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Prediction of the dynamic flow stress

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Abstract. This article explores a constitutive equation that is able to correlate stress, strain and strain rate. In order to show the advantages of the constitutive equation here proposed and how its material parameters are obtained, data extracted from the literature, for materials as different as polymers and metallic alloys, are used. Finite element simulation of the impact behaviour of a beam is presented to highlight the care one needs to exercise when using the more traditional Cowper-Symonds equation. The present constitutive equation has shown to be accurate for a wide range of strains, stresses and strain rates.

Key words: dynamic tensile test; constitutive equation; Cowper-Symonds equation.

1. Introduction

The response of structures subjected to severe dynamic loads, such as those found in earthquakes, collisions, explosions and manufacturing processes, depends on many factors. No doubt, the material behaviour plays a decisive role. Indeed, equilibrium equations and kinematic relations of a given structural problem are only related by the material constitutive law, which ultimately is a relation between load and displacement expressed in a stress-strain diagram. Despite the efforts to obtain constitutive equations from more basic physical constants, the material stress-strain behaviour must still be measured. Regardless of the sophistication of a theoretical model, the importance of an experimental procedure is determinant on the knowledge of the material behaviour (Knauss 2000).

Temperature, radiation, anisotropic effects and load path are important aspects which influence the material response. Of particular interest here is the stress level. It is imperative in the field of dynamic structural analysis in the presence of plastic strains, to take into consideration the effects the strain rate causes in the material response. Of course that the stress-strain-strain rate relationship engenders enormous difficulty for structural analysis due to its non-linear character. To overcome this non-linear feature, most of the structural impact analysis models the material as perfectly plastic so that a single parameter, namely the flow stress, σ_0, describes the material response. This flow stress should be corrected for dynamic effects and one way to do so is by using a material constitutive law due to Cowper-Symonds (Jones 1989), also called overstress power law.

The use of the Cowper-Symonds equation consists in performing a series of tensile tests at different velocities. One then specifies which stress level is going to be associated with the flow

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stress. If, for instance, the ultimate stress, σ_u, is to be equal the material flow stress, σ_0, then one needs to measure σ_u at different test velocities and, hence, strain rates. The set of dynamic ultimate stresses plotted against their respective strain rates allows the determination of the two material coefficients which form the basis of the Cowper-Symonds equation.

Thus, a set of pairs ($\sigma_u \times \dot{\varepsilon}_u$) leads to the material constants which, when applied in the Cowper-Symonds constitutive equation, allows the prediction of any dynamic ultimate stress for a given strain rate, $\dot{\varepsilon}$. This can be seen as a restrictive feature of this equation since any other dynamic stress level, e.g. the yielding stress, would call for new material constants. In other words, the so called material constants are, in fact, variables function of the stress level of interest.

It is important to understand that the material constants in the Cowper-Symonds equation are associated with one, and only one, strain level. Dynamic stresses for other strain levels can only be correctly forecast if another pair of constants are determined. This has great experimental consequences because in many cases the material response cannot be measured all the way through fracture due to, for instance, strain gauge failure. Also, it is rather troublesome to use the material parameters in the Cowper-Symonds equation as variables, function of the strain or stress level to be predicted, specially when dealing with theoretical models.

A remedy for this situation was suggested by Alves (2000), where only two material constants were effectively used to predict stresses at any strain and strain rate level. This new constitutive equation is here further explored by applying it to quite distinct strain rate sensitive materials, with the constants evaluated at different strain levels to show the usefulness of the equation.

Also, it is shown here that the total test time and the final specimen lenght measured after a tensile test can be used to obtain information for the constitutive law, which is quite useful given the elementary measurements to be performed. This is an important advantage of the constitutive law here discussed since local strain rates are difficult to measure. Finally, the limitations of the commonly used Cowper-Symonds constitutive law are highlighted by simulating the impact of a mass on a beam using a comercial finite element code.

2. Constitutive equation

The so called Cowper-Symonds equation (Jones 1989) is a known constitutive law in the field of structural impact and it relates material static, σ_{0_s}, and dynamic, σ_{0_d} flow stresses to the uniaxial strain rate, $\dot{\varepsilon}$, according to

$$\sigma_{0_d} = \sigma_{0_s}\left\{1 + \left(\frac{\dot{\varepsilon}}{C}\right)^{1/q}\right\} \tag{1}$$

where C and q are material parameters chosen in order to best describe the material sensitivity to strain rate.

