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An efficient high-order warping theory for laminated plates

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Abstract. The theory with hierarchical warping functions had been used to analyze composite thin-walled structure, laminated beam and had good results. In the present paper, a series of hierarchical warping functions are developed to analyze the cylindrical bending problems of composite lamina. These warping functions which refine through-the-thickness variation of displacements were composed of basic and corrective functions by taking into account of anisotropic, material discontinues, and transverse shear and normal strain. Then the hierarchical finite element method was used to form a numerical algorithm. The distribution of the displacements, in-plane stresses, transverse shear stresses and transverse normal stress for composite laminate were analyzed with the present model. The results show that the present model has precise mechanical response compared with the first deformation transverse theory and the corrective order affects the accuracy of result.

Keywords: composite; laminate; warping function; model; transverse stress.

1. Introduction

Layered composite structural elements have attracted considerable attention in primary structural component since they have excellent strength to weight and stiffness to weight ratio (Rolfe *et al.* 2003). But they suffer from relatively poor strength and stiffness for transverse shear. Engineers and researchers have paid much attention to the study on the structural mechanical behaviors at all times (Noor and Burton 1989). Many theories for laminated Composite Structures can be found in the composite mechanics literature. They are usually classified into four groups:

1) three-dimensional analytical solution (e.g., Pagano 1970), which is very difficult to get the solution for complex geometry of the structure, complex boundary condition and complex loads. 3D analytical solution is improper for the mechanics analysis of general laminate, but it is very important to offer a check for the other theories

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2) classical theories based on the Kirchhoff kinematic hypothesis, (e.g., Reissner and Stavsky 1961, Wang and Chou 1972, Reddy 1997), which are inaccurate for a moderately thick laminated plate and for highly anisotropic composites

3) first-order shear deformation theories in which shear strains are assumed to be constant in the thickness direction, and shear correction factors have to be incorporated to adjust the transverse shear stiffness for studying the mechanical response of plates (Whitney and Pagano 1970, Whitney 1972, Noor and Burton 1989, Reddy 1997).

4) high-order theories with refined through-the-thickness variation of displacements (e.g., Toledano and Murakami 1987, Valisetty and Rehfiedl 1987). Two different approaches have been basically explored for the higher-order theories: one based on an equivalent single-layer model (e.g., Sciuva 1987, Matsunaga 2002) and the other on a layerwise model (e.g., Mau 1973, Liu 2003, Kapuria and Archary 2004).

The high-order warping theory is capable of obtaining the accurate distribution of the displacements, by continuous correction of the warping displacements due to the stress of the structure (Zhu 1957). The theory has been used to analyze composite thin-walled structure, laminated beam and had good results (Deng and Zhu 1999, Zhu *et al.* 2001, Huang 2002). In this paper, a series of hierarchical warping functions were developed to analyze the cylindrical bending problems of composite lamina, and the effects on the displacements, in-plane stresses, and transverse stresses by the warping functions were studied. With the same accuracy, the present model is simple while the degree of freedom lower than the other high-order theories.

2. Displacement expressions and warping functions

Consider a composite plate (Fig. 1) of thickness h, length L, made of N perfectly bonded orthotropic layers with longitudinal axis x, subjected to load $q(x)$ with no variation along the width. The mid-plane of the plate is chosen as xy-plane. The planes $z = z_0 = -h/2$ and $z = z_N = h/2$ are the bottom and the top surfaces of the plate. Z-Coordinate of the bottom surface of the kth layer is denoted as z_{k-1}. For cylindrical bending problems, the displacements u, v and w of the plate are independent of y.

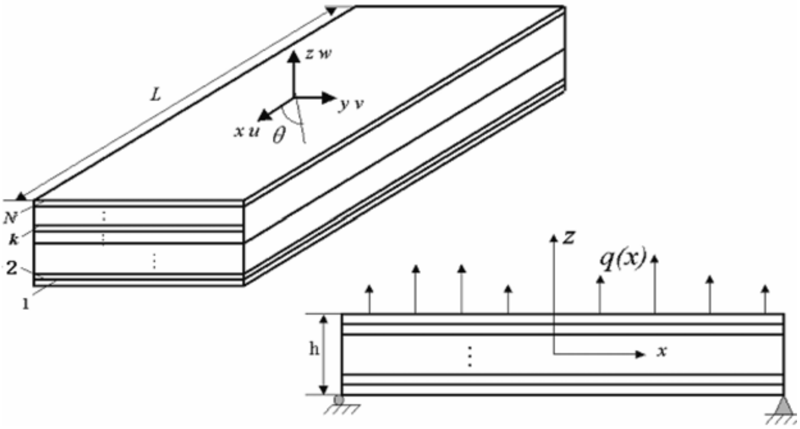


Fig. 1 Coordinate system

2.1 Assumed displacement

The displacement vector of laminate is expressed as:

