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Exact solutions of axisymmetric free vibration of transversely isotropic magnetoelectroelastic laminated circular plates

Jiangying Chen†

Faculty of Engineering, Ningbo University, Ningbo 315211, China
Ningbo Institute of Technology, Zhejiang University, Ningbo 315100, China

Rongqiao Xu‡

Department of Civil Engineering, Zhejiang University, Hangzhou 310027, China

Xusheng Huang‡†

Faculty of Engineering, Ningbo University, Ningbo 315211, China

Haojiang Ding‡‡

Department of Civil Engineering, Zhejiang University, Hangzhou 310027, China

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Abstract. The axisymmetric free vibrations of transversely isotropic magnetoelectroelastic laminated circular plates are studied. Based on the three-dimensional governing equations of magnetoelectroelastic medium, the state space equations of laminated circular plates are obtained. By using the finite Hankel transform and rendering the free terms left by the transform in terms of the boundary quantities, the solutions of the state space equations are given for two kinds of boundary conditions. The frequency equations of the free vibration are derived using the propagator matrix method and the boundary conditions at top and bottom surfaces. By virtue of the inverse Hankel transform, the mode shapes are also determined. Since the solutions strictly satisfy the governing equations in the region and the boundary conditions at the edges, they are the three-dimensionally exact. Finally, the natural frequencies of such plates are tabulated and compared with those of the piezoelectric and elastic plates in the numerical example.

Keywords: magnetoelectroelastic medium; laminated circular plates; free vibration; state space method.

† Assistant Professor, Corresponding author, E-mail: cjy@nit.net.cn or chenjiangying@nbu.edu.cn
‡ Assistant Professor, E-mail: xurongqiao@zju.edu.cn
‡† Assistant Professor, E-mail: huangxusheng@nbu.edu.cn
‡‡ Professor, E-mail: dinghj@zju.edu.cn

1. Introduction

As smart material, the magnetoelectroelastic medium had been paid more attentions by many researchers because of their characteristics of magnetic field coupled with electricity field and mechanics field. The state space method is widely used to investigate the behavior of those structures made by magnetoelectroelastic medium since they are usually in laminated forms (Pan 2001, Pan and Heyliger 2002, 2003, Chen and Lee 2003, Chen *et al.* 2005, Wang *et al*. 2003). Some interesting solutions are derived in the Cartesian coordinate system for the simply supported rectangular plates and the rectangular plates in cylindrical bending. Unfortunately, their methods cannot be applied in the cylindrical coordinate systems and are not suitable for circular plates.

The classical plate theory is widely used to investigate the static and dynamic behavior of thin isotropic circular plates. However, it has well-known limits in the case of anisotropic material, or laminated structure, or high-order vibration because of the assumption of planar section. Mindlin (1951) proposed a plate theory which considered the effects of shear deformation and rotating inertia and obtained the solutions of axisymmetric flexural vibration, symmetric flexural vibration and high-frequency extensional vibration of circular plates (Mindlin 1951a, 1951b, Mindlin and Deresiewicz 1954, Deresiewicz and Mindlin 1955, Dersiewice 1956, Kane and Mindlin 1956). These works are more exactly than the classic plate theory but they still are not exact solutions.

In the early of 1951, Timoshenko and Goodier (1951) gave an analytical solution of a simply supported isotropic circular plate, but it was not an exactly solution since the Saint-Venant principle was used in the boundary. Celep (1978, 1980) made three-dimensional investigation on the axisymmetric free vibration of circular plates using the method of initial functions. In order to overcome the difficulty of mathematics in solving equations in cylindrical coordinates, some assumption should be introduced. So the really exact solution of the anisotropic circular plate is few. Rao and Das (1977) introduced the state space method into three-dimensional elastic dynamics problem. Ding *et al.* (1999) first obtained the exact solutions of the axisymmetric free vibration of transversely isotropic piezoelectric circular plate by use of the state space method with the finite Hankel transform. Chen *et al*. (2003a) presented the solutions of stress field, electricity filed and magnetic field of simply supported magnetoelectroelastic circular plate under uniform loading. The three-dimensional analysis of rotating annular plate of magnetoelectroelastic medium is also given (Chen *et al.* 2003b).

In this paper, the state space equations of axisymmetric free vibration of laminated circular plates are obtained from the three-dimensional governing equations of the magnetoelectroelastic medium. By use of finite Hankel transform and let the free terms derived from the transform be zero, two boundary conditions, named as generalized elastic simply supported (GESS) and generalized rigid slipping supported (GRSS) are obtained. Based on the solutions of the state equations, the frequency equations are derived using the propagator matrix method and the boundary conditions on the bottom and top surfaces of the plates. The inverse Hankel transform is then used to determine the corresponding mode shapes. Since the solutions strictly satisfy the governing equations in the region and the boundary conditions at the edges, they are the three-dimensionally exact solutions of the free vibration of the magnetoelectroelastic circular plates.

