the response of cable C02 in the case of the previous periodic torsional excitation. Similar to the previous discussion, the parametric vibration in the principal unstable region occurs in cable C02. Comparing the cable response in Fig. 16 with that in Fig. 18, the result obtained from global and local vibration analysis is same as that obtained from direct dynamic analysis of cable-stayed bridges.

These results confirm that the concept of the global and local vibrations can be used to evaluate the parametric vibrations of stay cables in a MECS cable-stayed bridge model.

## 5. Conclusions

This paper proposes the use of cable finite element that takes into account the non-linearity and parametric vibration of sagging cables, from flat-sag cables to large-sag cables. The proposed method of removing the self-weight term of the cable element in the dynamic analysis is verified.

The applicability of this cable finite element for modeling the stay cables in cable-stayed bridges is discussed using a steel cable-stayed bridge. The following conclusions are reached.

1. The eigenvalue analysis process using dynamic condensation method is proposed for sorting out the natural modes of the girder and tower vibrations in MECS cable-stayed bridge model. The accuracy and efficiency are confirmed.

   Rayleigh damping that takes into account element damping is proposed as a method for creating damping matrix for MECS cable-stayed bridges. Extended Rayleigh damping can take into account the different damping constants of the stay cables, which have small structural damping, and the girder/towers, which have relatively greater damping.
2. The parametric-induced vibrations of stay cables in a MECS cable-stayed bridge model can be

taken into account by using cable finite elements.

The coupled cable-deck vibrations and the parametric vibrations of cables can be taken into account by using finite element method for dynamic analysis of cable-stayed bridges.

3. The concept of global and local vibrations can be used to evaluate the parametric vibrations of stay cables in a MECS cable-stayed bridge model.

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