$$u = [U(x,z)\ \ V(x,z)\ \ W(x,z)]^T = \left[\sum_{i=1}^{M_1} u_i(x)f_i(z)\ \ \sum_{i=1}^{M} v_i(x)g_i(z)\ \ \sum_{i=1}^{N} w_i(x)p_i(z)\right]^T \tag{1}$$

where $f_i(z)$, $g_i(z)$ and $p_i(z)$ in (1) are the hierarchical warping functions of the displacement u, v and w in terms of the thickness coordinate z, respectively. M_1, M and N refer the number of the warping functions.

Then, the equations of equilibrium without volume force are written as:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{2a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = 0 \tag{2b}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{2c}$$

The vector of the strain-displacement relations is given as:

$$\begin{aligned}\varepsilon &= \{\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\}^T \\ &= \left\{\frac{\partial U}{\partial x}, 0, \frac{\partial W}{\partial z}, \frac{\partial V}{\partial z}, \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}, \frac{\partial V}{\partial x}\right\}^T\end{aligned} \tag{3}$$

and the stress-strain relation of the kth layer is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}_k = \begin{Bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{21} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{31} & C_{32} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{54} & C_{55} & 0 \\ C_{61} & C_{62} & C_{63} & 0 & 0 & C_{66} \end{Bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{4}$$

2.2 The hierarchical warping functions

From the above equations, two series of warping functions of displacement u can be derived. One corresponds to the pair of stresses (σ_x, τ_{xz}), and the other relates to the other pair of stresses (τ_{xy}, τ_{yz}) (Huang 2002).

2.2.1 The first series of hierarchal warping functions of displacement U

The first series of hierarchal warping functions relates to the stresses $\sigma_x - \tau_{xz}$. We can set the basic distribution along the thickness direction of displacement U as $f_{11}(z) = 1 + a_1 z$ (where a_1 is pending coefficients). Then, from Eqs. (3) and (4), the distribution of σ_{x1} for the kth ply is given by $f_{11}(z)$ as:

$$\sigma_{x1}^{k} \propto C_{11}^{k} f_{11}(z) \tag{5}$$

where: "$\propto$" stands for direct proportion, σ_{x1}^{k} is the normal stress of the kth ply caused by the basic warping function of $f_{11}(z)$.

Eq. (5) refers that σ_{x1} may be discontinuous among laminates and its distribution is disconnected. The composition of σ_{x1} in the thickness direction is balanced:

$$\sum F_x = 0, \quad \sum_{1}^{N} \int_{z_k - t_k/2}^{z_k + t_k/2} \sigma_{x1}^{k} dz = 0 \tag{6}$$

where z_k is the middle plane coordinate and t_k is the k-th ply thickness, respectively.

Consequently, the coefficient a_1 in $f_{11}(z)$ can be obtained by Eq. (6), and the moment integration of the σ_{x1} in each ply through the thickness equals the extra moment:

$$\sum_{1}^{N} \int_{z_k - t_k/2}^{z_k + t_k/2} \sigma_{x1}^{k} z dz = M_y \tag{7}$$

By using Eq. (2a), the τ_{xz1} for the kth ply can be written as:

$$\tau_{xz1}^{k} = -\int_{-h/2}^{z_{k+1}} \frac{\partial \sigma_{x1}^{k}}{\partial x} dz \propto (F_{11}^{k}(z) + L_{11}^{k}) \tag{8}$$

where $F_{11}^{k}(z)$ are the integral source function of $C_{11}^{k} f_{11}(z)$, and L_{11}^{k} are pending constants, which can be obtained by the force boundary conditions and the inter-laminar continuity of the transverse stress.