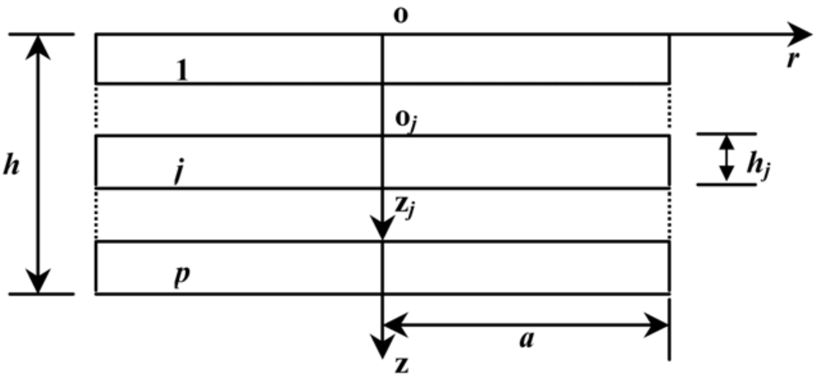

Fig. 1 The coordinate system of the laminated circular plates

2. State space equations and its solutions

Consider a p-ply laminated magnetoelectroelastic circular plate of radius a, thickness h. The cylindrical coordinate system (r, ϕ, z) is employed with z-axis being along the symmetry axis of the circular laminates (Fig. 1). The origin of the coordinates is the center of the top surface. The positive direction of the z-axis points the bottom surface from the top surface. Each layer may have its own constants of material. For the axisymmetric free vibration, without the body forces, free charges and magnetic inductions, the governing equations of the jth layer are given by

$$t_0\frac{\partial\bar{\sigma}_{rr}}{\partial\xi}+\frac{\partial\bar{\sigma}_{rz}}{\partial\zeta}+t_0\frac{\bar{\sigma}_{rr}-\bar{\sigma}_{\phi\phi}}{\xi} = -\bar{\rho}\Omega^2\bar{u}_r \tag{1}$$

$$t_0\frac{\partial\bar{\sigma}_{rz}}{\partial\xi}+\frac{\partial\bar{\sigma}_{zz}}{\partial\zeta}+t_0\frac{\bar{\sigma}_{rz}}{\xi} = -\bar{\rho}\Omega^2\bar{u}_z \tag{2}$$

$$t_0\left(\frac{\partial}{\partial\xi}+\frac{1}{\xi}\right)\bar{D}_r+\frac{\partial\bar{D}_z}{\partial\zeta} = 0 \tag{3}$$

$$t_0\left(\frac{\partial}{\partial\xi}+\frac{1}{\xi}\right)\bar{B}_r+\frac{\partial\bar{B}_z}{\partial\zeta} = 0 \tag{4}$$

$$\bar{\sigma}_{rr} = \bar{c}_{11}t_0\frac{\partial\bar{u}_r}{\partial\xi}+\bar{c}_{12}t_0\frac{\bar{u}_r}{\xi}+\bar{c}_{13}\frac{\partial\bar{u}_z}{\partial\zeta}+\bar{e}_{31}\frac{\partial\bar{\Phi}}{\partial\zeta}+\bar{d}_{31}\frac{\partial\bar{\Psi}}{\partial\zeta}$$

$$\bar{\sigma}_{\theta\theta} = \bar{c}_{12}t_0\frac{\partial\bar{u}_r}{\partial\xi}+\bar{c}_{11}t_0\frac{\bar{u}_r}{\xi}+\bar{c}_{13}\frac{\partial\bar{u}_z}{\partial\zeta}+\bar{e}_{31}\frac{\partial\bar{\Phi}}{\partial\zeta}+\bar{d}_{31}\frac{\partial\bar{\Psi}}{\partial\zeta}$$

$$\bar{\sigma}_{zz} = \bar{c}_{13}t_0\left(\frac{\partial\bar{u}_r}{\partial\xi}+\frac{\bar{u}_r}{\xi}\right)+\bar{c}_{33}\frac{\partial\bar{u}_z}{\partial\zeta}+\bar{e}_{33}\frac{\partial\bar{\Phi}}{\partial\zeta}+\bar{d}_{33}\frac{\partial\bar{\Psi}}{\partial\zeta}$$

$$\bar{\sigma}_{rz} = \bar{c}_{44}\left(\frac{\partial \bar{u}_r}{\partial \zeta} + t_0\frac{\partial \bar{u}_z}{\partial \xi}\right) + \bar{e}_{15}t_0\frac{\partial \bar{\Phi}}{\partial \xi} + \bar{d}_{15}t_0\frac{\partial \bar{\Psi}}{\partial \xi} \tag{5}$$

$$\bar{D}_r = \bar{e}_{15}\left(\frac{\partial \bar{u}_r}{\partial \zeta} + t_0\frac{\partial \bar{u}_z}{\partial \xi}\right) - \bar{\varepsilon}_{11}t_0\frac{\partial \bar{\Phi}}{\partial \xi} - \bar{g}_{11}t_0\frac{\partial \bar{\Psi}}{\partial \xi}$$

$$\bar{D}_z = \bar{e}_{31}t_0\left(\frac{\partial \bar{u}_r}{\partial \xi} + \frac{\bar{u}_r}{\xi}\right) + \bar{e}_{33}\frac{\partial \bar{u}_z}{\partial \zeta} - \bar{\varepsilon}_{33}\frac{\partial \bar{\Phi}}{\partial \zeta} - \bar{g}_{33}\frac{\partial \bar{\Psi}}{\partial \zeta} \tag{6}$$

$$\bar{B}_r = \bar{d}_{15}\left(\frac{\partial \bar{u}_r}{\partial \zeta} + t_0\frac{\partial \bar{u}_z}{\partial \xi}\right) - \bar{g}_{11}t_0\frac{\partial \bar{\Phi}}{\partial \xi} - \bar{\mu}_{11}t_0\frac{\partial \bar{\Psi}}{\partial \xi}$$

$$\bar{B}_z = \bar{d}_{31}t_0\left(\frac{\partial \bar{u}_r}{\partial \xi} + \frac{\bar{u}_r}{\xi}\right) + \bar{d}_{33}\frac{\partial \bar{u}_z}{\partial \zeta} - \bar{g}_{33}\frac{\partial \bar{\Phi}}{\partial \zeta} - \bar{\mu}_{33}\frac{\partial \bar{\Psi}}{\partial \zeta} \tag{7}$$