The prediction of this equation for the yield stress, σ_y, of a mild steel is shown in Fig. 1. In this case, the coefficients C and q in Eq. (1) are the ones in the last line of Table 1. Observe that this table lists different coefficients for the Cowper-Symonds equation according to the stress data used. The experimental data in Fig. 1 were obtained testing a series of tensile specimens using head and body gauges, as described elsewhere (Alves 1996, Alves and Jones 2002b).

Fig. 1 clearly shows that, if the coefficients in the Cowper-Symonds equation are kept the same

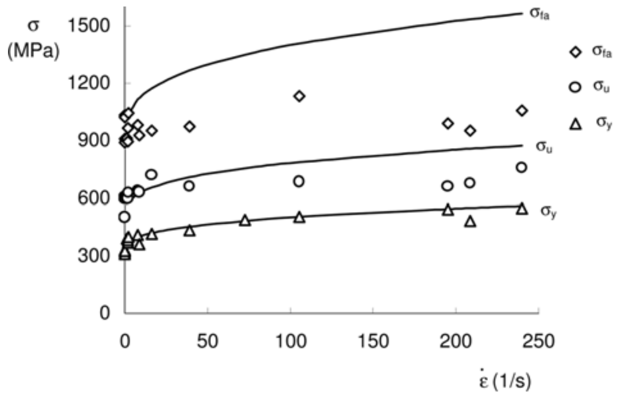

Fig. 1 Prediction of the dynamic yield, ultimate and failure stresses versus the strain rate, according to Eq. (1). The coefficients C and q for all the predictions are listed in the last line of Table 1. The symbols are experimental data for mild steel.

Table 1 Coeffcients of the Cowper-Symonds equation according to the stress used

Based on	C (s^{-1})	q
$\sigma_{y_{up}}$	360.07	3.428
$\sigma_{y_{low}}$	598.13	3.052
$\frac{1}{2}(\sigma_{y_{up}} + \sigma_{y_{low}})$	550.43	3.439

regardless of the stress to be predicted, this constitutive equation yields poor results. For instance, the failure stress, σ_{fa}, in Fig. 1 is not well predicted. A way to avoid this poor performance is to change the parameters C and q according to the stress to be forecast. Such a procedure, however, should be avoided in favor of a constitutive equation which uses the same constants regardless of the stress to be predicted.

An empirical equation which can fit well over the entire range of stresses, strains and strain rates for the present experimental data is (Alves 2000)

$$\frac{\sigma_{eq_d}}{\sigma_{eq_s}} = 1 + \frac{\sigma}{\sigma_{eq_s}}\left(\frac{\dot{\varepsilon}}{C}\right)^{1/q} \tag{2}$$

or

$$\sigma_{eq_d} = \sigma_{eq_s} + \overline{m}\,\dot{\varepsilon}^{\overline{n}} \tag{3}$$

where

$$\overline{m} = \frac{\sigma}{C^{1/q}} \quad \text{and} \quad \overline{n} = \frac{1}{q} \tag{4}$$

Note that in Eq. (3), σ_{eq_d} and σ_{eq_s} are the dynamic and static equivalent stresses and the coefficients C and q in Eq. (2) are determined using the same methodology as for the Cowper-Symonds'. An important remark here is that the parameters σ and $\dot{\varepsilon}$ can be freely chosen based on

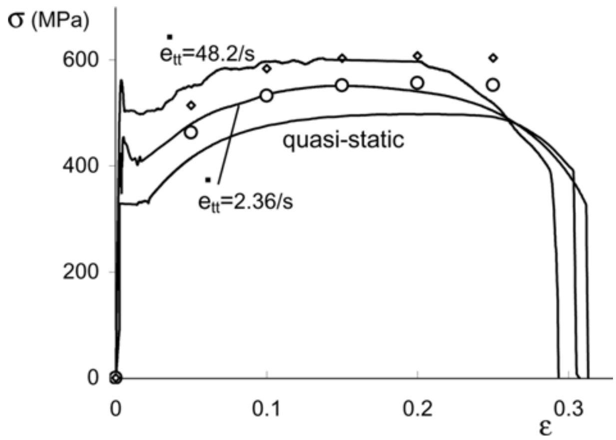

Fig. 2 Engineering stress-strain curve for a BS EN 10025 FE430A steel at various strain rates. The symbols are stress levels predicted according to Eq. (7).

