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A function space approach to study rank deficiency and spurious modes in finite elements

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Abstract. Finite elements based on isoparametric formulation are known to suffer spurious stiffness properties and corresponding stress oscillations, even when care is taken to ensure that completeness and continuity requirements are enforced. This occurs frequently when the physics of the problem requires multiple strain components to be defined. This kind of error, commonly known as *locking*, can be circumvented by using reduced integration techniques to evaluate the element stiffness matrices instead of the full integration that is mathematically prescribed. However, the reduced integration technique itself can have a further drawback - rank deficiency, which physically implies that spurious energy modes (e.g., hourglass modes) are introduced because of reduced integration. Such instability in an existing stiffness matrix is generally detected by means of an eigenvalue test. In this paper we show that a knowledge of the dimension of the solution space spanned by the column vectors of the strain-displacement matrix can be used to identify the instabilities arising in an element due to reduced/selective integration techniques *a priori*, without having to complete the element stiffness matrix formulation and then test for zero eigenvalues.

Key words: rank deficiency; zero energy mode; eigen value; function space; basis vectors; locking; reduced integration.

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1. Introduction

Reduced integration is a popularly adopted technique of using a lower order integration for evaluating the element stiffness matrix than is otherwise dictated by considerations of evaluating all the energy terms exactly. It was first devised to eliminate parasitic shear in plane stress elements (Doherty *et al.* 1969) and shear locking in plate bending (Zienkiewicz *et al.* 1971). Reduced integration is appealing for several reasons. In addition to reducing computational cost (fewer integration points are needed per element), it can eliminate locking (shear, membrane, volumetric), and further, it generally softens the element so that the predicted stress is more accurate. For example, full integration (quadrature) of the stiffness of a 4-node quadrilateral element requires four integration points while reduced integration only requires the evaluation of the matrices at one point - the element centroid. However reduced integration also makes the element too soft in the sense that spurious mechanisms other than rigid body modes are introduced. These modes or instabilities which were originally noticed in two dimensions in 1960s, are historically called hourglass or keystone modes, because of their shape. For other elements the modes are commonly referred to as zero energy modes. The *zero energy mode* refers to a nodal displacement vector that is not a physical rigid body motion but nevertheless produces zero strain energy modes as computational artefacts. Such spurious modes never arise if the element is integrated exactly.

2. Rank sufficiency and rank deficiency

Let the unconstrained stiffness matrix K_e of a finite element have the order N_f where N_f is the number of degrees of freedom per element. Let N_r^p be the number of independent rigid body modes that the element can physically have. Then the expected proper rank of the element stiffness matrix is given by,

$$\textit{Proper rank}\ (K_e) = N_f - N_r^p \tag{1}$$

Such a formulation, adhering to the canonical principles, is said to be *rank sufficient*. Due to various liberties one may take with the formulation (e.g., use of reduced integration, use of substitute functions), the actual rank obtained for the element stiffness matrix may be less than this expected proper rank. The finite element matrix is then said to be *rank deficient*. *Rank deficiency* implies the presence of spurious zero energy modes in addition to the physical rigid body modes. Rank deficiency can be checked by evaluating

$$\textit{rank deficiency}\ (K_e) = \textit{Proper rank}\ (K_e) - \textit{Actual rank}\ (K_e) \tag{2}$$

where the *Actual rank* (K_e) of the element stiffness matrix now varies, e.g., depending upon the quadrature rule used to evaluate it. When the stiffness matrix is evaluated by full integration, i.e., when we use a sufficient number of integration points, it always has the proper rank implying zero rank deficiency. Although rank-deficient elements may sometimes appear to work, they should not be used without an appropriate correction. Bathe (2003) has shown with examples that using a lower order integration for evaluation of element stiffness can introduce large errors. Therefore, to perform reliably, an element must have the proper rank and a sound FEM program should give a warning when a zero energy mode is detected.

Rank deficiency in an existing K_e can be detected by eigen-value analysis (Cook and Plesha 1989, Bathe 2003). If K_e has proper rank, then the number of zero eigen values is equal to the number of *physically admissible* rigid body modes. However, the eigenvalue test can be applied only after evaluating the stiffness matrix by various quadrature schemes. Belytschko *et al.* (2000) have discussed an explicit method to find the rank of the stiffness matrix which is evaluated by numerical quadrature. The rank-sufficiency of the QUAD4 element has been examined for various quadrature schemes.

In this paper we use the function space approach (Mukherjee and Prathap 2002) to detect the presence of zero-energy modes *a priori*, i.e., even before stiffness matrix evaluation. Examples, given below in section 4, suggest that this method can serve as a good practice to detect rank deficiency in the element stiffness matrix, arising due to various quadrature schemes.