where the common factor $e^{i\omega t}$ has been removed and the dimensionless coordinates and parameters are defined as

$$\xi = r/a, \quad \zeta = z_j/h, \quad \bar{\sigma}_{ij} = \sigma_{ij}/c_{11}^{(1)}, \quad \bar{u}_i = u_i/h, \quad \bar{D}_i = D_i/\sqrt{\varepsilon_{33}^{(1)}c_{11}^{(1)}}$$

$$\bar{B}_i = B_i/\sqrt{\mu_{33}^{(1)}c_{11}^{(1)}}; \quad \bar{\Phi} = \Phi\sqrt{\varepsilon_{33}^{(1)}/c_{11}^{(1)}}/h, \quad \bar{\Psi} = \Psi\sqrt{\mu_{33}^{(1)}/c_{11}^{(1)}}/h; \quad \bar{c}_{ij} = c_{ij}/c_{11}^{(1)}$$

$$\bar{e}_{ij} = e_{ij}/\sqrt{c_{11}^{(1)}\varepsilon_{33}^{(1)}}, \quad \bar{d}_{ij} = d_{ij}/\sqrt{c_{11}^{(1)}\mu_{33}^{(1)}}, \quad \bar{\varepsilon}_{ij} = \varepsilon_{ij}/\varepsilon_{33}^{(1)} \quad \bar{g}_{ij} = g_{ij}/\sqrt{\varepsilon_{33}^{(1)}\mu_{33}^{(1)}}$$

$$\bar{\mu}_{ij} = \mu_{ij}/\mu_{33}^{(1)}; \quad \bar{\rho} = \rho/\rho^{(1)}, \quad \Omega = \omega h\sqrt{\rho^{(1)}/c_{11}^{(1)}}, \quad t_0 = h/a, \quad d_j = h_j/h_o \tag{8a}$$

and

$$z_j = z - (h_1 + h_2 + \ldots + h_{j-1}) \tag{8b}$$

is the local coordinate in z-direction of the jth layer; h_j is the thickness of the jth layer; σ_{ij}, u_i, D_i and B_i are the components of stresses, displacements, electric displacements and magnetic inductions, respectively, Φ and Ψ are the electric potential and magnetic potential, respectively, c_{ij}, e_{ij}, d_{ij}, ε_{ij}, g_{ij} and μ_{ij} are elastic, piezoelectric, piezomagnetic, dielectric, electromagnetic and magnetic constants, respectively, ρ is the density of the material, ω denotes the circular frequency, t is the time; $c_{11}^{(1)}$, $\varepsilon_{33}^{(1)}$, $\mu_{33}^{(1)}$ and $\rho^{(1)}$ represent the material constants of the first layer, and t_0 is the ratio of thickness-to-radius.

If $\bar{u}_r$, $\bar{u}_z$, $\bar{\Phi}$, $\bar{\Psi}$, $\bar{\sigma}_{zr}$, $\bar{\sigma}_{zz}$, $\bar{D}_z$ and $\bar{B}_z$ are selected as the state variables, $\bar{\sigma}_{rr}$, $\bar{\sigma}_{\theta\theta}$, $\bar{D}_r$ and $\bar{B}_r$ as the derived variables, the state space equations can be obtained from Eqs. (1)-(7) as follows

$$\frac{\partial \mathbf{Y}}{\partial \zeta} = \mathbf{MY}, \quad \zeta \in [0, d_j], \ (j = 1, 2, \ldots, p) \tag{9}$$

$$\mathbf{X} = \mathbf{NY} \tag{10}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & \mathbf{M}_1 \\ \mathbf{M}_2 & 0 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & 0 \\ 0 & \mathbf{N}_2 \end{bmatrix} \tag{11}$$

$$\mathbf{Y}_1 = [\bar{u}_r(\xi,\zeta), \bar{\sigma}_{zz}(\xi,\zeta), \bar{D}_z(\xi,\zeta), \bar{B}_z(\xi,\zeta)]^T$$

$$\mathbf{Y}_2 = [-\bar{\sigma}_{zr}(\xi,\zeta), \bar{u}_z(\xi,\zeta), \bar{\Phi}(\xi,\zeta), \bar{\Psi}(\xi,\zeta)]^T \tag{12}$$

$$\mathbf{X}_1 = [\bar{\sigma}_{rr} + \bar{\sigma}_{\theta\theta}, \bar{\sigma}_{rr} - \bar{\sigma}_{\theta\theta}]^T, \quad \mathbf{X}_2 = [\bar{D}_r, \bar{B}_r]^T \tag{13}$$

$$\mathbf{M}_1 = \begin{bmatrix} -f_1 & -t_0\frac{\partial}{\partial\xi} & f_2\frac{\partial}{\partial\xi} & f_3\frac{\partial}{\partial\xi} \\ t_0\left(\frac{\partial}{\partial\xi}+\frac{1}{\xi}\right) & -\bar{\rho}\Omega^2 & 0 & 0 \\ -f_2\left(\frac{\partial}{\partial\xi}+\frac{1}{\xi}\right) & 0 & f_4\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial}{\xi\partial\xi}\right) & f_5\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial}{\xi\partial\xi}\right) \\ -f_3\left(\frac{\partial}{\partial\xi}+\frac{1}{\xi}\right) & 0 & f_5\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial}{\xi\partial\xi}\right) & f_6\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial}{\xi\partial\xi}\right) \end{bmatrix}$$