Warping deformation will arise in the laminates under the transverse shear stress τ_{xz1}. Based on the concepts of the separating variable, the effects on the warping deformation in the x direction by the displacement V and W in y and z direction respectively are not considered. Then, from Eq. (4), the transverse shear strain yielded by transverse stress $\tau_{xz(i-1)}^{k}$ can be expressed as:

$$\gamma_{xzi}^{k} = \frac{\tau_{xz(i-1)}^{k}}{C_{55}^{k}} = R_i^{k}(z) \tag{9}$$

Thus, the i-th warping function caused by $\tau_{xz(i-1)}^{k}$ is given as:

$$f_{1i}^{k} = a_i + b_i z + \int \gamma_{xzi}^{k} dz \tag{10}$$

where $\int \gamma_{xzi}^{k} dz \propto H_i^{k}(z) + I_i^{k}$, $H_i^{k}(z)$ are the integral source function of $R_i^{k}(z)$, and I_i^{k} are pending constants, respectively.

We can set $I_i^{1} = 0$, the others are obtained by the continue conditions among laminated. The a_i and b_i in Eq. (10) can be determined by the following balance conditions:

$$\sum_{1}^{N}\int_{z_k - t_k/2}^{z_k + t_k/2} f_{1i}^{k} dz = 0 \tag{11a}$$

$$\sum_{1}^{N}\int_{z_k - t_k/2}^{z_k + t_k/2} f_{1i}^{k} z dz = 0 \tag{11b}$$

Here, Eq. (11) means that the corrected stresses σ_x^k, τ_{xy}^k due to $f_{1i}^k(z)$ can form a self-equilibrating system of force. The corrected normal in-plane stresses can be written as:

$$\sigma_{(i+1)x}^{k} \propto C_{11}^{k} f_{1i}^{k}(z) \tag{12}$$

By assuming the basic warping function of the first series, we obtain the normal in-plane stress σ_{x1} by using displacement-strain and stress-strain relations. Then the successive warping shape functions can be derived by the above recurrence formulations step by step. These are the first series of warping functions of displacement U.

2.2.2 The second series of warping functions of displacement U

The second series of warping functions are correlative with the pair of stresses (τ_{xy}, τ_{yz}). The basic warping function, $f_{21}(z) = 1 + a_2 z$, also be set, where a_2 is a pending constant.

Then:

$$\tau_{xy1}^{k} \propto C_{16}^{k} f_{21}(z) \tag{13}$$

Set the shear stresses τ_{xy1} are self-equilibrium under the basic function $f_{21}(z)$, namely, $\sum_{1}^{N}\int_{z_k - t_k/2}^{z_k + t_k/2} \tau_{xy1}^{k} dz$ = 0, and the pending constant a_2 can be obtained.

By using Eq. (2), the transverse shear stresses τ_{yz1} distribution in thickness direction are:

$$\tau_{yz1}^{k} = -\int \frac{\partial \tau_{xy1}^{k}}{\partial x} dz \propto (F_{21}^{k}(z) + L_{21}^{k}) \tag{14}$$

where $F_{21}^k(z)$ are the integral source functions of $C_{16}^k f_{21}(z)$, and L_{21}^k are pending constants, which can be obtained by inter-laminar continuity of the τ_{yz1} and the boundary conditions.

The warping deformation in the section can be originated under the transverse shear stress τ_{yz}. Assuming the f_{2i}^{k*} is the warping displacement due to τ_{yzi}^k, then we can get through the stress-strain and displacement-strain relations:

$$\gamma_{xzi}^{k} = \frac{\tau_{yz(i-1)}^{k}}{C_{45}^{k}} = \frac{\partial f_{2i}^{k*}}{\partial z} \propto R_{2(i-1)}^{k}(z) \tag{15}$$

So, the *i*-th transverse warping function of the second series of the section is:

$$f_{2i}^{k} = a_{2i} + b_{2i} z + \int \frac{\tau_{yz(i-1)}^{k}}{G_{45}^{k}} dz \tag{16}$$

where $\int \frac{\tau_{yz(i-1)}^{k}}{G_{45}^{k}} dz \propto (H_{2i}^{k}(z) + I_{2i}^{k})$, H_{2i}^k are the integral source functions of $R_{2(i-1)}^k(z)$, and I_{2i}^k are pending constants. Set the $I_{2i}^1 = 0$, the other I_{2i}^k can be obtained by the continuity conditions in the layer.