3. Rank deficiency from function space approach

From the orthogonality condition following the generalized form of the Hu Washizu theorem, it has been suggested by (Prathap 1996) that the determination of the approximate solutions of differential equations of conservative systems using the finite element method is actually tantamount to generating best-fit solutions to the analytical solution. In terms of linear algebra, these best-fits can also be looked upon as orthogonal projections upon function subspaces. This view is also inherent in the seminal work of (Strang and Fix 1966) on the finite element method. Recently, using the function space of linear algebra, (Mukherjee and Prathap 2001, 2002, 2003) and (Sangeeta *et al.* 2003) have shown that even at the element level, the finite element strain vector can be viewed as an orthogonal projection of the analytical strain vector, provided no variational crimes have been incorporated in the formulation. In (Mukherjee and Prathap 2001, 2002, 2003) it has been shown that the dimension of the function subspace $\boldsymbol{B}$ for the orthogonal projection of the strain vector at the element level agreed with the proper rank for a simple one-dimensional bar, Euler beam and Timoshenko beam elements. The reduced integration technique to eliminate shear locking has been interpreted as a replacement of the original field inconsistent strain-displacement matrix $[B]$ (and its subspace) by a corresponding field consistent strain-displacement matrix $[B^*]$ (and its subspace) *through removal of the appropriate Legendre polynomials*. Though the reduced integration preserved proper rank and thus did not introduce any artificial mechanism for simple Timoshenko beam elements, it cannot be generalized that reduced integration preserved proper rank for arbitrary elements. The present section shows, using the function space approach, how reduced integration in two-dimensional elements can fail to preserve proper rank by introducing additional artificial mechanisms, or rigid body motions that do not physically exist.

The finite element strain vector $\{\varepsilon^{he}\}$ is given by

$$\{\varepsilon^{he}\} = [B]\{\delta^e\} \tag{3}$$

where $[B]$ is the strain-displacement matrix and $\{\delta^e\}$ is the element nodal displacement vector. This suggests that the finite element strain is a linear combination of the column vectors of the strain-displacement $[B]$ matrix (Strang and Fix 1966). In general, for a strain vector involving r components (i.e., r rows) the $\boldsymbol{B}$ space (arising out of the columns of the $[B]$ matrix of r rows) is a subspace of the (rxn)-dimensional polynomial space $P_n^r(\xi)$ of ordered r-tuples of polynomials in ξ, upto degree $n-1$, bounded within the closed domain $(-1, 1)$. The space $P_n^r(\xi)$ is represented by

$$P_n^r(\xi) = \left\{ \{p\}:\{p\} = \sum_{i=1}^{n} \{\alpha_i\}\xi^{i-1}, \quad -1 \le \xi \le 1, \quad \{\alpha_i\} \in R^r \right\} \tag{4}$$

and

$$(B \subset P_n^r(\xi)) \tag{5}$$

Here R^r is the r-dimensional space of real numbers. The column vectors of $[B]$ are not all linearly independent, showing that there are inherent rigid body motions in the element. The dimension m of the subspace $\boldsymbol{B}$ is defined as the number of linearly independent vectors spanning B. The dimension of the $\boldsymbol{B}$ space can also be ascertained through the number of non-zero orthogonal vectors $\{u_i\}$, $(i = 1, 2\ldots, m)$, spanning $\boldsymbol{B}$ determined by the *Gram-Schmidt procedure,* applied to the column vectors of the matrix $[B]$ (Mukherjee and Prathap 2001, 2002). The initial basis vector can be taken as any of the column vectors of the matrix $[B]$,

$$\{u_1\} = \{b_1\} \tag{6}$$

The other $(m - 1)$ non-zero orthogonal basis vectors can be obtained from the general formula for the $(k + 1)^{\text{th}}$ basis vector as

$$\{u_{k+1}\} = \{b_{k+1}\} - \sum_{j=1}^{k} \frac{\langle u_j, b_{k+1}\rangle}{\langle u_j, u_j\rangle}\{u_j\} \tag{7}$$

Here the inner product of two vectors, $\{a\}$ and $\{b\}$, each of r rows, is given by

$$\langle a, b\rangle = \int_e \{a\}^T[D]\{b\}dx \tag{8}$$

For a strain vector of r rows or components, the material stiffness matrix $[D]$ is of size rxr.

