Structural Engineering and Mechanics, Vol. 21, No. 1 (2005) 19-33
DOI: http://dx.doi.org/10.12989/sem.2005.21.1.019

Free vibration and elastic analysis of shear-deformable non-symmetric thin-walled curved beams: A centroid-shear center formulation

Nam-Il Kim†

Department of Civil and Environmental Engineering, University of Maryland, College Park, MD 20742, USA

Moon-Young Kim‡

Department of Civil and Environmental Engineering, Sungkyunkwan University, Cheoncheon-Dong, Jangan-Ku, Suwon 440-746, Korea

(Received September 17, 2004, Accepted May 30, 2005)

Abstract. An improved *shear deformable* thin-walled curved beam theory to overcome the drawback of currently available beam theories is newly proposed for the spatially coupled free vibration and elastic analysis. For this, the displacement field considering the shear deformation effects is presented by introducing displacement parameters defined at the centroid and shear center axes. Next the elastic strain and kinetic energies considering the shear effects due to the shear forces and the restrained warping torsion are rigorously derived. Then the equilibrium equations are consistently derived for curved beams with non-symmetric thin-walled sections. It should be noticed that this formulation can be easily reduced to the warping-free beam theory by simply putting the sectional properties associated with warping to zero for curved beams with L- or T-shaped sections. Finally in order to illustrate the validity and the accuracy of this study, finite element solutions using the isoparametric curved beam elements are presented and compared with those in available references and ABAQUS's shell elements.

Key words: free vibration; elastic analysis; curved beam; thin-walled; shear deformation; warping.

1. Introduction

Generally it is well known that the vibration and elastic behavior of thin-walled curved beam structures are very complex because the axial, flexural and torsional deformations are coupled due to the curvature effects as well as non-symmetry of cross section. Investigation into the behavior of thin-walled straight and curved members with open and closed cross sections has been carried out extensively since the early works of Vlasov (1961) and Timoshenko and Gere (1961).

Up to the present, for free in-plane vibration of curved beams, considerable research (Raveendranath *et al*. 2000, Wilson and Lee 1995, Wilson *et al*. 1994, Gupta and Howson 1994)

† Ph.D., Faculty Research Associate
‡ Ph.D., Professor, Corresponding author, E-mail: kmye@skku.ac.kr

have been done considering various parameters such as boundary conditions, shear deformation, rotary inertia, variable curvatures and variable cross sections. Several authors (Chucheepsakul and Saetiew 2002, Piovan *et al.* 2000, Cortínez and Piovan 1999, Howson and Jemah 1999) have studied the decoupled free out-of-plane vibration behavior of curved beams. Also Choi and Hong (2001) performed the static analysis of single- and multi-span curved box girder bridges using the modified finite strip method.

It is well known that the centroid-shear center formulation for thin-walled straight beam with non-symmetric cross sections is established assuming that the flexural and warping-torsional deformations are decoupled. Hence the warping-free theory for beams with non-symmetric cross section is easily obtained by simply putting the warping moment of inertia to zero.

On the other hand, for the vibration and elastic theories of curved beams based on the centroid-shear center formulation, most of previous research has been restricted to doubly symmetric thin-walled sections. Furthermore it has been reported by Gendy and Saleeb (1992) that the curved beam theory based on the centroid-shear center formulation is valid only for a cross section having doubly symmetry or one axis of symmetry which lies in the plane of beam curvature, otherwise, coupling terms exist. For this reason, it appears that most of thin-walled curved beam theories with non-symmetric cross sections have been developed based on displacement parameters which are all defined at the centroid axis (Kim and Kim 2004, Kim *et al.* 2002, Gendy and Saleeb 1994, 1992). To the authors' knowledge, Tong and Xu's study (2002) was the only recent attempt reported on the curved beam theory with non-symmetric cross section based on the centroid-shear center formulation in the literature. However they did not consider the shear deformation effect and only restricted to the static analysis of curved beam. Also the thickness-curvature effect which made the difference larger in curved beam with large subtended angle and small radius was not considered in their formulation.

It should be noted that in case of curved beams with non-symmetric thin-walled cross sections such as L- or T-shaped sections, it is impossible to present the curved beam theory neglecting the restrained warping torsion from the centroid formulation because the sectional properties associated with warping of cross section become non-zero with respect to the centroid axis. Accordingly, several sectional properties associated with warping should be evaluated additionally. Therefore, it is evident that the vibration and elastic theories of *shear-deformable* curved beams neglecting the restrained warping torsion need to be developed in case of these curved beams.

The aim of this study is to propose the centroid-shear center formulation for the spatially coupled free vibration and elastic analysis of shear deformable curved beam with non-symmetric cross sections. In this formulation, for beams with L- or T-shaped sections, one can obtain the curved beam theory easily neglecting the restrained warping torsion by simply putting the sectional properties associated with warping defined at the shear center to zero. Also for the curved beam with non-symmetric closed sections, this beam theory may be reduced naturally to that with warping deformation neglected because the values of sectional properties associated with warping at the shear center become extremely large. The important points presented are summarized as follows

1. The displacement field for *shear-deformable non-symmetric* thin-walled curved beams with constant curvature is introduced, in which the axial displacement and two flexural rotations are defined at the centroid and the torsional rotation including the normalized warping function and two lateral displacements are defined at the shear center, respectively.
2. Force-deformation relationships due to the normal stress considering the thickness-curvature effect and due to the simple shear and *warping-torsional shear* stresses are accurately derived.

3. The elastic strain and kinetic energies based on the centroid-shear center formulation are newly derived for the free vibration and elastic analysis of non-symmetric curved beams having shear-deformable thin-walled cross sections.
4. In addition, finite element (FE) procedure using the isoparametric curved beam elements is presented for the analysis of non-symmetric curved beams. Finally numerical solutions are presented and compared with results by available references and ABAQUS's shell elements.