the available experimental data or application. Upon this choice, the coefficients $\overline{m}$ and $\overline{n}$ will assume specific values. Once the coefficients are determined, they are kept constant and yet stresses can be predicted for any strain level provided strain hardening is not sensitive to the strain rate, a common characteristics for many materials. Since $\overline{m}$ and $\overline{n}$ are constants, load-displacement data at large strains can be used to infer the material behaviour even at low strains for a given strain rate, e.g. at the yielding point.

Equally important, the strain rate does not need to be the one at yielding, $\dot{\varepsilon}$. It can be, for instance, the overall strain rate, easily calculated (and measured) from

$$\dot{e}_V = \frac{V}{L_0} \tag{5}$$

where V is the test velocity and L_0 the initial specimen length.

An alternative of obtaining a strain rate measure is to use the total test time, t_t, and the initial and final gauge lengths, such that

$$\dot{e}_{t_t} = \frac{e}{t} = \frac{L - L_0}{L_0 t_t} \tag{6}$$

where e is the engineering strain.

To show that this crude strain rate definition is suitable for the prediction of a material behaviour at any strain or strain rate, consider the steel BS EN 10025 FE430A in Fig. 2. The continuous line represents experimental data, while ○ and △ are the prediction according to equation

$$s_d = s_s(1 + 49.12\dot{e}_{t_t}^{0.21}) \tag{7}$$

where s_d, s_s are the dynamic and quasi-static engineering stress.

The constants in this equation were obtained by plotting $\ln(s_d - s_s) \times \ln(\dot{e}_{t_t})$ and fitting the data with a straight line, from which the coefficients 49.12 $\mathrm{Ns}^{1/q}/\mathrm{m}^2$ and 0.21 were obtained. Note that the stress level used in $\ln(s_d - s_s)$ were arbitrarily chosen to be associated with the strain $e = 0.10$; yet dynamic stresses at any strain level can be predict for a given engineering strain rate, $\dot{e}_{t_t}$.

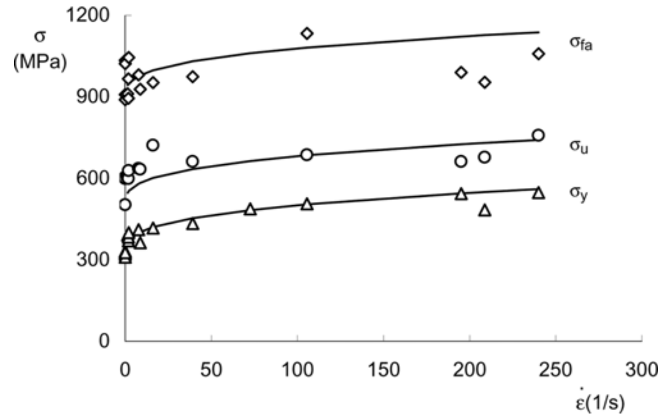

Fig. 3 Experimental results for lower yield, ultimate and failure stresses versus strain rate (Alves and Jones 2002b). The continuous lines are the prediction of Eqs. (9)-(11), with $\overline{m}$ = 38.4 $Ns^{1/q}/m^2$ and $\overline{n}$ = 0.328.