The element stiffness matrix $[K_e]$ is obtained as

$$[K_e] = \int_e [B]^T[D][B]dx \tag{9}$$

When $[K_e]$ is evaluated by full integration no additional mechanisms are invoked. Hence the number of rigid body motions computationally available (N_r) is the same as those physically permissible $(N_r^p = N_r)$ (Cook and Plesha 1989) which is also equal to the number of linearly independent vectors in $[B]$. This implies,

$$rank\ (K_e) = dim\ B = N_f - N_r^p \tag{10}$$

From Eq. (1) and Eq. (10) it follows that when full integration is used for stiffness

$$\begin{gathered} Proper\ rank\ (K_e) = rank\ (K_e) = dim\ \boldsymbol{B} \\ \Rightarrow \boldsymbol{rank\ deficiency = 0} \end{gathered} \tag{11}$$

When reduced integration is used to evaluate the stiffness matrix, it may invoke some additional mechanisms. For such cases,

$$\textbf{\textit{rank deficiency}} = \textit{Number of spurious mechanisms from reduced integration} \quad (12)$$
$$= N_r - N_r^p$$

It has been shown in (Mukherjee and Prathap 2001, 2002, 2003) that a reduced integration scheme adopted to eliminate locking situations replaces the original stiffness matrix $[K_e]$ by the modified, element stiffness matrix $[K_e^*]$. This is equivalent to replacing the original matrix $[B]$ in Eq. (3) by a lower order matrix $[B^*]$, thereby eliminating spurious strain oscillations, i.e.,

$$[K_e^*] = \int_e [B^*]^T [D][B^*] dV \quad (13)$$

It follows that,

$$\textit{Reduced integration of } [K_e] \Leftrightarrow \textit{Full integration of } [K_e^*] \Leftrightarrow \textit{Replacing } [B] \textit{ by } [B^*]$$

This suggests that the rank deficiency in the stiffness matrix, arising due to reduced integration, can be detected by finding the dimension of the $\boldsymbol{B}^*$ space. Thus,

$$Rank\ [K_e^*] = dim[B^*] \quad (14)$$

From Eq. (10) and Eq. (14) it follows that when

$$dim\ [B] = dim[B^*] \quad (15)$$

reduced integration does not lead to rank deficiency. The $[B^*]$ matrix is obtained from the $[B]$ matrix by expressing the highest order term in terms of Legendre polynomials and then dropping the highest Legendre polynomial. This is explained with an example in section 4. The dimension of the $\boldsymbol{B}^*$ space is then obtained by using the *Gram-Schmidt algorithm.*

It is thus proposed that the presence of an instability or mechanism that arises in an element due to the use of a lower order integration can be identified from the knowledge of the dimension of $\boldsymbol{B}^*$ space. A loss in dimension of space $\boldsymbol{B}^*$ corresponds to the rank deficiency that equals the number of spurious mechanisms invoked in the stiffness matrix Eq. (13). If the dimension of the space spanned by the column vectors of $[B]$ and $[B^*]$ is the same, the use of lower order integration does not give rise to zero energy modes. In other words, such a special situation does not lead to rank deficiency. The procedure to identify the $\boldsymbol{B}^*$ space and to find its dimension can be easily implemented with any finite element code. As will be shown in the following section this is also useful in identifying an optimal integration strategy to eliminate locking.

4. Examples

In this section we examine the rank deficiency of some elements for various quadrature schemes.

4.1 The two noded Timoshenko beam element has 2 nodes, with 2 dof at each node. The number of rigid body modes physically admissible is 2 so that proper rank of K_e = 2. A 2-point Gauss quadrature preserves the rank of the stiffness matrix, but causes locking. A one point Gauss quadrature eliminates locking and brings in no zero energy mode, which is also evident from the dimension of the $\boldsymbol{B}^*$ space. The $[B]$ and $[B^*]$ matrices and their corresponding basis vectors are presented in Appendix A. The basis vectors (A.2) correspond to the $[B]$ matrix (A.1) of the subspace $\boldsymbol{B}$ while the basis vectors (A.4) correspond to the $[B^*]$ matrix (A.3) for the subspace $\boldsymbol{B}^*$.

Table 1 Two-noded Timoshenko beam element ($N_f = 4$, $N_r^p = 2$, $N_r = 2$, *Rank deficiency* = 0, $dimB = dimB^* = 2$)

Element type	Integration rule	Rank of K_e (no. of nonzero eigen values)	No. of Mechanisms	Dimension of B or B^*
●——● two noded (4 d.o.f)	Full	2	0	2
	Reduced	2	0	2

Note: Basis vectors of $\boldsymbol{B}/\boldsymbol{B}^*$ given in Appendix A

4.2 The three noded Timoshenko beam element has 3 nodes with 2 dof at each node. The number of rigid body modes is 2, so that proper rank of $K_e = 4$. A 2-point Gauss quadrature preserves the rank of the stiffness matrix but causes locking. A one point Gauss quadrature eliminates locking without bringing in any zero energy mode. This is also evident from the dimension of the $\boldsymbol{B}^*$ space. The $[B]$ and $[B^*]$ matrices and their corresponding basis vectors are presented in Appendix B. The basis vectors (A.6) correspond to the $[B]$ matrix defined by (A.5) of the subspace $\boldsymbol{B}$ while the basis vectors (A.9) correspond to the $[B^*]$ matrix (A.8) for the subspace $\boldsymbol{B}^*$.