2. Shear-deformable curved beam theory

2.1 Total potential energy

To derive a general theory for the free vibration and elastic analysis of *shear-deformable* thin-walled curved beams consistently, two curvilinear coordinate systems are adopted. The first coordinate system (x_1, x_2, x_3) is shown in Fig. 1, in which the x_1 axis coincides with the curved centroid axis having the radius of curvature R but x_2, x_3 axes are not necessarily principal inertia axes, while the second coordinate system (x_1^s, x_2^s, x_3^s) is constituted by the shear center axis and two orthogonal axes running parallel with the direction of x_2, x_3 axes (see Fig. 2). (e_2, e_3) denotes the

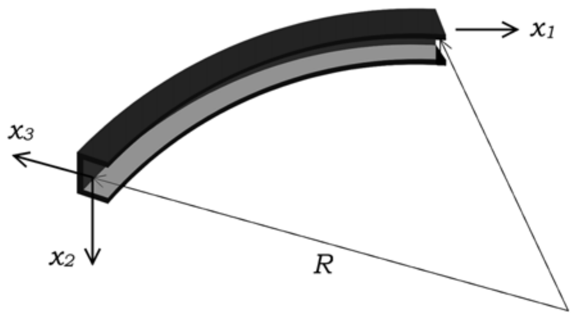
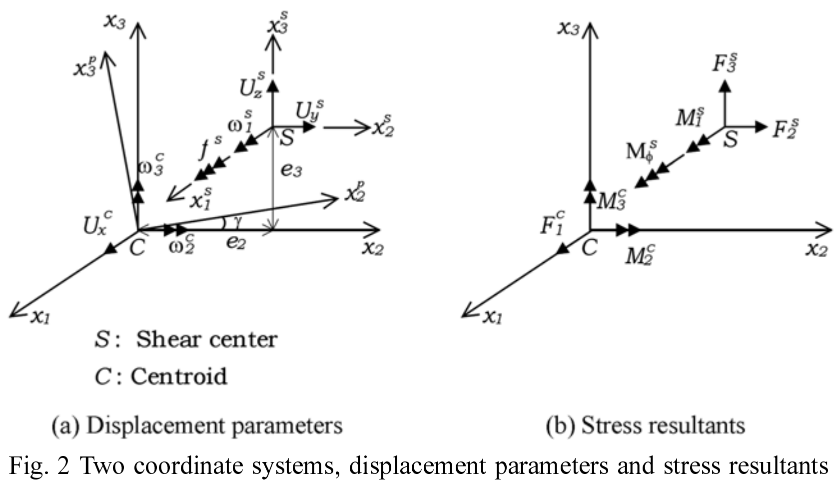


Fig. 1 A curvilinear coordinate system for thin-walled curved beam



(a) Displacement parameters (b) Stress resultants

Fig. 2 Two coordinate systems, displacement parameters and stress resultants position vector of the shear center and γ is the angle between x_2 and the x_2^p axis.

To introduce the displacement field for the non-symmetric thin-walled cross section, seven displacement parameters and stress resultants are used as shown in Figs. 2(a) and 2(b), respectively. Assuming that the distorsional deformation of cross section is neglected, the longitudinal displacement U_1 and the transverse displacements U_2 and U_3 at the arbitrary point can be written as follows

$$U_1 = U_x + x_3\omega_2 - x_2\omega_3 + f\phi(x_2, x_3) \tag{1a}$$

$$U_2 = U_y - \omega_1(x_3 - e_3) \tag{1b}$$

$$U_3 = U_z + \omega_1(x_2 - e_2) \tag{1c}$$

where U_x, ω_2, ω_3 are the rigid body translation and two rotations with respect to x_1, x_2, x_3 axes; ω_1, U_y, U_z are the rigid body rotation and two translations with respect to x_1^s, x_2^s, x_3^s axes; f, ϕ are the displacement parameter measuring warping deformation and the normalized warping function defined at the shear center, respectively. F_1 is the axial force acting at the centroid; F_2 and F_3 are the shear forces acting at the shear center; M_1 is the total twisting moment with respect to the shear center axis; M_2 and M_3 are the bending moments with respect to x_2 and x_3 axes, respectively. M_ϕ and M_R are the bimoment and the restrained torsional moment about the shear center axis, respectively. And the detailed definitions of these stress resultants are

$$F_1 = \int_A \tau_{11} dA,\ F_2 = \int_A \tau_{12} dA,\ F_3 = \int_A \tau_{13} dA,\ M_1 = \int_A [\tau_{13}(x_2 - e_2) - \tau_{12}(x_3 - e_3)] dA$$

$$M_2 = \int_A \tau_{11} x_3 dA,\ M_3 = -\int_A \tau_{11} x_2 dA,\ M_\phi = \int_A \tau_{11} \phi dA \tag{2a-h}$$

$$M_R = \int_A \left[\left\{ \tau_{12}\phi_{,2} + \tau_{13}\left(\phi_{,3} - \frac{\phi}{R + x_3}\right) \right\} \frac{R + x_3}{R} + \frac{e_3}{R}(\tau_{13}x_2 - \tau_{12}x_3) \right] dA$$

Now the total potential energy of thin-walled curved beam for the free vibration analysis under stationary harmonic conditions vibrating with the circular frequency ω be expressed as

$$\Pi = \Pi_E - \Pi_M - \Pi_{ext} \tag{3}$$

where Π_E, Π_M and Π_{ext} are the elastic strain energy, the kinetic energy and the energy due to the external force. The detailed expressions for each term of Π are

$$\Pi_E = \frac{1}{2}\int_o^l \int_A [\tau_{11}e_{11} + 2\tau_{12}e_{12} + 2\tau_{13}e_{13}] \frac{R + x_3}{R} dA dx_1 \tag{4a}$$

$$\Pi_M = \frac{1}{2}\rho\omega^2 \int_o^l \int_A [U_1^2 + U_2^2 + U_3^2] \frac{R + x_3}{R} dA dx_1 \tag{4b}$$

$$\Pi_{ext} = \frac{1}{2}\mathbf{U_e^T F_e} \tag{4c}$$

where $\mathbf{U_e}$, $\mathbf{F_e}$ are the nodal displacement and nodal force vectors, respectively.