Assume the distribution τ_{xyi}^{k} due to the i-th corrective warping function are in direct proportion to f_{2i}^{k}, they are defined as: $\tau_{xyi}^{k} \propto C_{16}^{k} f_{2i}^{k}(z)$. Set the section force and moment arose by the corrective shear stress τ_{xyi} are a self-equilibrating system, namely:

$$\sum_{1}^{N} \int_{z_k - t_k/2}^{z_k + t_k/2} \tau_{xyi}^{k} dz = 0 \tag{17a}$$

$$\sum_{1}^{N} \int_{z_k - t_k/2}^{z_k + t_k/2} \tau_{xyi}^{k} z dz = 0 \tag{17b}$$

The constants a_{2i} and b_{2i} can be determined by Eq. (17). The physical meaning of f_{2i}^{k} is the attached part of the displacement u by the transverse stress τ_{xyi}^{k}. According to the obtained warping functions and the distribution of the τ_{xyi}^{k} in the z direction, the i-th corrective through-the-thickness transverse shear stress τ_{yzi}^{k} can be derived using the balance equations, then the (i+1)-th warping function and corrective $\tau_{xy(i+1)}^{k}$ can be obtained. Step by step, the second series of warping shape functions in z direction can be derived.

For symmetric laminates with multiple specially orthotropic layer, because the $C_{16}^{k} = 0$, the second series warping functions do not exist.

The hierarchal warping functions $g_i(z)$ for displacement V can be derived following the same above steps.

2.2.3 The series of warping functions of displacement W

According to balance equations, the displacement w has a series of warping functions which relate to the pair of stresses (σ_z, τ_{xz}). Same as the above derived, the warping functions can be obtained by using the following recurrence formulations step by step.

$$p_1(z) = 1 \tag{18a}$$

$$\tau_{xz1}^{k} \propto C_{55}^{k} p_1(z) \tag{18b}$$

$$\sigma_{z1}^{k} = -\int \frac{\partial \tau_{xz1}^{k}}{\partial x} dz \propto (F_{z1}^{k}(z) + C_{z1}^{k}) \tag{18c}$$

$$p_i^{k}(z) = a_i + \int \frac{\sigma_{z(i-1)}^{k}}{C_{55}} dz \tag{18d}$$

$$\sigma_{zi}^{k} \propto \int C_{33}^{k} p_i^{k}(z) dz = F_{zi}^{k}(z) + C_{zi}^{k} \tag{18e}$$

where the pending constants in Eq. (18), such as C_{zi}^{k}, can be determined by the inter-laminar continuity and boundary conditions, and the a_i are obtained by the balance equation:

$$\sum_{1}^{N} \int_{z_k - t_k/2}^{z_k + t_k/2} \tau_{xzi}^{k} dz = 0 \quad \text{where} \quad \tau_{xzi}^{k} \propto C_{55}^{k} p_i^{k}(z) \tag{19}$$

3. The hierarchical finite element formulations

C_0 hierarchical finite element method is recommended in the present paper. All of the argument functions $(u_{1i}, u_{2i}, v_{1i}, v_{2i}, w)$ are expanded into the same series but each has its own generalized nodal parameters, for example:

$$u_{1i} = [a_{1i1} \ \ a_{1i2} \ \ \dots \ \ a_{1ij}][h_1 \ \ h_2 \ \ \dots \ \ h_j]^T = A_{1i}H^T \tag{20}$$

where: J is the maximum order of the hierarchical shape function, A_{1i} is the vector of the nodal parameters, H^T is the matrix of hierarchical shape functions. The expressions of the C_0 shape function are

$$h_1(\xi) = \frac{1}{2}(1-\xi) \quad h_2(\xi) = \frac{1}{2}(1+\xi) \quad \xi = \frac{2x-L}{L}$$

$$h_k(\xi) = \sum_{l=0}^{(k-1)/2} \frac{(-1)^l(2k-2l-5)!!}{2^l l!(k-2l-1)!}\xi^{k-2l-1} \quad (k = 3, 4, \dots, j)$$

For more details about the shape functions, namely H^T, see the references (Zhu 1998, Zhu and Deng 2001).

Substituted Eqs. (11) and (20) into Eq. (3), the strain vector of the laminate can be written as:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{M} \frac{du_i}{dx} f_i \\ 0 \\ \sum_{i=1}^{M} \frac{dp_i}{dz} w_i \\ \sum_{i=1}^{M} \frac{dg_i}{dz} v_i \\ \sum_{i=1}^{M} \frac{df_i}{dz} u_i + \sum_{i=1}^{M} \frac{dw_i}{dx} p_i \\ \sum_{i=1}^{M} \frac{dv_i}{dx} g_i \end{bmatrix} = \begin{bmatrix} U_{x0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W_{0z} \\ 0 & V_{0z} & 0 \\ U_{0z} & 0 & W_{x0} \\ 0 & V_{x0} & 0 \end{bmatrix} \delta^T = B\delta^T \tag{21}$$