$$\mathbf{M}_2 = \begin{bmatrix} \bar{\rho}\Omega^2 + f_7\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial}{\xi\partial\xi}-\frac{1}{\xi^2}\right) & f_8\frac{\partial}{\partial\xi} & f_9\frac{\partial}{\partial\xi} & f_{10}\frac{\partial}{\partial\xi} \\ -f_8\left(\frac{\partial}{\partial\xi}+\frac{1}{\xi}\right) & \alpha_1 & \alpha_2 & \alpha_3 \\ -f_9\left(\frac{\partial}{\partial\xi}+\frac{1}{\xi}\right) & \alpha_2 & \beta_2 & \beta_3 \\ -f_{10}\left(\frac{\partial}{\partial\xi}+\frac{1}{\xi}\right) & \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \tag{14}$$

$$\mathbf{N}_1 = \begin{bmatrix} f_{11}\left(\frac{\partial}{\partial\xi}+\frac{1}{\xi}\right) & 2\alpha & 2\beta & 2\gamma \\ f_{12}\left(\frac{\partial}{\partial\xi}-\frac{1}{\xi}\right) & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} f_{13} & 0 & f_{14}\frac{\partial}{\partial\xi} & f_{15}\frac{\partial}{\partial\xi} \\ f_{16} & 0 & f_{15}\frac{\partial}{\partial\xi} & f_{17}\frac{\partial}{\partial\xi} \end{bmatrix} \tag{15}$$

in which

$$f_1 = \frac{1}{\bar{c}_{44}}, \quad f_2 = -\frac{\bar{e}_{15}}{\bar{c}_{44}}t_0, \quad f_3 = -\frac{\bar{d}_{15}}{\bar{c}_{44}}t_0, \quad f_4 = -t_0(\bar{e}_{15}f_2 - \bar{\varepsilon}_{11}t_0), \quad f_5 = -t_0(\bar{e}_{15}f_3 - \bar{g}_{11}t_0)$$

$$f_6 = -t_0(\bar{d}_{15}f_3 - \bar{\mu}_{11}t_0), \quad f_7 = t_0^2(\bar{c}_{11} - \bar{c}_{13}\alpha - \bar{e}_{31}\beta - \bar{d}_{31}\gamma), \quad f_8 = t_0\alpha, \quad f_9 = t_0\beta$$

$$f_{10} = t_0\gamma,\ f_{11} = (\bar{c}_{11}+\bar{c}_{12})t_0 - 2(\bar{c}_{13}f_8 + \bar{e}_{31}f_9 + \bar{d}_{31}f_{10}),\ f_{12} = (\bar{c}_{11}-\bar{c}_{12})t_0,\ f_{13} = \bar{e}_{15}f_1$$

$$f_{14} = \bar{e}_{15}f_2 - \bar{\varepsilon}_{11}t_0,\ f_{15} = \bar{d}_{15}f_2 - \bar{g}_{11}t_0,\ f_{16} = \bar{d}_{15}f_1,\ f_{17} = \bar{d}_{15}f_3 - \bar{\mu}_{11}t_0$$

$$\delta = \bar{c}_{33}\bar{\varepsilon}_{33}\bar{\mu}_{33} - 2\bar{d}_{33}\bar{e}_{33}\bar{g}_{33} + \bar{\varepsilon}_{33}\bar{d}_{33}^2 + \bar{\mu}_{33}\bar{e}_{33}^2 - \bar{c}_{33}\bar{g}_{33}^2$$

$$\alpha_1 = \frac{1}{\delta}(\bar{\varepsilon}_{33}\bar{\mu}_{33} - \bar{g}_{33}^2),\quad \alpha_2 = \frac{1}{\delta}(\bar{e}_{33}\bar{\mu}_{33} - \bar{d}_{33}\bar{g}_{33}),\ \alpha_3 = \frac{1}{\delta}(\bar{d}_{33}\bar{\varepsilon}_{33} - \bar{e}_{33}\bar{g}_{33})$$

$$\beta_2 = -\frac{1}{\delta}(\bar{c}_{33}\bar{\mu}_{33} + \bar{d}_{33}^2),\ \beta_3 = \frac{1}{\delta}(\bar{c}_{33}\bar{g}_{33} + \bar{e}_{33}\bar{d}_{33}),\ \gamma_3 = -\frac{1}{\delta}(\bar{c}_{33}\bar{\varepsilon}_{33} + \bar{e}_{33}^2)$$

$$\alpha = \bar{c}_{13}\alpha_1 + \bar{e}_{31}\alpha_2 + \bar{d}_{31}\alpha_3,\quad \beta = \bar{c}_{13}\alpha_2 + \bar{e}_{31}\beta_2 + \bar{d}_{31}\beta_3,\quad \gamma = \bar{c}_{13}\alpha_3 + \bar{e}_{31}\beta_3 + \bar{d}_{31}\gamma_3 \tag{16}$$

Also the following relations can be obtained,

$$\bar{e}_{15}f_3 = \bar{d}_{15}f_2,\quad f_2 = -f_{13}t_0,\ f_4 = -f_{14}t_0,\ f_5 = -f_{15}t_0,\ f_6 = -f_{17}t_0 \tag{17}$$