On the other hand, strain-displacement relations due to the first order displacements are expressed as

$$e_{11} = \left(U_{1,1} + \frac{U_3}{R}\right)\frac{R}{R+x_3} = \left[\left(U_x' + \frac{U_z}{R} - \frac{e_2}{R}\omega_1\right) - x_2\left(-\frac{\omega_1}{R} + \omega_3'\right) + x_3\omega_2' + \phi f'\right]\frac{R}{R+x_3} \quad (5a)$$

$$2e_{12} = \frac{U_{2,1}R}{R+x_3} + U_{1,2} = [U_y' - \omega_1'(x_3 - e_3)]\frac{R}{R+x_3} - \omega_3 + f\phi_{,2} \quad (5b)$$

$$2e_{13} = \left(U_{3,1} - \frac{U_1}{R}\right)\frac{R}{R+x_3} + U_{1,3}$$

$$= \left[-\frac{U_x}{R} + U_z' + \omega_1'(x_2 - e_2) - \frac{x_3}{R}\omega_2 + \frac{x_2}{R}\omega_3 - \frac{f}{R}\phi\right]\frac{R}{R+x_3} + \omega_2 + f\phi_{,3} \quad (5c)$$

For the thin-walled curved beam subjected to distributed loadings, substituting linear strains (5a-c) into Eq. (4a) and integrating over the cross sectional area, Eq. (4a) is reduced to the following equation.

$$\begin{aligned}\Pi_E = \frac{1}{2}\int_o^l\Bigg[& F_1\left(U_x' + \frac{U_z}{R} - \frac{e_2}{R}\omega_1\right) + M_2\omega_2' + M_3\left(-\frac{\omega_1}{R} + \omega_3'\right) + M_\phi f' \\ & + F_2\left(U_y' - \omega_3 - \frac{e_3}{R}\omega_3 + \frac{e_3^2}{R}f\right) + F_3\left(-\frac{U_x}{R} + U_z' + \omega_2 + \frac{e_2}{R}\omega_3 - \frac{e_2 e_3}{R}f\right) \\ & + (M_1 - M_R)\left(\omega_1' + \frac{\omega_3}{R} - \frac{e_3}{R}f\right) + M_R\left(\omega_1' + \frac{\omega_3}{R} + f - \frac{e_3}{R}f\right)\Bigg]dx_1\end{aligned} \quad (6)$$

And Eq. (4c) can be expressed as

$$\Pi_{ext} = \int_o^l [p_1U_x + p_2U_y + p_3U_z + m_1\omega_1 + m_2\omega_2 + m_3\omega_3 + m_\phi f]dx_1 \quad (7)$$

where p_1, p_2, p_3 are the distributed forces in the direction of x_1, x_2, x_3 axes and m_1, m_2, m_3, m_ϕ denote distributed moments.

Now invoking the stationary condition of the total potential energy, equilibrium equations are obtained as

$$F_1' + \frac{F_3}{R} = -p_1 \quad (8a)$$

$$F_2' = -p_2 \quad (8b)$$

$$-\frac{F_1}{R} + F_3' = -p_3 \quad (8c)$$

$$\frac{e_2}{R}F_1 + M_1' + \frac{M_3}{R} = -m_1 \quad (8d)$$

$$-F_3 + M_2' = -m_2 \quad (8e)$$

$$F_2 + \frac{e_3}{R}F_2 - \frac{e_2}{R}F_3 - \frac{M_1}{R} + M_3' = -m_3 \quad (8f)$$

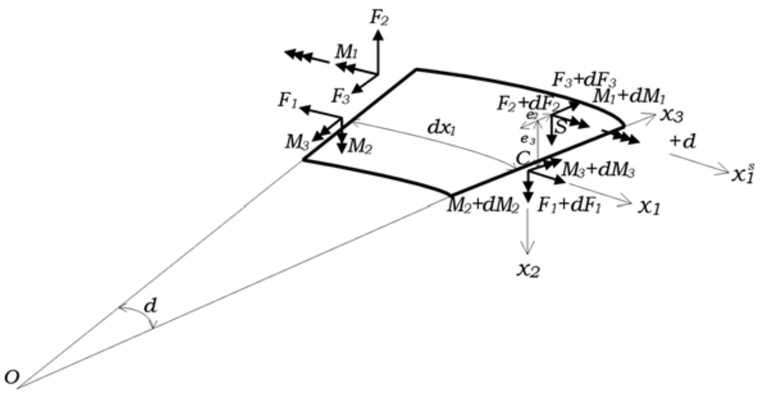


Fig. 3 Forces and moments of small segment of curved beam

$$-\frac{e_3^2}{R}F_2 + \frac{e_2 e_3}{R}F_3 + \frac{e_3}{R}M_1 - M_R + M_\phi' = -m_\phi \tag{8g}$$

Also equilibrium equations can be derived by using the equilibrium conditions of forces and moments of a small segment of curved beam as shown in Fig. 3. Detailed expressions are presented in Appendix.

2.2 Force-deformation relations

Force-deformation relations due to the normal and shear stresses are derived. First by substituting Eq. (5a) into Eqs. (2a), (2e), (2f), (2g) and integrating over the cross section, the following relations are obtained.

$$\begin{Bmatrix} F_1 \\ M_2 \\ M_3 \\ M_\phi \end{Bmatrix} = E \begin{bmatrix} A + \frac{\hat{I}_2}{R^2} & -\frac{\hat{I}_2}{R} & \frac{\hat{I}_{23}}{R} & \frac{I_{\phi 22}}{R^2} \\ -\frac{\hat{I}_2}{R} & \hat{I}_2 & -\hat{I}_{23} & -\frac{I_{\phi 22}}{R} \\ \frac{\hat{I}_{23}}{R} & -\hat{I}_{23} & \hat{I}_3 & \frac{I_{\phi 23}}{R} \\ \frac{I_{\phi 22}}{R^2} & -\frac{I_{\phi 22}}{R} & \frac{I_{\phi 23}}{R} & \hat{I}_\phi \end{bmatrix} \begin{Bmatrix} U_x' + \frac{U_z}{R} - \frac{e_2}{R}\omega_1 \\ \omega_2' \\ -\frac{\omega_1}{R} + \omega_3' \\ f' \end{Bmatrix} \tag{9a-d}$$

where E is the Young's modulus and

$$\hat{I}_2 = I_2 - \frac{I_{222}}{R}, \quad \hat{I}_3 = I_3 - \frac{I_{233}}{R}, \quad \hat{I}_{23} = I_{23} - \frac{I_{223}}{R}, \quad \hat{I}_\phi = I_\phi - \frac{I_{\phi\phi 2}}{R} \tag{10a-d}$$