where:

$$U_{x0} = \left[\frac{dH}{dx}f_{11}, \frac{dH}{dx}f_{12}, \dots, \frac{dH}{dx}f_{21}, \frac{dH}{dx}f_{22}, \dots\right] \tag{22a}$$

$$U_{0z} = \left[H\frac{df_{11}}{dz}, H\frac{df_{12}}{dz}, \dots, H\frac{df_{21}}{dz}, H\frac{df_{22}}{dz}, \dots\right] \tag{22b}$$

$$V_{x0} = \left[\frac{dH}{dx}g_{11}, \frac{dH}{dx}g_{12}, \ldots, \frac{dH}{dx}g_{21}, \frac{dH}{dx}g_{22}, \ldots\right] \tag{22c}$$

$$V_{0z} = \left[H\frac{dg_{11}}{dz}, H\frac{dg_{12}}{dz}, \ldots, H\frac{dg_{21}}{dz}, H\frac{dg_{22}}{dz}, \ldots\right] \tag{22d}$$

$$W_{x0} = \left[\frac{dH}{dx}p_1, \frac{dH}{dx}p_1, \ldots, \frac{dH}{dx}p_N\right] \tag{22e}$$

$$W_{0z} = \left[H\frac{dp_1}{dz}, H\frac{dp_2}{dz}, \ldots, H\frac{dp_N}{dz}\right] \tag{22f}$$

and the extend-displacement vector δ of the element is defined as:

$$\delta = [A, B, C] = [A_1, A_2, B_1, B_2, C] \tag{23}$$

where: $A = [A_1, A_2],\ A_1 = [A_{11}, A_{12}, \ldots, A_{1M_1}],\ A_2 = [A_{21}, A_{22}, \ldots, A_{2M_2}]$

$B = [B_1, B_2],\ B_1 = [B_{11}, B_{12}, \ldots, B_{1M_1}],\ B_2 = [B_{21}, B_{22}, \ldots, B_{2M_2}]$

$C = [C_{11}, C_{12}, \ldots, C_{1N_1}]$ $\qquad M_1 + M_2 = M$

By using the above symbols, the functional of potential energy of the bending laminate can be written as:

$$\Pi = \iint_A [\sigma]^T[\varepsilon]dxdz - \int_L qW|_{top}dx \tag{24}$$

where q is the distribution press which is on the upper surface. After variation, we can get:

$$[K]\delta = [P] \tag{25}$$

where: $[K] = \iint_A [B]^T[D][B]dxdz$ and

$$[B]^T[D][B] = \begin{bmatrix} C_{11}U_{x0}^T & C_{12}U_{x0}^T & C_{13}U_{x0}^T & C_{45}U_{0z}^T & C_{55}U_{0z}^T & C_{16}U_{x0}^T \\ C_{16}V_{x0}^T & C_{26}V_{x0}^T & C_{36}V_{x0}^T & C_{44}V_{0z}^T & C_{45}V_{0z}^T & C_{66}V_{x0}^T \\ C_{13}W_{0z}^T & C_{23}W_{0z}^T & C_{33}W_{0z}^T & C_{45}W_{x0}^T & C_{55}W_{x0}^T & C_{36}W_{0z}^T \end{bmatrix}^{(k)} \begin{bmatrix} U_{x0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W_{0z} \\ 0 & V_{0z} & 0 \\ U_{0z} & 0 & W_{x0} \\ 0 & V_{x0} & 0 \end{bmatrix}^{(k)}$$

The force vector can be written as: $[P] = \int_L q[0\ \ 0\ \ W_{00}]_{top}^T dx$.

where $W_{00} = [Fw_0, Fw_1, \ldots, Fw_N]$

From Eqs. (24) and (25), the displacement field of laminate can be solved, and the stress distribution can be get by using geomertry and stress-strain relations.