Define the finite Hankel transform as

$$J_\mu[f(\xi,\zeta)] = \int_0^1 \xi f(\xi,\zeta)J_\mu(k\xi)d\xi,\ \ \zeta\in[0,d_j] \tag{18}$$

where $J_\mu(k\xi)$ is the μth order Bessel function of the first kind. The following transform of the state variables are introduced,

$$u(k,\zeta) = J_1[\bar{u}_r(\xi,\zeta)],\ \tau(k,\zeta) = J_1[\bar{\sigma}_{zr}(\xi,\zeta)]$$

$$\sigma(k,\zeta) = J_0[\bar{\sigma}_{zz}(\xi,\zeta)],\ D(k,\zeta) = J_0[\bar{D}_z(\xi,\zeta)],\ B(k,\zeta) = J_0[\bar{B}_z(\xi,\zeta)]$$

$$w(k,\zeta) = J_0[\bar{u}_z(\xi,\zeta)],\ \Phi(k,\zeta) = J_0[\bar{\Phi}(\xi,\zeta)],\ \Psi(k,\zeta) = J_0[\bar{\Psi}(\xi,\zeta)] \tag{19}$$

Applying the Hankel transform (19) to Eq. (9) yields

$$\frac{\partial \mathbf{Y}_j(k,\zeta)}{\partial\zeta} = \mathbf{M}_j(k)\mathbf{Y}_j(k,\zeta) + \mathbf{Q}_j(k,\zeta),\ \ \zeta\in[0,d_j] \tag{20}$$

where

$$\mathbf{M}_j(k) = \begin{bmatrix} 0 & \mathbf{M}_{1j}(k) \\ \mathbf{M}_{2j}(k) & 0 \end{bmatrix},\ \ \mathbf{Y}_j(k,\zeta) = \begin{bmatrix} \mathbf{Y}_{1j}(k,\zeta) \\ \mathbf{Y}_{2j}(k,\zeta) \end{bmatrix} \tag{21a}$$

$$\mathbf{M}_{1j}(k) = \begin{bmatrix} -f_1 & kt_0 & -kf_2 & -kf_3 \\ kt_0 & -\bar{\rho}\Omega^2 & 0 & 0 \\ -kf_2 & 0 & -k^2f_4 & -k^2f_5 \\ -kf_3 & 0 & -k^2f_5 & -k^2f_6 \end{bmatrix}$$

$$\mathbf{M}_{2j}(k) = \begin{bmatrix} \bar{\rho}\Omega^2 - k^2 f_7 & -kf_8 & -kf_9 & -kf_{10} \\ -kf_8 & \alpha_1 & \alpha_2 & \alpha_3 \\ -kf_9 & \alpha_2 & \beta_2 & \beta_3 \\ -kf_{10} & \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \tag{21b}$$

$$\mathbf{Y}_{1j}(k, \zeta) = [u(k, \zeta), \sigma(k, \zeta), D(k, \zeta), B(k, \zeta)]_j^T$$

$$\mathbf{Y}_{2j}(k, \zeta) = [-\tau(k, \zeta), w(k, \zeta), \Phi(k, \zeta), \Psi(k, \zeta)]_j^T \tag{21c}$$

$$\mathbf{Q}_j(k, \zeta) = \begin{bmatrix} -t_0\bar{u}_z(1, \zeta)J_1(k) + f_2\bar{\Phi}(1, \zeta)J_1(k) + f_3\bar{\Psi}(1, \zeta)J_1(k) \\ -t_0 J_0(k)\bar{\sigma}_{zr}(1, \zeta) \\ -t_0\bar{D}_r(1, \zeta)J_0(k) + kf_4\bar{\Phi}(1, \zeta)J_1(k) + kf_5\bar{\Psi}(1, \zeta)J_1(k) \\ -t_0\bar{B}_r(1, \zeta)J_0(k) + kf_5\bar{\Phi}(1, \zeta)J_1(k) + kf_6\bar{\Psi}(1, \zeta)J_1(k) \\ -kf_7\bar{u}_r(1, \zeta)J_0(k) + \left[\dfrac{c_{11}-c_{12}}{c_{11}^{(1)}}t_0\bar{u}_r(1, \zeta) + \bar{\sigma}_{rr}(1, \zeta)\right]t_0 J_1(k) \\ -f_8\bar{u}_r(1, \zeta)J_0(k) \\ -f_9\bar{u}_r(1, \zeta)J_0(k) \\ -f_{10}\bar{u}_r(1, \zeta)J_0(k) \end{bmatrix} \tag{21d}$$

If let $\mathbf{Q}_j(k, \zeta)$ be zero, it can be easily observed that the following two boundary conditions, named as the generalized elastic simply supported (GESS) and generalized rigid slipping supported (GRSS), will be obtained.

(1) GESS conditions

$$\bar{u}_z(1, \zeta) = 0, \quad \frac{c_{11}-c_{12}}{c_{11}^{(1)}}t_0\bar{u}_r(1, \zeta) + \bar{\sigma}_{rr}(1, \zeta) = 0, \quad \bar{\Phi}(1, \zeta) = 0, \quad \bar{\Psi}(1, \zeta) = 0 \tag{22a}$$

$$J_0(k) = 0 \tag{22b}$$

(2) GRSS conditions

$$\bar{u}_r(1, \zeta) = 0, \quad \bar{\sigma}_{zr}(1, \zeta) = 0, \quad \bar{D}_r(1, \zeta) = 0, \quad \bar{B}_r(1, \zeta) = 0 \tag{23a}$$

$$J_1(k) = 0 \tag{23b}$$

Thus Eq. (20) becomes homogeneous and the solution is

$$\mathbf{Y}_j(k, \zeta) = \mathbf{T}_j(k, \zeta)\mathbf{Y}_j(k, 0), \quad \zeta \in [0, d_j] \tag{24}$$