In Eq. (10), the sectional properties with respect to the centroid-shear center are defined as

$$I_2 = \int_A x_3^2 dA, \qquad I_3 = \int_A x_2^2 dA, \qquad I_{23} = \int_A x_2 x_3 dA$$

$$I_\phi = \int_A \phi^2 dA, \qquad I_{\phi 2} = \int_A \phi x_3 dA, \qquad I_{\phi 3} = \int_A \phi x_2 dA$$

$$I_{222} = \int_A x_3^3 dA, \qquad I_{223} = \int_A x_2 x_3^2 dA, \qquad I_{233} = \int_A x_2^2 x_3 dA$$

$$I_{\phi 22} = \int_A \phi x_3^2 dA, \qquad I_{\phi 23} = \int_A \phi x_2 x_3 dA, \qquad I_{\phi\phi 2} = \int_A \phi^2 x_3 dA \tag{11a-l}$$

where A, I_2, I_3 and I_{23} are the cross sectional area, the second moments of inertia and the product moment of inertia about x_2 and x_3 axes, respectively. I_ϕ is the warping moment of inertia and $I_{\phi 2}$, $I_{\phi 3}$ are always equal to zero. I_{222}, I_{223}, I_{233}, $I_{\phi 22}$, $I_{\phi 23}$, $I_{\phi\phi 2}$ are the third order inertia moments to take into account the thickness-curvature effect consistently.

For the curved beams, we can easily obtain the force-deformation relations for the shear forces, the restrained torsional and the St. Venant torsional moments from Eq. (6) as follows

$$\begin{Bmatrix} F_2 \\ F_3 \\ M_R \end{Bmatrix} = G \begin{bmatrix} A_2 & A_{23} & o \\ A_{23} & A_3 & o \\ o & o & A_r^s \end{bmatrix} \begin{Bmatrix} U_y' - \omega_3 - \dfrac{e_3}{R}\omega_3 + \dfrac{e_3^2}{R}f \\ -\dfrac{U_x}{R} + U_z' + \omega_2 + \dfrac{e_2}{R}\omega_3 - \dfrac{e_2 e_3}{R}f \\ \omega_1' + \dfrac{\omega_3}{R} + f - \dfrac{e_3}{R}f \end{Bmatrix} \tag{12a-c}$$

$$M_{st} = M_1 - M_R = GJ\left(\omega_1' + \frac{\omega_3}{R} - \frac{e_3}{R}f\right) \tag{12d}$$

where G is the shear modulus and J is the torsional constant and

$$A_2 = A_2^s \cos^2\gamma + A_3^s \sin^2\gamma \tag{13a}$$

$$A_3 = A_3^s \cos^2\gamma + A_2^s \sin^2\gamma \tag{13b}$$

$$A_{23} = (A_2^s + A_3^s)\cos\gamma \sin\gamma \tag{13c}$$

where A_2^s, A_3^s and A_r^s are the effective shear areas defined by

$$\frac{1}{A_2^s} = \frac{1}{I_{3p}^2}\int_A Q_3^2 \frac{ds}{t}, \quad \frac{1}{A_3^s} = \frac{1}{I_{2p}^2}\int_A Q_2^2 \frac{ds}{t}, \quad \frac{1}{A_r^s} = \frac{1}{I_\phi^2}\int_A Q_r^2 \frac{ds}{t} \tag{14a-c}$$

And

$$I_{2p} = \int_A (x_3^p)^2 dA, \qquad I_{3p} = \int_A (x_2^p)^2 dA, \qquad I_\phi = \int_A \phi^2 dA$$

$$Q_2 = \int_o^s x_3^p t ds, \qquad Q_3 = \int_o^s x_2^p t ds, \qquad Q_r = \int_o^s \phi t ds \tag{15a-f}$$

Consequently Eqs. (9a-d) and (12a-d) constitute the force-deformation relations of *shear-deformable* thin-walled curved beam.

Finally substitution of force-deformation relations (9a-d) and (12a-d) into Eq. (6) leads to the elastic strain energy for the shear deformable thin-walled curved beam with non-symmetric cross section.

$$\begin{aligned}\Pi_E = \frac{1}{2}\int_o^l\Big[& EA\Big(U_x' + \frac{U_z}{R} - \frac{e_2}{R}\omega_1\Big)^2 + E\hat{I}_2\Big(-\frac{U_x'}{R} - \frac{U_z}{R^2} + \frac{e_2}{R^2}\omega_1 + \omega_2'\Big)^2 + E\hat{I}_3\Big(-\frac{\omega_1}{R} + \omega_3'\Big)^2 + E\hat{I}_\phi f'^2 \\ & - 2E\hat{I}_{23}\Big(-\frac{\omega_1}{R} + \omega_3'\Big)\Big(-\frac{U_x'}{R} - \frac{U_z}{R^2} + \frac{e_2}{R^2}\omega_1 + \omega_2'\Big) - 2\frac{EI_{\phi 22}}{R}\Big(-\frac{U_x'}{R} - \frac{U_z}{R^2} + \frac{e_2}{R^2}\omega_1 + \omega_2'\Big)f' \\ & + 2\frac{EI_{\phi 23}}{R}\Big(-\frac{\omega_1}{R} + \omega_3'\Big)f' + GJ\Big(\omega_1' + \frac{\omega_3}{R} - \frac{e_3}{R}f\Big)^2 + GA_2\Big(U_y' - \omega_3 - \frac{e_3}{R}\omega_3 + \frac{e_3^2}{R}f\Big)^2 \\ & + GA_3\Big(-\frac{U_x}{R} + U_z' + \omega_2 + \frac{e_2}{R}\omega_3 - \frac{e_2 e_3}{R}f\Big)^2 + GA_r^s\Big(\omega_1' + \frac{\omega_3}{R} + f - \frac{e_3}{R}f\Big)^2 \\ & + 2GA_{23}\Big(U_y' - \omega_3 - \frac{e_3}{R}\omega_3 + \frac{e_3^2}{R}f\Big)\Big(-\frac{U_x}{R} + U_z' + \omega_2 + \frac{e_2}{R}\omega_3 - \frac{e_2 e_3}{R}f\Big)\Big]dx_1\end{aligned} \tag{16}$$