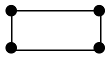
Table 2 Three-noded Timoshenko beam element ($N_f = 6$, $N_r^p = 2$, $N_r = 2$, *Rank deficiency* = 0, $dimB = dimB^* = 4$)

Element type	Integration rule	Rank of K_e (no. of nonzero eigen values)	No. of Mechanisms	Dimension of B or B^*
●—●—● three noded (6 d.o.f)	Full	4	0	4
	Reduced	4	0	4

Note: Basis vectors of $\boldsymbol{B}/\boldsymbol{B}^*$ given in Appendix B

4.3 Here we examine the rank sufficiency of the QUAD4 element for various quadrature schemes. This element has 4 nodes with 2 dof at each node, so that $N_f = 8$. The number of rigid body modes physically permissible is three i.e., $N_r^p = 3$. Therefore the proper rank of $K_e = 5$. When a 2×2 Gauss quadrature is used to evaluate the stiffness matrix the dimension of B space is 5, which is equal to the proper rank of K_e. In case of one point Gauss quadrature, the rank of $K_e = 3$, indicating that the element has a rank deficiency of 2. The dimension of $\boldsymbol{B}^*$ space is now 3, which is obtained by finding the basis vectors of the $[B^*]$ matrix. The $[B^*]$ matrix is obtained from the $[B]$ matrix by dropping the higher order polynomial terms. These results have been summarized in Table 3 and are in agreement with those given in Belytschko *et al.* (2000). The $[B]$ and $[B^*]$ matrices and their corresponding basis vectors are presented in Appendix C. The basis vectors (A.11) correspond to the $[B]$ matrix defined by (A.10) of the subspace $\boldsymbol{B}$ while the basis vectors (A.13) correspond to the $[B^*]$ matrix (A.12) for the subspace $\boldsymbol{B}^*$.

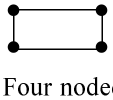
Table 3 QUAD4 element ($N_f = 8$, $N_r^p = 3$, $N_r = 4$, *Rank deficiency* = 2, $dimB \neq dimB^*$)

Element type	Integration rule	Rank of K_e (no. of nonzero eigen values)	No. of Mechanisms	Dimension of B or B^*
 Four noded (8 d.o.f)	Full	5	0	5
	Reduced	3	2	3

Note: Basis vectors of ***B/B***** given in Appendix C

4.4 Mindlin plate elements account for bending deformation and for transverse shear deformation, so that the stiffness matrix [K] can be regarded as being composed of a bending stiffness [K_b] and a transverse shear stiffness [K_s]. The locking of Mindlin plate elements caused by too many transverse shear constraints can be avoided by adopting a reduced or selective integration rule to generate [K]. We consider a 4-noded bilinear element which is the simplest element based on Mindlin theory. It has been established that, a fully integrated bilinear element, even in its rectangular form, would lock when used to analyze thin plates. Locking was seen to vanish if a 2×2 Gauss rule was used to evaluate [K_b] and a reduced 1-point rule was used to evaluate [K_s]. An analysis of the element, from the function space approach, reveals a loss in dimension of the ***B*** space. This deficiency is due to the zero energy mechanisms of the bilinear element arising due to reduced integration. As shown in the Table 4, *full integration* is sufficient to avoid element mechanisms, but causes locking. A 1-point integration of the bending and stiffness matrices eliminates locking but has 4 additional mechanisms. This is also revealed by the loss in dimension of the solution space (from 9 to 4) spanned by the column vectors of the ***B***-matrix. A selective integration of the bending and stiffness matrix also eliminates locking but brings in two mechanisms in addition to the usual three rigid body modes. Thus both reduced and selective quadrature rules fail to eliminate locking without introducing other deficiencies. An optimal integration strategy as suggested by the field consistency method (Prathap 1993) is to use a 2×2 Gauss rule to evaluate [k_b], a 1×2 Gauss rule to evaluate stiffness due to γ_{xz} and a 2×1 Gauss rule to evaluate stiffness due to γ_{yz}. An analysis of this element by the function space approach reveals no reduction in the dimension of the space spanned by the column vectors of the ***B***-matrix. This suggests that the field consistency arguments lead to optimal

Table 4 $N_f = 12$, $N_r^p = 2$, $N_r^a = 4$, $dimB \neq dimB^*$

Element type	Integration rule			Rank deficiency = No. of mechanisms	Rank of K_e (no. of nonzero eigen values)	Dimension of B space	Dimension of B^* space
	Type	[k_b]	[k_s]				
 Four noded (12 d.o.f)	Full	2×2	2×2	0	9	9	9
	Reduced	1×1	1×1	4	5	9	5
	Selective	2×2	1×1	2	7	9	7
	Shear Selective	2×2	2×1 1×2	0	9	9	9