By virtue of the Cayley-Hamilton theorem, the matrix $\mathbf{T}_j(k, \zeta)$ can be written as

$$\mathbf{T}_j(k, \zeta) = \exp[\mathbf{M}_j(k)\zeta] = a_0(\zeta)\mathbf{I} + \sum_{i=1}^{7} a_i(\zeta)\mathbf{M}_j(k)^i \tag{25}$$

in which $\mathbf{I}$ is an identity matrix of the eight-order, the matrix $\mathbf{M}_j(k)$ is defined in Eqs. (21) and the coefficients $a_i(\zeta)(i = 0, 1, \ldots, 7)$ are determined by

$$\begin{pmatrix} a_0(\zeta) \\ a_1(\zeta) \\ a_2(\zeta) \\ a_3(\zeta) \\ a_4(\zeta) \\ a_5(\zeta) \\ a_6(\zeta) \\ a_7(\zeta) \end{pmatrix} = \begin{bmatrix} 1 & \eta_1 & \eta_1^2 & \eta_1^3 & \eta_1^4 & \eta_1^5 & \eta_1^6 & \eta_1^7 \\ 1 & \eta_2 & \eta_2^2 & \eta_2^3 & \eta_2^4 & \eta_2^5 & \eta_2^6 & \eta_2^7 \\ 1 & \eta_3 & \eta_1^2 & \eta_1^3 & \eta_1^4 & \eta_1^5 & \eta_1^6 & \eta_1^7 \\ 1 & \eta_4 & \eta_4^2 & \eta_4^3 & \eta_4^4 & \eta_4^5 & \eta_4^6 & \eta_4^7 \\ 1 & \eta_5 & \eta_5^2 & \eta_5^3 & \eta_5^4 & \eta_5^5 & \eta_5^6 & \eta_5^7 \\ 1 & \eta_6 & \eta_6^2 & \eta_6^3 & \eta_6^4 & \eta_6^5 & \eta_6^6 & \eta_6^7 \\ 1 & \eta_7 & \eta_7^2 & \eta_7^3 & \eta_7^4 & \eta_7^5 & \eta_7^6 & \eta_7^7 \\ 1 & \eta_8 & \eta_8^2 & \eta_8^3 & \eta_8^4 & \eta_8^5 & \eta_8^6 & \eta_8^7 \end{bmatrix}^{-1} \begin{pmatrix} \exp(\eta_1\zeta) \\ \exp(\eta_2\zeta) \\ \exp(\eta_3\zeta) \\ \exp(\eta_4\zeta) \\ \exp(\eta_5\zeta) \\ \exp(\eta_6\zeta) \\ \exp(\eta_7\zeta) \\ \exp(\eta_8\zeta) \end{pmatrix} \tag{26}$$

where $\eta_i(i = 1, 2, \ldots, 8)$ are the distinct eigenvalues of the matrix $\mathbf{M}_j(k)$. Eq. (26) will have other forms when multiple equal eigenvalues are occurred.

Setting $\zeta = d_j$ in Eq. (24) yields

$$\mathbf{Y}_j(k, d_j) = \mathbf{T}_j(k, d_j)\mathbf{Y}_j(k, 0), \quad (j = 1, 2, \ldots, p) \tag{27}$$

Eq. (27) establishes the relation between the state variables of the jth layer at upper and lower surfaces by the transfer matrix $\mathbf{T}_j(k, d_j)$.

Using of the continuity conditions of the state variables $\bar{u}_r$, $\bar{u}_z$, $\bar{\Phi}$, $\bar{\Psi}$, $\bar{\sigma}_{zr}$, $\bar{\sigma}_{zz}$, $\bar{D}_z$ and $\bar{B}_z$ at each interface, i.e.,

$$\mathbf{Y}_{j+1}(k, 0) = \mathbf{Y}_j(k, d_j), \quad (j = 1, 2, \ldots, p-1) \tag{28}$$

yields

$$\mathbf{Y}_p(k, d_p) = \mathbf{F}(k)\mathbf{Y}_1(k, 0) \tag{29}$$

where

$$\mathbf{F}(k) = [F_{kl}] = \prod_{j=p}^{1} \mathbf{T}_j(k, d_j) \tag{30}$$

3. Frequency equations and mode shapes

For the free vibration problem, the mechanical boundary conditions at the top and bottom surfaces of the laminated circular plates are vanishing of the normal and shear stresses, namely,

$$\bar{\sigma}_{zr}(\xi, 0) = 0, \bar{\sigma}_{zr}(\xi, d_p) = 0, \bar{\sigma}_{zz}(\xi, 0) = 0, \bar{\sigma}_{zz}(\xi, d_p) = 0 \tag{31}$$

Furthermore, the electric boundary conditions are given by

$$\bar{D}_z(\xi, 0) = 0, \ \bar{D}_z(\xi, d_p) = 0 \tag{32a}$$

or $$\bar{D}_z(\xi, 0) = 0, \ \bar{\Phi}(\xi, d_p) = 0 \tag{32b}$$

or $$\bar{\Phi}(\xi, 0) = 0, \ \bar{D}_z(\xi, d_p) = 0 \tag{32c}$$

or $$\bar{\Phi}(\xi, 0) = 0, \ \bar{\Phi}(\xi, d_p) = 0 \tag{32d}$$

and the magnetoelectric conditions are

$$\bar{B}_z(\xi, 0) = 0, \ \bar{B}_z(\xi, d_p) = 0 \tag{33a}$$

or $$\bar{B}_z(\xi, 0) = 0, \ \bar{\Psi}(\xi, d_p) = 0 \tag{33b}$$

or $$\bar{\Psi}(\xi, 0) = 0, \ \bar{B}_z(\xi, d_p) = 0 \tag{33c}$$

or $$\bar{\Psi}(\xi, 0) = 0, \ \bar{\Psi}(\xi, d_p) = 0 \tag{33d}$$

Thus the Eqs. (31)-(33) have 16 combinations of boundary condition on the bottom and top surfaces. For example, the case 1 is defined as