And by substituting the displacement field in Eqs. (1a-c) into Eq. (4b), the kinetic energy Π_M including the rotary inertia based on the centroid-shear center formulation can be obtained as

$$\begin{aligned}\Pi_M = \frac{1}{2}\rho\omega^2\int_o^l[& A\{U_x^2 + U_y^2 + U_z^2 + \omega_1^2(e_2^2 + e_3^2) + 2\omega_1(e_3 U_y - e_2 U_z)\} + \tilde{I}_2\omega_2^2 + \tilde{I}_3\omega_3^2 \\ & + \tilde{I}_\phi f^2 + \Big(\tilde{I}_o - \frac{2e_3}{R}I_2 - \frac{2e_2}{R}I_{23}\Big)\omega_1^2 + 2\frac{I_2}{R}(U_x\omega_2 - U_y\omega_1) - 2\frac{I_{23}}{R}(U_x\omega_3 - U_z\omega_1) \\ & - 2\tilde{I}_{23}\omega_2\omega_3 + 2\frac{I_{\phi 22}}{R}\omega_2 f - 2\frac{I_{\phi 23}}{R}\omega_3 f\Big]dx_1\end{aligned} \tag{17}$$

where ρ is the density and

$$\tilde{I}_2 = I_2 + \frac{I_{222}}{R}, \qquad \tilde{I}_3 = I_3 + \frac{I_{233}}{R}, \qquad \tilde{I}_{23} = I_{23} + \frac{I_{223}}{R}$$

$$\tilde{I}_\phi = I_\phi + \frac{I_{\phi\phi 2}}{R}, \qquad \tilde{I}_o = I_2 + I_3 + \frac{I_{222} + I_{233}}{R} \tag{18a-e}$$

2.3 Shear-deformable curved beam with sections neglecting warping torsion

As mentioned previously, for the curved beams with *open* cross sections neglecting the warping torsional effect at the shear center such as L- or T-shaped sections, the sectional properties $(\hat{I}_\phi, I_{\phi 22}, I_{\phi 23}, \tilde{I}_\phi)$ in Eqs. (16) and (17) associated with warping become obviously zero. Also for the curved beams with non-symmetric *closed* sections, these properties become very large values so they can be interpreted as penalty numbers in the strain energy of curved beam. Consequently, this means that strain and kinetic energy terms related to warping should vanish in the centroid-shear center formulation for the curved beams with L- or T-shaped cross sections or closed sections. It is remarkable to note that when Eq. (16) and Eq. (17) compare with Eq. (16) and Eq. (17) in paper by Kim and Kim (2004), one can find the main drawback in previous study. The elastic strain and kinetic energies obtained from the centroid formulation should retain the additional sectional properties (i.e., I_ϕ, $I_{\phi 2}$, $I_{\phi 3}$, $I_{\phi 22}$, $I_{\phi 23}$, $I_{\phi\phi 2}$, A_{2r}, A_{3r} in Eqs. (16) and (17) in study by Kim and Kim (2004) due to the restrained warping which does not become zero at the centroid. Also this drawback is revealed in the study by Gendy and Saleeb (1994, 1992).

3. Finite element formulation

In this study, the isoparametric curved beam element having arbitrary thin-walled cross sections is used. It is based on the elastic strain and kinetic energy expressions derived in the previous Section and the reduced integration scheme is adopted to avoid the shear locking phenomena. Resultantly, the coordinate and all the displacement parameters of the curved beam element can be interpolated with respect to the nodal coordinates and displacements. Substituting the shape functions and the cross-sectional properties into Eqs. (16) and (17) and integrating along the element length, the equilibrium equation of thin-walled curved beam element is obtained in matrix form as

$$(\mathbf{K_e} - \omega^2 \mathbf{M_e})\mathbf{U_e} = \mathbf{F_e} \tag{19}$$

where $\mathbf{K_e}$ and $\mathbf{M_e}$ are the element elastic stiffness and mass matrices in local coordinate, respectively.

$$\mathbf{U_e} = [U^1, U^2, \ldots, U^n] \tag{20a}$$

$$\mathbf{U}^\alpha = [U_x^\alpha, U_y^\alpha, U_z^\alpha, \omega_1^\alpha, \omega_2^\alpha, \omega_3^\alpha, f^\alpha]^T \tag{20b}$$

$$\mathbf{F_e} = [F^1, F^2, \ldots, F^n] \tag{20c}$$

$$\mathbf{F}^\alpha = [F_1^\alpha, F_2^\alpha, F_3^\alpha, M_1^\alpha, M_2^\alpha, M_3^\alpha, M_\phi^\alpha]^T, \quad \alpha = 1, 2, \ldots, n \tag{20d}$$

Now using direct stiffness method, the matrix equilibrium equation for the free vibration analysis of non-symmetric thin-walled curved beam is obtained as

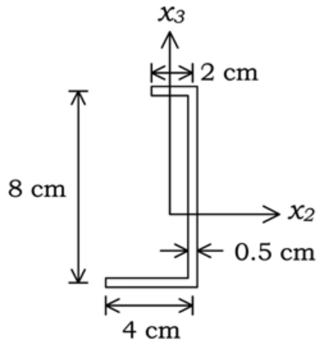
$$\mathbf{K_E U} = \omega^2 \mathbf{M_E U} \tag{21}$$

where $\mathbf{K_E}$ and $\mathbf{M_E}$ are global elastic stiffness and mass matrices, respectively.