Note: No. of basis vectors of ***B/B***** given in Appendix D integration strategies, without introducing any zero energy mechanisms. This element would be the optimal rectangular bilinear element. This also suggests that knowledge of the dimension of the solution space spanned by the column vectors of $\boldsymbol{B}$ can not only detect the presence of zero energy modes, but can be of help in choosing an optimal integration strategy to get a lock free element. The $[B]/[B^*]$ matrices corresponding to different integration rules are given in Appendix D.

5. Conclusions

The dimension of the $\boldsymbol{B}$ space is a good measure to detect rank deficiency in an element stiffness matrix. This can help in deciding an optimal integration strategy to eliminate locking. Integrating this test with a finite element program, the use of an element that contains a possible instability can be avoided and the robustness or usability of finite element analysis can be increased.

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Appendix A

For the simple Timoshenko beam element (Fig. 1a) the element strain vector is given by

$$\{\varepsilon^{he}\} = [B]\{\delta^e\} = \begin{bmatrix} 0 & -1/L & 0 & 1/L \\ 1/L & (1-\xi)/2 & -1/L & (1+\xi)/2 \end{bmatrix} \{\delta^e\} \tag{A.1}$$

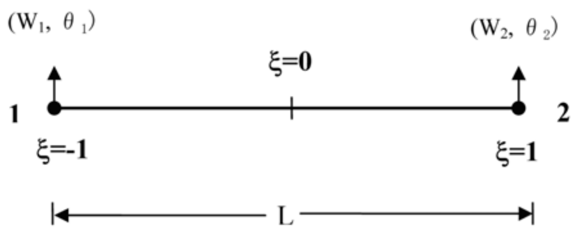

Fig. 1 (a) Isoparametric two-noded Timoshenko beam element

Here L is the element length and $\{\delta^e\} = [w_1, \theta_1, w_2, \theta_2]^T$ is the nodal displacement vector. The space ***B*** is evidently a subspace of the polynomial space P_2^2 (linear in ξ). Applying the Gram-Schmidt process on the column vectors of $[B]$, we get the normalized orthogonal basis vectors $\{u_i\}$ for the subspace ***B*** (of two dimensions) as

$$\{u_1\} = [0 \ \ 1]^T \quad \text{and} \quad \{u_2\} = [2/L \ \ \xi]^T \tag{A.2}$$

The function space $\boldsymbol{B}^*$ is a subspace of the space P_1^2 which is actually the space R^2. It is obtained from $[B]$, by dropping the highest Legendre polynomial, i.e., the ξ term. Thus,

$$[B^*] = \begin{bmatrix} 0 & -1/L & 0 & 1/L \\ 1/L & 1/2 & -1/L & 1/2 \end{bmatrix} \tag{A.3}$$

The normalized basis vectors for the subspace $\boldsymbol{B}^*$ (again of two dimensions) are given by

$$\{u_1^*\} = [0 \ \ 1]^T \quad \text{and} \quad \{u_2^*\} = [2/L \ \ 0]^T \tag{A.4}$$

So, in this example, using a lower order integration does not bring in a change in the dimension of the $[B]$ matrix.

Appendix B

The three noded Timoshenko beam element (Fig. 1b) uses quadratic Lagrangian interpolation functions for displacement and geometry. The element strain vector is given by

$$\{\varepsilon^{he}\} = [B]\{\delta^e\} = \begin{bmatrix} 0 & (2\xi-1)/L & 0 & -4\xi/L & 0 & (2\xi+1)/L \\ -(2\xi-1)/L & -\xi(1-\xi)/2 & 4\xi/L & (1-\xi^2) & -(2\xi+1)/L & \xi(1+\xi)/2 \end{bmatrix}\{\delta^e\} \tag{A.5}$$

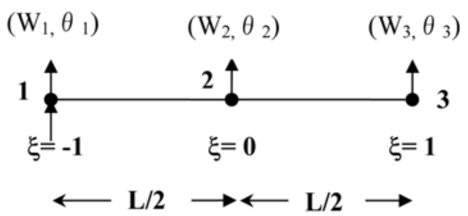

Fig. 1(b) Isoparametric three-noded Timoshenko beam element

Here L is the element length and $\{\delta^e\} = [w_1, \theta_1, w_2, \theta_2, w_3, \theta_3]^T$ is the nodal displacement vector. Using the Gram-Schmidt procedure on the column vectors of the above matrix, the four orthogonal basis vectors spanning the four dimensional subspace $\boldsymbol{B}(B \subset P_3^2)$ are determined as