$$\sigma(k, 0) = \sigma(k, d_p) = \tau(k, 0) = \tau(k, d_p) = 0$$

$$D(k, 0) = D(k, d_p) = 0, \ B(k, 0) = B(k, d_p) = 0 \tag{34}$$

Substituting Eqs. (34) into (29), yields

$$\begin{pmatrix} F_{21} & F_{26} & F_{27} & F_{28} \\ F_{31} & F_{36} & F_{37} & F_{38} \\ F_{41} & F_{46} & F_{47} & F_{48} \\ F_{51} & F_{56} & F_{57} & F_{58} \end{pmatrix} \begin{pmatrix} u(k, 0) \\ w(k, 0) \\ \Phi(k, 0) \\ \Psi(k, 0) \end{pmatrix} = 0 \tag{35}$$

The corresponding frequency equation can be obtained from Eq. (35) because of the requirement of non-trivial solution

$$\begin{vmatrix} F_{21} & F_{26} & F_{27} & F_{28} \\ F_{31} & F_{36} & F_{37} & F_{38} \\ F_{41} & F_{46} & F_{47} & F_{48} \\ F_{51} & F_{56} & F_{57} & F_{58} \end{vmatrix} = 0 \tag{36}$$

The frequency equation is transcendental with respect to Ω_{mm} and gives infinite number of frequencies for each k.

For the GESS conditions, the parameter k must satisfy Eq. (22b) and a series of roots $k_m(m = 1, 2, ...)$ can be obtained. Substitution of $k_m(m = 1, 2, ...)$ one by one into Eq. (36) determines a series of the non-dimensional frequencies $\Omega_{mn}(m = 1, 2, ...)$. Substituting Ω_{mm} and the corresponding k_m into Eq. (35) yields the ratios of $u(k_m, 0)$, $w(k_m, 0)$, $\Phi(k_m, 0)$ and $\Psi(k_m, 0)$, respectively. Consider the boundary conditions Eqs. (34) and using Eq. (24) along with Eq. (28), $u(k_m, \zeta)$, $w(k_m, \zeta)$, $\Phi(k_m, \zeta)$, $\Psi(k_m, \zeta)$, $\sigma(k_m, \zeta)$, $\tau(k_m, \zeta)$, $D(k_m, \zeta)$ and $B(k_m, \zeta)$ can readily be determined. By virtue of the inverse Hankel transform (Sneddon 1970), the corresponding mode shapes can finally be derived as

$$\bar{u}_r(\xi, \zeta) = 2u(k_m, \zeta)\frac{J_1(k_m\xi)}{[J_1(k_m)]^2}, \quad \bar{u}_z(\xi, \zeta) = 2w(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_1(k_m)]^2}$$

$$\bar{\sigma}_{zz}(\xi, \zeta) = 2\sigma(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_1(k_m)]^2}, \quad \bar{\sigma}_{zr}(\xi, \zeta) = 2\tau(k_m, \zeta)\frac{J_1(k_m\xi)}{[J_1(k_m)]^2}$$

$$\bar{\Phi}(\xi, \zeta) = 2\Phi(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_1(k_m)]^2}, \quad \bar{D}_z(\xi, \zeta) = 2D(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_1(k_m)]^2}$$

$$\bar{\Psi}(\xi, \zeta) = 2\Psi(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_1(k_m)]^2}, \quad \bar{B}_z(\xi, \zeta) = 2B(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_1(k_m)]^2} \tag{37}$$

Substituting Eq. (37) into Eq. (10), the corresponding variables in the **X** can be determined.

For the GRSS conditions, the parameter k must satisfy Eq. (23b) and a series of roots $k_m(m = 1, 2, ...)$ can be obtained. The non-dimensional frequency $\Omega_{mn}(n = 1, 2, ...)$ and the corresponding mode shapes and the components of the vector **X** in Eq. (10) can be determined In a similar manner, the mode shapes for the above-mentioned GESS conditions is given by

$$\bar{u}_r(\xi, \zeta) = 2u(k_m, \zeta)\frac{J_1(k_m\xi)}{[J_0(k_m)]^2}, \quad \bar{u}_z(\xi, \zeta) = 2w(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_0(k_m)]^2}$$

$$\bar{\sigma}_{zz}(\xi, \zeta) = 2\sigma(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_0(k_m)]^2}, \quad \bar{\sigma}_{zr}(\xi, \zeta) = 2\tau(k_m, \zeta)\frac{J_1(k_m\xi)}{[J_0(k_m)]^2}$$

$$\bar{\Phi}(\xi, \zeta) = 2\Phi(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_0(k_m)]^2}, \quad \bar{D}_z(\xi, \zeta) = 2D(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_0(k_m)]^2}$$

$$\bar{\Psi}(\xi, \zeta) = 2\Psi(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_0(k_m)]^2}, \quad \bar{B}_z(\xi, \zeta) = 2B(k_m, \zeta)\frac{J_0(k_m\xi)}{[J_0(k_m)]^2} \tag{38}$$

For the other cases of the conditions different from the case 1, the corresponding frequency and mode shapes can be derived using the same method. It also should be noted that the presented method can be reduced to for the corresponding cases of the axisymmetric piezoelectric or pure elastic circular plates.