4. Numerical examples

4.1 Curved beam with non-symmetric cross section

Fig. 4 shows the non-symmetric cross section of curved beam clamped at both ends and its material and sectional properties. In Table 1, the lowest ten spatially coupled natural frequencies of beam by this study using 20 three-noded isoparametric curved beam elements are presented with respect to subtended angles 10 and 90. For comparison, the solutions obtained from single reference



(a) Non-symmetric cross section

$$E = 30000\ \mathrm{N/cm^2},\ \ G = 11500\ \mathrm{N/cm^2},\ \ \rho = 0.00785\ \mathrm{N/cm^3},\ \ A = 7\ \mathrm{cm^2},\ \ J = 0.58333\ \mathrm{cm^4}$$

$$e_2 = 1.44846\ \mathrm{cm},\ \ e_3 = -2.04461\mathrm{cm},\ \ I_2 = 67.04762\ \mathrm{cm^4},\ \ I_3 = 8.42857\ \mathrm{cm^4},\ \ I_{23} = 9.14286\ \mathrm{cm^4}$$

$$I_{222} = 52.24490\ \mathrm{cm^5},\ \ I_{223} = -20.02721\mathrm{cm^5},\ \ I_{233} = -17.41497\ \mathrm{cm^5},\ \ I_{333} = -13.38776\ \mathrm{cm^5},\ \ I_\phi = 42.48664\ \mathrm{cm^6}$$

$$I_{\phi 22} = 24.48383\ \mathrm{cm^6},\ \ I_{\phi 23} = -42.48664\ \mathrm{cm^6},\ \ I_{\phi 33} = -10.53165\ \mathrm{cm^6},\ \ I_{\phi\phi 2} = 117.44909\ \mathrm{cm^7},\ \ l = 200\ \mathrm{cm}$$

(b) Material and sectional properties

Fig. 4 Clamped curved beam with a non-symmetric cross section

Table 1 Natural frequency of clamped curved beam with non-symmetric section, (rad./sec)

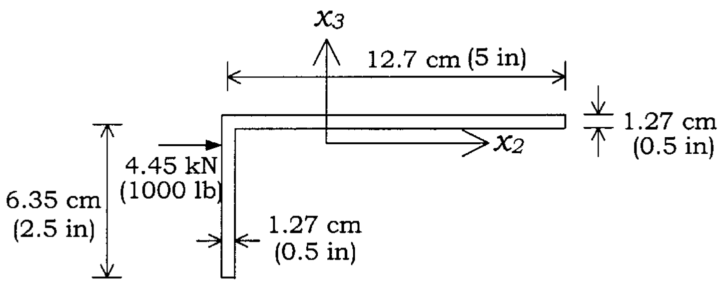
θ_o	Method	Vibration mode									
		1	2	3	4	5	6	7	8	9	10
10	This study	0.9689	2.092	2.490	4.109	4.285	4.543	6.917	7.566	9.753	10.29
	Kim and Kim (2004)	0.9689	2.092	2.490	4.109	4.285	4.543	6.917	7.566	9.753	10.29
	Kim *et al*. (2002)	0.9741	2.101	2.515	4.211	4.333	4.615	7.045	7.716	9.989	10.94
	ABAQUS	0.9838	2.087	2.531	4.117	4.309	4.623	7.087	7.654	10.02	10.25
90	This study	0.8446	1.982	3.646	5.497	5.840	6.409	8.231	8.727	11.34	11.72
	Kim and Kim (2004)	0.8446	1.982	3.646	5.497	5.840	6.409	8.231	8.727	11.34	11.72
	Kim *et al*. (2002)	0.8499	1.998	3.684	5.642	5.935	6.469	8.429	8.981	11.76	12.20
	ABAQUS	0.8379	1.977	3.659	5.553	5.904	6.148	8.357	8.869	10.73	11.86

line (the line of centroid) formulations presented by Kim and Kim (2004) considering shear deformation effect and by Kim *et al*. (2002) neglecting it and the results by 300 nine-noded shell elements (S9R5) of ABAQUS which is the commercial finite element analysis program are presented. From Table 1, it can be found that the natural frequencies by this study are in greatly agreement with the centroid formulation solution considering shear deformation and are in good agreement with those by ABAQUS's shell elements. Also it is observed that maximum difference due to shear deformation effect is 6.3% at the tenth mode for the subtended angle 10^o.

4.2 Curved beam with L-shaped cross section

We concern the vibration and elastic analysis of the L-shaped curved beam clamped at both ends as shown in Fig. 5. The purpose of this example is to show the usefulness of the proposed curved beam theory with non-symmetric cross section neglecting warping deformation and to verify how it predicts well the behavior of structure by comparing the present solutions with those by ABAQUS's shell elements and the previous research.

First, the natural frequencies analyzed using beam elements with 6 DOF per node by this study are compared with the solutions using 300 nine-noded shell elements of ABAQUS in Table 2, where excellent agreement is observed with less than 2.5% as maximum of difference. Next, the lateral displacement U_y at the corner of the L-shaped cross section along the curved beam subjected



(a) Non-symmetric L-shaped cross section

$E = 20684.28$ kN/cm^2, $G = 7955.49$ kN/cm^2, $\rho = 0.077009$ N/cm^3, $A = 24.1935$ cm^2

$J = 13.00723$ cm^4, $e_2 = -4.23333$ cm, $e_3 = 1.05833$ cm, $I_2 = 81.29520$ cm^4, $I_3 = 433.57440$ cm^4

$I_{23} = 108.39360$ cm^4, $I_{222} = -229.43312$ cm^5, $I_{223} = -229.43312$ cm^5, $R = 914.4$ cm, $l = 609.6$ cm

(b) Material and sectional properties

Fig. 5 Clamped curved girder with a non-symmetric L-shaped section

Table 2 Natural frequency of clamped curved beam with L-shaped section, (rad./sec)

Mode	This study	ABAQUS
1	5.564	5.593
2	6.013	6.164
3	10.74	11.00
4	17.17	17.22
5	18.98	19.46
6	23.57	23.92
7	28.41	28.33
8	30.11	30.59
9	34.62	34.71
10	36.02	36.13