$$\{u_1\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad \{u_2\} = \begin{Bmatrix} 0 \\ \xi \end{Bmatrix}, \quad \{u_3\} = \begin{Bmatrix} (2\xi-1)/L \\ (3\xi^2-1)/6 \end{Bmatrix} \quad \text{and} \quad u_4 = \begin{Bmatrix} (2\xi+\kappa)/6 \\ (3\xi^2-1)/6 \end{Bmatrix} \tag{A.6}$$

where $\kappa = \dfrac{4(e+5)}{15}$, $e = \dfrac{kGAL^2}{12EI}$

The strain displacement matrix $[B^*]$ that emerges from using a two-point Gaussian quadrature rule instead of the necessary three point rule for integration for the stiffness matrix is obtained by first expressing ξ^2 in terms of the Legendre quadratic polynomial as

$$\xi^2 = (3\xi^2 - 1)/3 + 1/3 = P_3 + 1/3 \tag{A.7}$$

and then dropping the Legendre polynomial $P_3 = 3\xi^2 - 1$. Thus the matrix $[B^*]$ is obtained from the $[B]$ matrix by replacing ξ^2 by (1/3) as follows

$$\boldsymbol{B}^* = \begin{bmatrix} 0 & (2\xi-1)/L & 0 & -4\xi/L & 0 & (2\xi+1)/L \\ -\dfrac{(2\xi-1)}{L} & \dfrac{\{\xi-(1/3)\}}{2} & \dfrac{4\xi}{L} & \dfrac{2}{3} & -\dfrac{(2\xi+1)}{L} & \dfrac{\{\xi+(1/3)\}}{2} \end{bmatrix} \tag{A.8}$$

The normalized basis vectors for subspace $\boldsymbol{B}^*$ (of dimension 4), as obtained by the Gram-Schmidt process are

$$\{u_1^*\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \{u_2^*\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad \{u_3^*\} = \begin{Bmatrix} \xi \\ 0 \end{Bmatrix}, \quad \{u_4^*\} = \begin{Bmatrix} 0 \\ \xi \end{Bmatrix} \tag{A.9}$$

So, in this example too, using a lower order integration does not bring in a change in the dimension of the $[B]$ matrix.

Appendix C

For the QUAD4 element (Fig. 2) for plane stress/strain the element strain vector is given by

$$\{\varepsilon^{he}\} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}^T = [B]\{\delta^e\}$$

$$\{\varepsilon^{he}\} = [B]\{\delta^e\}$$

$$= \begin{bmatrix} \dfrac{(\eta-1)}{4a} & 0 & \dfrac{(1-\eta)}{4a} & 0 & \dfrac{(1+\eta)}{4a} & 0 & -\dfrac{(1+\eta)}{4a} & 0 \\ 0 & \dfrac{(\xi-1)}{4b} & 0 & -\dfrac{(1+\xi)}{4b} & 0 & \dfrac{(1+\xi)}{4b} & 0 & \dfrac{(1-\xi)}{4b} \\ \dfrac{(\xi-1)}{4b} & \dfrac{(\eta-1)}{4a} & -\dfrac{(1+\xi)}{4b} & \dfrac{(1-\eta)}{4a} & \dfrac{(1+\xi)}{4b} & \dfrac{(1+\eta)}{4a} & \dfrac{(1-\xi)}{4b} & -\dfrac{(1+\eta)}{4a} \end{bmatrix} \{\delta^e\} \tag{A.10}$$

Here $2a$ and $2b$ are the sides of the rectangle and $\{\delta^e\} = \{u_x \;\; v_y \;\; u_y + v_x\}^T$. The space $\boldsymbol{B}$ is evidently a subspace of the space of polynomials (linear in ξ and η). Applying the Gram-Schmidt process on the column vectors of $[B]$, we get the normalized orthogonal basis vectors $\{u_i\}$ for subspace $\boldsymbol{B}$ as

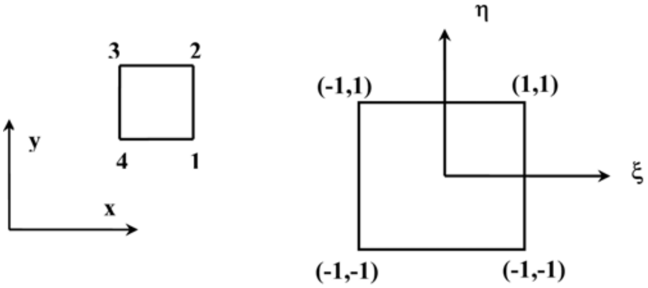

Fig. 2 Isoparametric 4-noded rectangular element

$$u_1 = \begin{Bmatrix} \dfrac{\eta-1}{a} \\ 0 \\ \dfrac{\xi-1}{b} \end{Bmatrix} \quad u_2 = \begin{Bmatrix} b(\eta-1) \\ \dfrac{\xi-1}{b}t_1 \\ \dfrac{\eta-1}{a}t_1 - a(\xi-1) \end{Bmatrix} \quad u_3 = \begin{Bmatrix} (80a^2-11b^2)(\eta-1) \\ (80b^2-11a^2)(\xi-1) \\ \dfrac{t_2}{3a} \end{Bmatrix}$$