4. Calculation of transverse stresses

The present formulation can be extended to include an accurate through-the-thickness description of both transverse shear and normal stresses (Brank 2003) but it should take a lot of calculation time. In this paper, for degrading the order of the calculated model, the transverse stresses are obtained by the post-process method using balance equations:

$$\tau_z^{(k)} = \begin{bmatrix} \tau_{xz}^{(k)} \\ \tau_{yz}^{(k)} \end{bmatrix} = -\int_{\xi=0}^{\xi=z} \begin{bmatrix} \sigma_{x,x}^{(k)} + \tau_{xy,y}^{(k)} \\ \sigma_{y,y}^{(k)} + \tau_{xy,x}^{(k)} \end{bmatrix} d\xi \tag{26}$$

and the transverse normal stress can be expressed as:

$$\sigma_z^k = -\int_{\xi=-h/2}^{z=z_k+t_k/2} (\tau_{xz,x}^k + \tau_{yz,y}^k) d\xi \tag{27}$$

5. Discussions of numerical results

The calculation model is shown in the Fig. 1, the material properties are:

$$E_1 = 25 \times 10^6,\ E_2 = 10^6,\ G_{12} = 0.5 \times 10^6,\ G_{23} = 0.2 \times 10^6,\ \nu_{12} = \nu_{23} = 0.25$$

the other parameters: length of the laminate $L = 4$, the ratio of length to thickness $s = L/h = 4$, Laminate have simple supports on the two bottom.

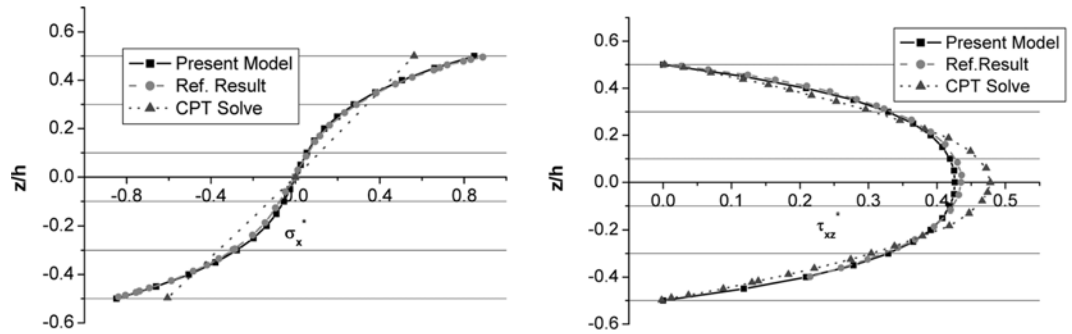
The distributed loads $q(x) = \cos(\pi x/L)$ are applied on the upper surface. Three style kind of laminates are analyzed:

1) A single-layer plate, ply angle $\theta = 15^\circ$
2) A simply supported, three-layered plate, cross-ply sequence is [30°/–30°/30°], the thicknesses of each ply are (h/4, h/2, h/4).
3) Four-layered, crossply [30°/–30°/30°/–30°], equal thickness of each ply and simply supported.

The maximum order of hierarchal shape functions J is 5. The different names of the model are defined by the number of the warping functions of U and V in Z direction. One is called the first-order shear deformation model which only consider one warping function of two series, the other is named as one-order correction, namely, the present model in the following figures which have two functions for each series, and it is called as one order bending correction model which select two functions of the first series and one function of the second series. The displacement W has one warping function in the numerical simulations. The non-dimensional quantities shown in the figures are defined as:

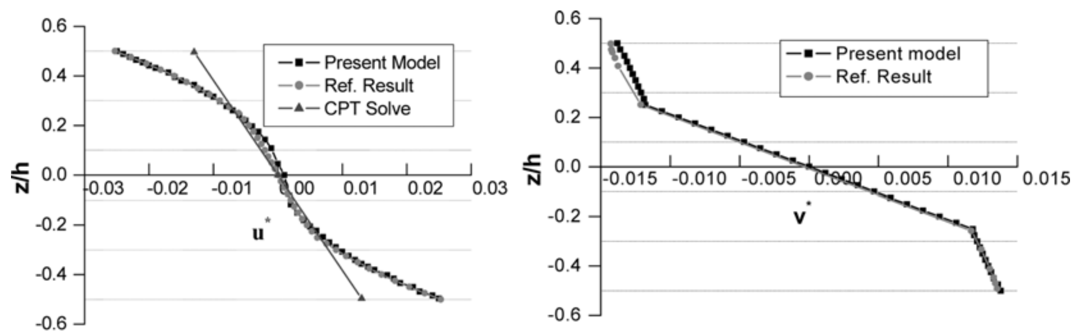
$$(\sigma_x^*, \tau_{xy}^*) = \frac{(\sigma_x, \tau_{xy})}{s^2},\quad \tau_{xz}^* = \frac{\tau_{xz}}{s},\quad (u^*, v^*) = \frac{(u, v)}{(hs^3)},\quad \text{where } s = L/h$$