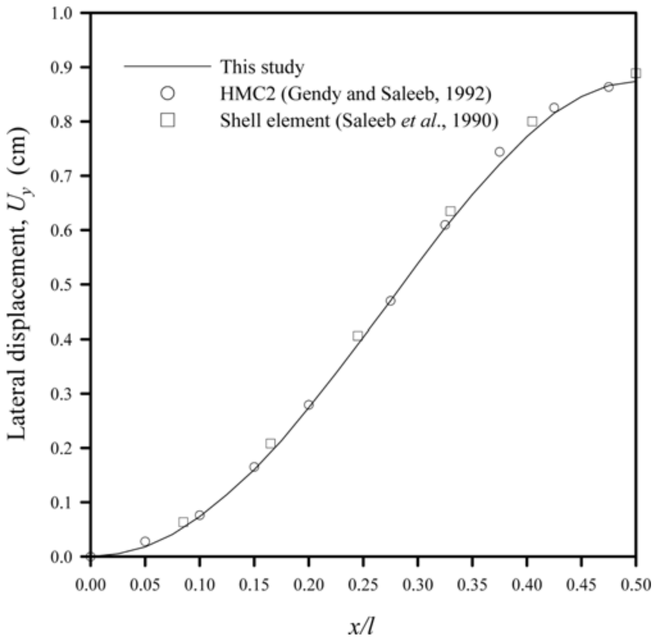


Fig. 6 Lateral displacement at the shear center of L-shaped girder

to out-of-plane lateral force 4.45 kN (1000lb) acting at the mid-span is evaluated and plotted in Fig. 6. By considering the symmetry, 10 three-noded isoprametric curved beam elements with 6 DOF per node are used. For comparison, the results using 8 HMC2 curved beam elements with 7 DOF per node by Gendy and Saleeb (1992) based on a centroid formulation and the solutions using 24 quadrilateral shell elements developed by Saleeb *et al*. (1990) are presented. Investigation of Fig. 6 reveals that the solutions by this study are in good agreement with those obtained from HMC2 elements and shell elements. It should be noted that the present curved beam with non-symmetric cross section which the warping function is zero at the shear center eliminates the total DOF of structures for the dynamic and elastic analysis of curved structures.

5. Conclusions

An improved formulation for the spatially coupled free vibration and elastic analysis of *shear-deformable* thin-walled curved beam is newly proposed. This study is a first attempt to deal with the vibration and elastic analysis of curved beam with non-symmetric cross sections based on the centroid-shear center formulation. Also the isoparametric curved beam element is developed for two cases in which the restrained warping torsion is considered or not, respectively. Through the numerical examples, FE solutions by this study are compared with those from the centroid formulation and the results by available references and ABAQUS's shell elements. Consequently, the following conclusions may be drawn.

1. The vibration and elastic theories of shear-deformable warping-free curved beams may be easily derived from the thin-walled curved beam theory based on the centroid-shear center formulation

by simply putting the sectional properties associated with warping to zero.
2. For vibration and elastic analysis of curved beams with non-symmetric cross sections, the solutions by this study in greatly agreement with those obtained from the centroid formulation.
3. For curved beam with L-shaped cross section, the natural frequencies and displacements obtained from this curved beam with 6 DOF per node are in excellent agreement with those from ABAQUS's shell elements and curved beam elements with 7 DOF per node including warping, respectively.
4. Resultantly it is believed that this study based on the centroid-shear center formulation overcomes the drawbacks of the centroid formulation which has some difficulties in formulating a non-symmetric curved beam neglecting the restrained warping torsion.

Acknowledgements

This work is a part of a research project supported by a grant (R01-2002-000-00265-0) from the Korea Science and Engineering Foundation and Korea Ministry of Construction & Transportation through Korea Bridge Design & Engineering Research Center at Seoul National University. The authors wish to express their gratitude for the financial support.

References

Choi, C.K. and Hong, H.S. (2001), "Finite strip analysis of multi-span box girder bridges by using non-periodic B-spline interpolation", *Struct. Eng. Mech.*, **12**(3), 313-328.
Chucheepsakul, S. and Saetiew, W. (2002), "Free vibrations of inclined arches using finite elements", *Struct. Eng. Mech.*, **13**(6), 713-730.
Cortínez, V.H. and Piovan, M.T. (1999), "Out of plane vibrations of thin-walled curved beams considering shear flexibility", *Struct. Eng. Mech.*, **8**(3), 257-272.
Gendy, A.S. and Saleeb, A.F. (1994), "Vibration analysis of coupled extensional/flexural/torsional modes of curved beams with arbitrary thin-walled sections", *J. Sound Vib.*, **174**, 261-274.
Gendy, A.S. and Saleeb, A.F. (1992), "On the finite element analysis of the spatial response of curved beams with arbitrary thin-walled sections", *Comput. Struct.*, **44**, 639-652.
Gupta, A.K. and Howson, W.P. (1994), "Exact natural frequencies of plane structures composed of slender elastic curved members", *J. Sound Vib.*, **175**, 145-157.
Howson, W.P. and Jemah, A.K. (1999), "Exact out-of-plane natural frequencies of curved Timoshenko beams", *J. Eng. Mech.*, **125**, 19-25.
Kim, M.Y., Kim, N.I. and Min, B.C. (2002), "Analytical and numerical study on spatial free vibration of non-symmetric thin-walled curved beams", *J. Sound Vib.*, **258**(4), 595-618.
Kim, N.I. and Kim, M.Y. (2004), "Spatial free vibration of shear deformable circular curved beams with non-symmetric thin-walled sections", *J. Sound Vib.*, **276**, 245-271.
Piovan, M.T., Cortínez, V.H. and Rossi, R.E. (2000), "Out-of-plane vibrations of shear deformable continuous horizontally curved thin-walled beams", *J. Sound Vib.*, **237**, 101-118.
Raveendranath, P., Singh, G. and Pradhan, B. (2000), "Free vibration of arches using a curved beam element based on a coupled polynomial displacement field", *Comput. Struct.*, **78**, 583-590.
Saada, Adel S. (1974), *Elasticity, Theory and Applications*, Pergamon Press Inc.
Saleeb, A.F., Chang, T.Y., Graf, W. and Yingyeunyoung, S. (1990), "A hybrid/mixed model for nonlinear shell analysis and its applications to large-rotation problems", *Int. J. Numer. Meth. Eng.*, **29**, 407-446.
Timoshenko, S.P. and Gere, J.M. (1961), *Theory of Elastic Stability*. 2nd Ed., McGraw-Hill, N.Y.
Tong, G. and Xu, Q. (2002), "An exact theory for curved beams with any thin-walled open sections", *Advanced*

in Struct. Eng., **5**(4), 195-209.
Vlasov, V.Z. (1961), *Thin-walled Elastic Beams*, 2nd Ed., National Science Foundation, Washington, D.C.
Wilson, J.F. and Lee, B.K. (1995), "In-plane free vibrations of catenary arches with unsymmetric axes", *Struct. Eng. Mech.*, **3**(5), 511-525.
Wilson, J.F., Lee, B.K. and Oh, S.J. (1994), "Free vibrations of circular arches with variable cross-section", *Struct. Eng. Mech.*, **2**(4), 345-357.