Table 1 Dimensionless frequencies of the magnetoelectroelastic plate for GESS of case 1

t_0	$k_1 = 2.40483$			$k_2 = 5.52008$			$k_3 = 8.65373$		
	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.1	0.01811	0.22914	1.47748	0.08892	0.51993	1.57973	0.19723	0.80227	1.73981
0.2	0.06885	0.45589	1.55052	0.29428	1.00410	1.88564	0.57653	1.45422	2.30724
0.3	0.14406	0.67486	1.66074	0.54169	1.40799	2.25361	0.99204	1.84660	2.85683
0.4	0.23538	0.88549	1.79696	0.80334	1.69864	2.62267	1.42580	2.12940	3.28808
0.5	0.33658	1.08321	1.94950	1.07280	1.90173	2.94806	1.87859	2.44855	3.63150
0.6	0.44370	1.26432	2.11079	1.34956	2.07991	3.22180	2.34891	2.83238	3.95243
0.7	0.55442	1.42517	2.27497	1.63404	2.27017	3.45443	2.83040	3.27401	4.29191
0.8	0.66750	1.56344	2.43749	1.92632	2.48511	3.66470	3.30864	3.75729	4.66959
0.9	0.78232	1.67964	2.59485	2.22567	2.72689	3.86871	3.77626	4.26536	5.09741
1.0	0.89858	1.77769	2.74447	2.53049	2.99362	4.07701	4.22046	4.77772	5.57189

Table 2 Dimensionless frequencies of the magnetoelectroelastic plate for GRSS of case 1

t_0	$k_1 = 3.83171$			$k_2 = 7.01559$			$k_3 = 10.17350$		
	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
0.1	0.04475	0.36395	1.51569	0.13712	0.65696	1.65049	0.25800	0.93233	1.83097
0.2	0.16010	0.71500	1.68452	0.42568	1.23567	2.08378	0.72023	1.61957	2.57472
0.3	0.31370	1.04101	1.91496	0.75369	1.65270	2.55644	1.21827	1.99596	3.09900
0.4	0.48470	1.32691	2.17202	1.09568	1.91685	2.97292	1.74188	2.34699	3.53428
0.5	0.66371	1.55920	2.43215	1.44953	2.14500	3.30805	2.29081	2.78224	3.91289
0.6	0.84726	1.73643	2.67973	1.81599	2.40147	3.58744	2.85641	3.29910	4.31115
0.7	1.03437	1.87588	2.90533	2.19439	2.70065	3.84752	3.41778	3.87246	4.76337
0.8	1.22495	2.00023	3.10554	2.58188	3.04046	4.11333	3.95914	4.47342	5.28446
0.9	1.41916	2.12506	3.28244	2.97417	3.41391	4.39966	4.46716	5.06980	5.86344
1.0	1.61715	2.25839	3.44160	3.35972	3.81090	4.71304	4.94565	5.63574	6.46737

4. Numerical examples

Consider a three-layered transversely isotropic magnetoelectroelastic circular plate. The first and third layer material are $CoFe_2O_4$ and the core layer is $BaTiO_3$. The material constants show in Li (2000). The thickness ratio of the plate is $h_1 : h_2 : h_3 = 1 : 2 : 1$.

The first three dimensionless frequencies and the corresponding k_1, k_2 and k_3 for boundary conditions GESS and GRSS of cases 1 with different thickness-to-radius ratios t_0 are shown in Tables 1-2. It shows that the dimensionless frequency increase with the thickness-to-radius ratios t_0 both for the boundary conditions GESS and GRSS. Tables 3 shows the lowest dimensionless frequencies of the magnetoelectroelastic plate for the boundary conditions GESS and GRSS of cases 1 as well as those of the piezoelectric and elastic plate. It is shown that the lowest dimensionless frequencies of three mediums increase with the thickness-to-radius ratio t_0. Furthermore, the dimensionless frequency of the magnetoelectroelastic plate is the largest and one of the elastic plates is the smallest for same t_0. It is readily observed that magnetoelectric effects lead to the increase of

Table 3 The lowest dimensionless frequency of three kinds of materials for GESS and GRSS conditions of case 1

t_0	GESS (k_1 = 2.40483)			GRSS (k_2 = 3.83171)		
	Magneto-electroelastic	Piezo-electric	Elastic	Magneto-electroelastic	Piezo-electric	Elastic
0.1	0.01811	0.01804	0.01293	0.04475	0.04459	0.03219
0.2	0.06885	0.06861	0.04984	0.16010	0.15964	0.11849
0.3	0.14406	0.14363	0.10623	0.31370	0.31306	0.23904
0.4	0.23538	0.23481	0.17691	0.48470	0.48400	0.37799
0.5	0.33658	0.33592	0.25740	0.66371	0.66306	0.52602
0.6	0.44370	0.44300	0.34436	0.84726	0.84670	0.67820
0.7	0.55442	0.55373	0.43547	1.03437	1.03390	0.83200
0.8	0.66750	0.66685	0.52917	1.22495	1.22456	0.98617
0.9	0.78232	0.78172	0.62443	1.41916	1.41883	1.14013
1.0	0.89858	0.89805	0.72057	1.61715	1.61686	1.29361

the frequency of free vibration. However, this influence of the frequency is not significant for the material in the numerical example.

5. Conclusions

1. For the axisymmetric free vibration problem, the state space equations of laminated transversely isotropy circular plates are obtained from the three-dimensional governing equations of magnetoelectroelastic medium. The frequencies equations of two different boundary conditions are then derived as well as the corresponding mode shapes.
2. The frequencies increase with the ratios of thickness to radius.
3. The present method can be applied in the corresponding cases of the axisymmetric piezoelectric or pure elastic circular plates.
4. The calculating results show the magnetoelectric effects increase the dimensionless frequencies, i.e., the magnitude of the frequency of the magnetoelectroelastic medium is the biggest and one of the elastic mediums is the smallest for same t_0 and k.

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