$$u_4 = \begin{Bmatrix} 9bt_3 \\ \dfrac{t_4}{2b} \\ 3at_5 \end{Bmatrix} \quad u_5 = \begin{Bmatrix} 35a^2 + b^2(273\eta + 100) \\ -\dfrac{3}{2}(7a^2 + 20b^2) \\ 273ab\xi \end{Bmatrix} \tag{A.11}$$

where

$$t_1 = -\frac{(80b^2+28a^2)}{39}, \qquad t_2 = 240(a^4\xi + b^4\eta) - 33a^2b^2(\xi+\eta) + 80a^4 + 875a^2b^2 + 80b^4$$

$$t_3 = (\eta-1)(20a^2+7b^2), \quad t_4 = a^4(420\xi+140) + a^2b^2(1649+4476\xi) + 560b^4$$

$$t_5 = a^2(70\eta+60\xi) + b^2(746\eta+21\xi)$$

The function space B^* is a subspace of the space B_1^2, which is actually the space R^2. It is obtained from $[B]$, by dropping the highest Legendre polynomials, i.e., the ξ and η terms. Note that this must strictly be the higher order term. Equivalently, this means that the number of points required for optimal integration is reduced by one. Thus

$$\boldsymbol{B}^* = \begin{bmatrix} -1/4a & 0 & 1/4a & 0 & 1/4a & 0 & -1/4a & 0 \\ 0 & -1/4b & 0 & -1/4b & 0 & 1/4b & 0 & 1/4b \\ -1/4b & -1/4a & -1/4b & 1/4a & 1/4b & 1/4a & 1/4b & -1/4a \end{bmatrix} \{\delta^e\} \tag{A.12}$$

The normalized basis vectors for the subspace B^* are given by

$$\{u_1^*\} = \begin{Bmatrix} a \\ 0 \\ b \end{Bmatrix} \quad \{u_3^*\} = \begin{Bmatrix} b \\ -1/b \\ -1/a + r_1 a \end{Bmatrix} \quad \{u_3^*\} = \begin{Bmatrix} (10a^2-3b^2) \\ (10b^2-3a^2) \\ -26ab \end{Bmatrix} \tag{A.13}$$

where

$$r_1 = \frac{13}{4} * \left(\frac{b}{2b^2 + 7a^2} \right)$$

So, in this example, using a lower order integration reduces the number of nonzero vectors by 2, as is reflected in the dimension of the $[B^*]$ matrix.

Appendix D

For the Mindlin plate element (Fig. 2) the element strain vector is given by

$$\{\varepsilon^{he}\} = \{\theta_{x,x} \ \ \theta_{y,y} \ \ \theta_{x,y} + \theta_{y,x} \ \ \theta_y - w_{,y} \ \ \theta_x - w_{,x}\}^T = [B]\{\delta^e\}$$

$\{\varepsilon^{he}\} = [B]\{\delta^e\} =$

$$\begin{bmatrix} 0 & \frac{\eta-1}{4a} & 0 & 0 & \frac{1-\eta}{4a} & 0 & 0 & \frac{1+\eta}{4a} & 0 & 0 & -\frac{1+\eta}{4a} & 0 \\ 0 & 0 & \frac{\xi-1}{4b} & 0 & 0 & -\frac{1+\xi}{4b} & 0 & 0 & \frac{1+\xi}{4b} & 0 & 0 & \frac{1-\xi}{4b} \\ 0 & \frac{\xi-1}{4b} & \frac{\eta-1}{4a} & 0 & -\frac{1+\xi}{4b} & \frac{1-\eta}{4a} & 0 & \frac{1+\xi}{4b} & \frac{1+\eta}{4a} & 0 & \frac{1-\xi}{4b} & -\frac{1+\eta}{4a} \\ \frac{1-\xi}{4b} & 0 & \frac{(1-\xi)(1-\eta)}{4} & \frac{1+\xi}{4b} & 0 & \frac{(1+\xi)(1-\eta)}{4} & -\frac{1+\xi}{4b} & 0 & \frac{(1+\xi)(1+\eta)}{4} & \frac{\xi-1}{4b} & 0 & \frac{(1-\xi)(1+\eta)}{4} \\ \frac{1-\eta}{4a} & \frac{(1-\xi)(1-\eta)}{4} & 0 & \frac{\eta-1}{4a} & \frac{(1+\xi)(1-\eta)}{4} & 0 & -\frac{1+\eta}{4a} & \frac{(1+\xi)(1+\eta)}{4} & 0 & \frac{1+\eta}{4a} & \frac{(1-\xi)(1+\eta)}{4} & 0 \end{bmatrix} \{\delta^e\} \tag{A.14}$$