Appendix. Equilibrium equations of curved beam

Equilibrium equations in Eqs. (8a-g) can be derived by using the equilibrium conditions of forces and moments of a small segment of curved beam as shown in Fig. 3. First, in the x_1 direction, the components of the axial forces are equal to

$$-F_1 + (F_1 + dF_1)\cos(d\theta) + p_1 dx_1 + (F_3 + dF_3)\sin(d\theta) = 0 \tag{A-1}$$

where $d\theta = dx_1/R$ is a small value and resultantly Eq. (A-1) leads to

$$F_1' + \frac{F_3}{R} = -p_1 \tag{A-2}$$

Similarly, the x_2 and x_3 direction components of the shear forces acting on the element are

$$-F_2 + (F_2 + dF_2) + p_2 dx_1 = 0 \tag{A-3}$$

$$-(F_1 + dF_1)\sin(d\theta) - F_3 + (F_3 + dF_3)\cos(d\theta) + p_3 dx_1 = 0 \tag{A-4}$$

And respectively, Eq. (A-3) and Eq. (A-4) can be rewritten as

$$F_2' = -p_2 \tag{A-5}$$

$$-\frac{F_1}{R} + F_3' = -p_3 \tag{A-6}$$

Additionally the moment equilibrium about the x_1^s, x_2 and x_3 axes, respectively are

$$-M_1 + (M_1 + dM_1) + e_2 F_1 \sin(d\theta) + M_3 \sin(d\theta) + m_1 dx_1 = 0 \tag{A-7}$$

$$-F_3 dx_1 - M_2 + (M_2 + dM_2) + m_2 dx_1 = 0 \tag{A-8}$$

$$-M_3 + (M_3 + dM_3) - M_1^c d\theta + F_2 dx_1 + m_3 dx_1 = 0 \tag{A-9}$$

In Eq. (A-9), M_1^c denotes the twisting moment with respect to the centroid axis and is expressed as

$$M_1^c = M_1 + e_2 F_3 - e_3 F_2 \tag{A-10}$$

Resultantly, Eqs. (A-7), (A-8) and (A-9) are represented, respectively as

$$\frac{e_2}{R} F_1 + M_1' + \frac{M_3}{R} = -m_1 \tag{A-11}$$

$$-F_3 + M_2' = -m_2 \tag{A-12}$$

$$F_2 + \frac{e_3}{R} F_2 - \frac{e_2}{R} F_3 - \frac{M_1}{R} + M_3' = -m_3 \tag{A-13}$$

Finally, to obtain the restrained torsional moment equilibrium about the x_1^s axis, we consider the stress resultant M_R in Eq. (2h).

$$M_R = M_R^* + \frac{e_3}{R}\int_A (\tau_{13}x_2 - \tau_{12}x_3)dA = M_R^* - \frac{e_3^2}{R}F_2 + \frac{e_2 e_3}{R}F_3 + \frac{e_3}{R}M_1 \tag{A-14}$$

where

$$M_R^* = \int_A \left\{ \tau_{12}\phi_{,2} + \tau_{13}\left(\phi_{,3} - \frac{\phi}{R + x_3}\right)\right\}\frac{R + x_3}{R}dA \tag{A-15}$$

Eq. (A-15) can be expressed as

$$M_R^* = \int_A \left\{ \left(\tau_{12}\phi\frac{R + x_3}{R}\right)_{,2} + \left(\tau_{13}\phi\frac{R + x_3}{R}\right)_{,3} - \tau_{12,2}\phi\frac{R + x_3}{R} - \tau_{12,3}\phi\frac{R + x_3}{R} - 2\tau_{13}\phi\frac{1}{R}\right\}dA \tag{A-16}$$

In the linear geometric case, we can consider following relation

$$\int_A \left\{ \left(\tau_{12}\phi\frac{R + x_3}{R}\right)_{,2} + \left(\tau_{13}\phi\frac{R + x_3}{R}\right)_{,3}\right\}dA = \oint\left(\tau_{12}\frac{R + x_3}{R}\alpha_{n2} + \tau_{13}\frac{R + x_3}{R}\alpha_{n3}\right)\phi dS \tag{A-17}$$

where α_{n2} and α_{n3} denote the direction cosine for the initial direction of the exterior normal. The exterior normal n for the cylindrical surface is perpendicular to the x_1 direction. And the stress boundary condition is

$$p_{n1} = \tau_{12}\frac{R + x_3}{R}\alpha_{n2} + \tau_{13}\frac{R + x_3}{R}\alpha_{n3} = 0 \tag{A-18}$$

Based on Eqs. (A-17) and (A-18), Eq. (A-16) leads to

$$M_R^* = -\int_A \left(\tau_{12,2}\frac{R + x_3}{R} + \tau_{12,3}\frac{R + x_3}{R} + 2\tau_{13}\frac{1}{R}\right)\phi dA \tag{A-19}$$

Also using the cylindrical coordinate system, the equilibrium equation is presented by Saada (1974).

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{z\theta}}{\partial z} + \frac{2}{r}\sigma_{r\theta} = 0 \tag{A-20}$$

We can rewrite Eq. (A-20) in the following form

$$\frac{\tau_{11,1}R}{R + x_3} + \tau_{12,2} + \tau_{13,2} + \frac{2\tau_{13}}{R + x_3} = 0 \tag{A-21}$$

Resultantly, Eq. (A-19) is written

$$M_R^* = -\int_A \left(\tau_{12,2}\frac{R + x_3}{R} + \tau_{12,3}\frac{R + x_3}{R} + 2\tau_{13}\frac{1}{R}\right)\phi dA = \int_A \tau_{11,1}\phi dA = M_\phi' \tag{A-22}$$

If we consider the uniformly distributed bimoment, we can obtain the following equilibrium equation.

$$-M_R^* + M_\phi' = -m_\phi \tag{A-23}$$