The figures show though-the-thickness distribution of displacement and stress. The results compared with the accurate solutions (Pagano 1970) (namely, Ref. Result in figures) and the classical plate theory (CPT) solution curves are showed in Figs. 2-5. Figs. 2-5 show that the



(a) σ_x^* (at $x = 0$) (b) τ_{xz}^* ($x = L/2$)

Fig. 2 The σ_x^* and τ_{xz}^* distribution of the single-layered



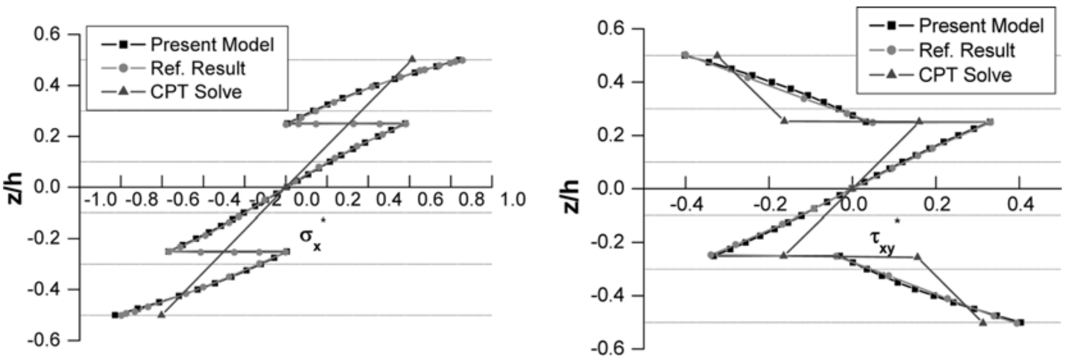
(a) u^* (b) v^*

Fig. 3 The in-plane displacement distribution of the three-layered at $x = L/2$

Table 1 The maximum stress, displacement results of two models

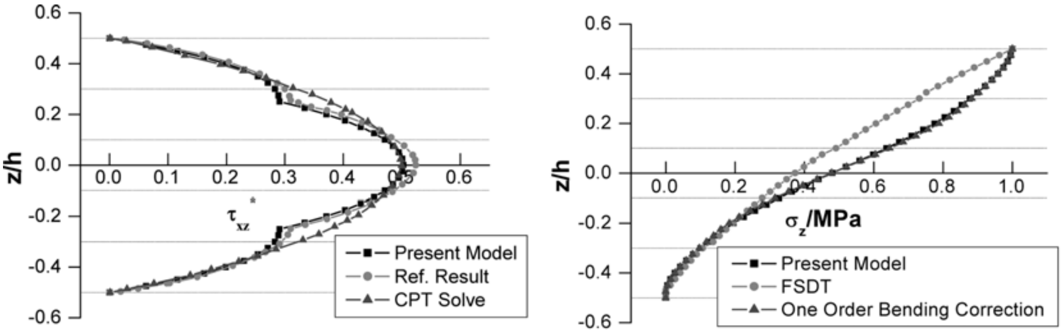
Results		σ_x	τ_{xy}	τ_{xz}	u	v
$N = 1$ ($\theta = 15^\circ$)	Present model	0.848		0.426		
	Accurate Solution (Pagano 1970)	0.889		0.435		
	Difference	4.6%		2.1%		
$N = 3$ ($30^\circ/-30^\circ/30^\circ$)	Present model	0.84	0.402	0.503	0.253	0.138
	Accurate Solution (Pagano 1970)	0.857	0.394	0.524	0.25	0.14
	Difference	2.00%	2.00%	4.00%	1.20%	1.40%

presented model with hierarchical warping functions are certainly an improvement over CPT and are in good agreement with analytical solutions. It means that the though-the-thickness distribution of the displacement, in-plane stresses, transverse shear stress of the present model are close to the accurate solution. The maximum difference between present model and accurate solution is shown in Table 1.