Here $2a$ and $2b$ are the sides of the rectangle and $\{\delta^e\} = \{w_1 \ \ \theta_{x1} \ \ \theta_{y1} \ \ldots \ w_4 \ \ \theta_{x4} \ \ \theta_{y4}\}^T$. When the stiffness matrix is evaluated with full integration, the number of basis vectors of the $[B]$ matrix is 9. Using a selective integration strategy (2×2 for bending and 1×1 for shear) to evaluate the stiffness matrix, is equivalent to replacing the $[B]$ matrix by the following $[B^*]$ matrix in Eq. (9).

$$[B^*] = \begin{bmatrix} 0 & \frac{\eta-1}{4a} & 0 & 0 & \frac{1-\eta}{4a} & 0 & 0 & \frac{1+\eta}{4a} & 0 & 0 & -\frac{1+\eta}{4a} & 0 \\ 0 & 0 & \frac{\xi-1}{4b} & 0 & 0 & -\frac{1+\xi}{4b} & 0 & 0 & \frac{1+\xi}{4b} & 0 & 0 & \frac{1-\xi}{4b} \\ 0 & \frac{\xi-1}{4b} & \frac{\eta-1}{4a} & 0 & -\frac{1+\xi}{4b} & \frac{1-\eta}{4a} & 0 & \frac{1+\xi}{4b} & \frac{1+\eta}{4a} & 0 & \frac{1-\xi}{4b} & -\frac{1+\eta}{4a} \\ \frac{1}{4b} & 0 & \frac{1}{4} & \frac{1}{4b} & 0 & \frac{1}{4} & -\frac{1}{4b} & 0 & \frac{1}{4} & \frac{1}{4b} & 0 & \frac{1}{4} \\ \frac{1}{4a} & \frac{1}{4} & 0 & -\frac{1}{4a} & \frac{1}{4} & 0 & -\frac{1}{4a} & \frac{1}{4} & 0 & \frac{1}{4a} & \frac{1}{4} & 0 \end{bmatrix} \tag{A.15}$$

The subspace B^*, spanned by the column vectors of the $[B^*]$ matrix, has 7 basis vectors so that this integration rule reduces the dimension of the $\boldsymbol{B}^*$ space and hence is not optimal. A shear selective integration rule corresponds to the following $[B^*]$ matrix,

$$[B^*]=\begin{bmatrix} 0 & \frac{\eta-1}{4a} & 0 & 0 & \frac{1-\eta}{4a} & 0 & 0 & \frac{1+\eta}{4a} & 0 & 0 & -\frac{1+\eta}{4a} & 0 \\ 0 & 0 & \frac{\xi-1}{4b} & 0 & 0 & -\frac{1+\xi}{4b} & 0 & 0 & \frac{1+\xi}{4b} & 0 & 0 & \frac{1-\xi}{4b} \\ 0 & \frac{\xi-1}{4b} & \frac{\eta-1}{4a} & 0 & -\frac{1+\xi}{4b} & \frac{1-\eta}{4a} & 0 & \frac{1+\xi}{4b} & \frac{1+\eta}{4a} & 0 & \frac{1-\xi}{4b} & -\frac{1+\eta}{4a} \\ \frac{1-\xi}{4b} & 0 & \frac{(1-\xi)}{4} & \frac{1+\xi}{4b} & 0 & \frac{(1+\xi)}{4} & -\frac{1+\xi}{4b} & 0 & \frac{(1+\xi)}{4} & \frac{\xi-1}{4b} & 0 & \frac{(1-\xi)}{4} \\ \frac{1-\eta}{4a} & \frac{(1-\eta)}{4} & 0 & \frac{\eta-1}{4a} & \frac{(1-\eta)}{4} & 0 & -\frac{1+\eta}{4a} & \frac{(1+\eta)}{4} & 0 & \frac{1+\eta}{4a} & \frac{(1+\eta)}{4} & 0 \end{bmatrix} \tag{A.16}$$

The corresponding $\boldsymbol{B}^*$ space is 9-dimensional, which is equal to the dimension of the $\boldsymbol{B}$ space used to evaluate the stiffness matrix in Eq. (9) by full integration. Thus, a shear selective integration strategy eliminates locking, without reducing the dimension of the $\boldsymbol{B}^*$ space.