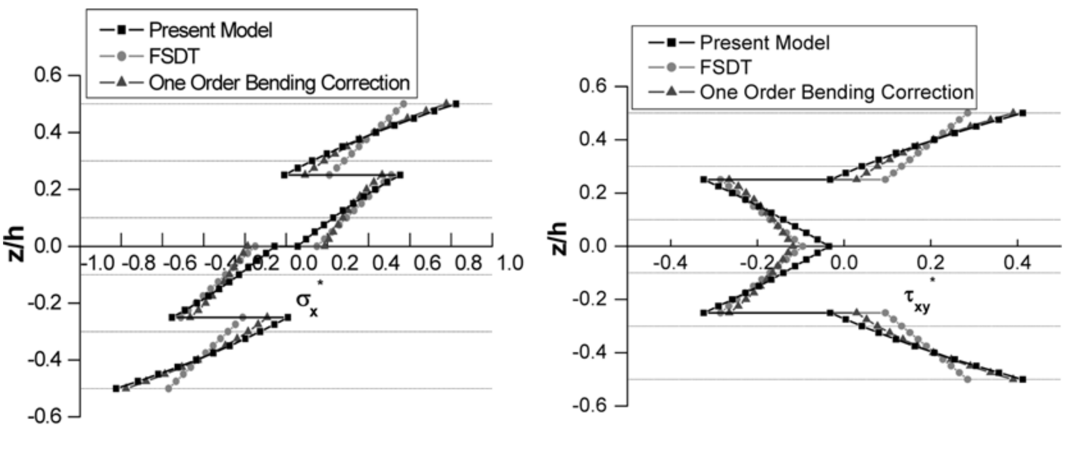
(a) σ_x^* (b) τ_{xy}^*

Fig. 4 The in-plane stresses distribution of the three-layered at x = 0



(a) τ_{xz} (b) σ_z

Fig. 5 Transverse stresses distribution of the three-layered at $x = L/2$



(a) σ_x^* (b) τ_{xy}^*

Fig. 6 The stresses distribution of the four-layered at x = 0

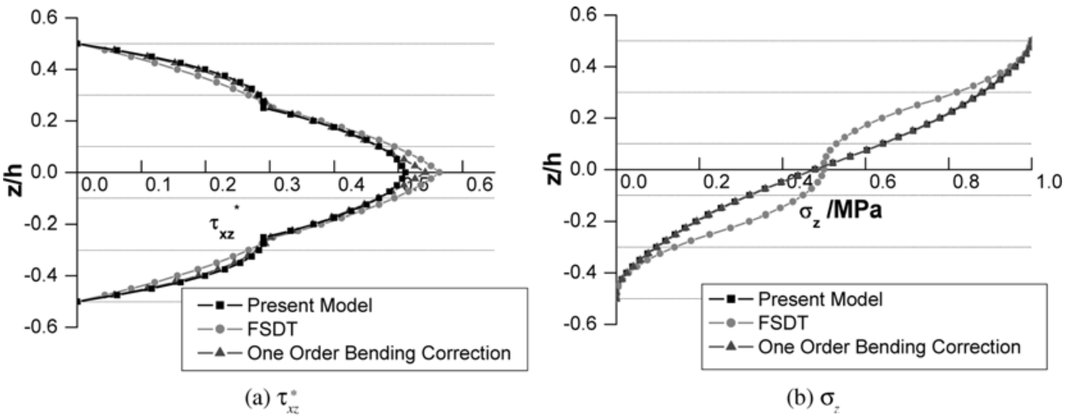


Fig. 7 Transverse stresses distribution of four-layered at $x = L/2$

The Fig. 6 and Fig. 7 are the calculation results of the four-layered laminate. The results illustrate that the present model (with two warping functions of each series) has more better accuracy than the first shear deformation model (with one warping function of each series), moreover, the one bending correction has better distribution than the first shear deformation model.

Fig. 5(b) and Fig. 7 show that the present model and the one order bending model have a small difference in calculating transverse stresses, it means that the second series warping function have some effects on the calculate results. However, the results illustrate that there is a greater difference between the present model, which have select primary two order for every series corrected hieriarcial functions, and the first shear deformation model results, especially for the transverse normal stress results. It means that the correct order has great effect on the transverse stresses calculation.

6. Conclusions

The present study has dealed with the cylindrical bending problem of a composite laminate. A series of warping functions are derived for correcting the displacement distribution along the thickness direction. The warping functions can be obtained by two steps. First, a basic through-the-thickness displacement field was assumed in such a way that the corresponding composition force and moment of stress as well are equal to exterior load. Second, by using constitutive relation, the relation among equilibrium equation, displacement kinematics, and distribution displacement function is corrected successively while the corresponding stress is self-equilibrium. Base on the numerical studies conducted:

1. The present model is of good convergence and accuracy. One order correction with two functions in each series is well consistent with analytical solutions.
2. The present model exhibits the precise mechanical response compared with the first deformation transverse theory which has one function in two series, especially in the normal transverse stress. Compared with the present model, the results calculated using the one order bending model have some differences.

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