Now, if the yield stress, σ_y, and the strain rate at yielding, $\dot{\varepsilon}$, are used for the evaluation of C and q, Eq. (2) becomes

$$\frac{\sigma_{eq_d}}{\sigma_{eq_s}} = 1 + \frac{\sigma_{y_s}}{\sigma_{eq_s}}\left(\frac{\dot{\varepsilon}}{C}\right)^{1/q} \tag{8}$$

and allows the prediction of, for instance, the dynamic yield, ultimate and failure stresses according to

$$\sigma_{y_d} = \sigma_{y_s} + \sigma_{y_s}\left(\frac{\dot{\varepsilon}}{C}\right)^{1/q} \tag{9}$$

$$\sigma_{u_d} = \sigma_{u_s} + \sigma_{y_s}\left(\frac{\dot{\varepsilon}}{C}\right)^{1/q} \tag{10}$$

and

$$\sigma_{fa_d} = \sigma_{fa_s} + \sigma_{y_s}\left(\frac{\dot{\varepsilon}}{C}\right)^{1/q} \tag{11}$$

respectively. The predictions of these equations are plotted in Fig. 3 together with the experimental data for the steel in Fig. 2.

3. Exploring the constitutive equation

It is important to assure that the constitutive equation here suggested is valid for a relatively wide range of materials. On this ground, more confidence is gained on its use for the typical complex structural problems one is met in design and analysis of structures under severe dynamic loads.

Table 2 gives details about the strength of some important materials tested by different researchers. The stress × strain × strain rate curves will be used here to access the accuracy of the present constitutive equation.

Table 2 Materials used for testing the constitutive law

Material	Ref.	$\overline{m}$ (Ns$^{1/q}$/m^2)	$\overline{n}$	Constants obtained at
Al-6061-T6	(Lee *et al.* 2000)	0.00102	1.22	$\varepsilon = 0.25$
Vanadium	(Nemat-Nasser and Guo 2000)	41.63	0.20	$\varepsilon = 0.50$
304L stainless steel	(Lee and Lin 2001)	6.96	0.44	$\varepsilon = 0.10$
Titanium	(Nemat-Nasser *et al.* 1999)	96.81	0.07	$\varepsilon = 0.10$
Nylon		0.30	0.20	$\varepsilon = 0.18$

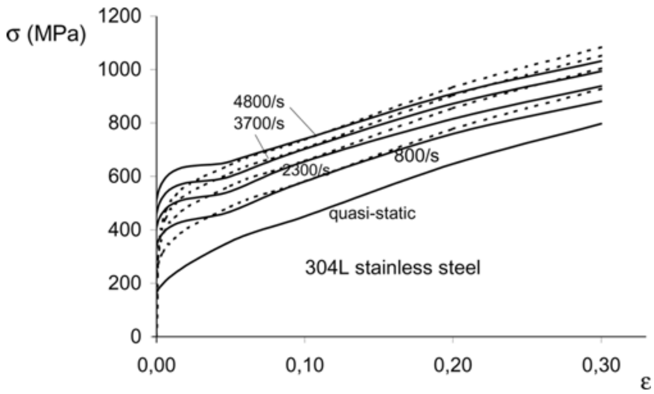

Fig. 4 Experimental results for 304L stainless steel from Lee and Lin (2001) (thick line) together with the new constitutive equation prediction (dashed line)

Consider the 304L stainless steel characterised in Lee and Lin (2001) and shown in Fig. 4. The constitutive law here discussed had its coefficients calculated for a stress level associated with a strain of 0.10. At this point, the constitutive law here coincides with the one from Cowper-Symonds and offers the best prediction for the stresses. For strains different from 0.10, the Cowper-Symonds equation would fail completely in predicting the stresses, as already anticipated in Fig. 1. On the other hand, the present equation uses just the two parameters listed in Table 2 and yet predicts stresses for any strain and strain rate with a reasonable accuracy.

For further illustration purposes, Fig. 5 presents by thick lines the experimental data from various materials, whose behaviour were extracted from the references listed in Table 2. It also shows by the dashed lines the stress response as predicted by the present constitutive equation when using the material constants in Table 2. It is clear that the constitutive Eq. (3) is able to predict well the material response for different strain rates and within a broad range of strains.

In general, the constitutive law here presented provides a reasonable estimation for the stress level by using only two material constants. Hence, Eq. (3) has an important advantage over the Cowper-Symonds as much as any stress level can be predicted using a single pair of constants, $\overline{m}$ and $\overline{n}$. This is in contradistinction to the Cowper-Symonds, which should have different values for C and q in Eq. (1), according to the stress of interest, being it the yielding, the ultimate or the fracture one.

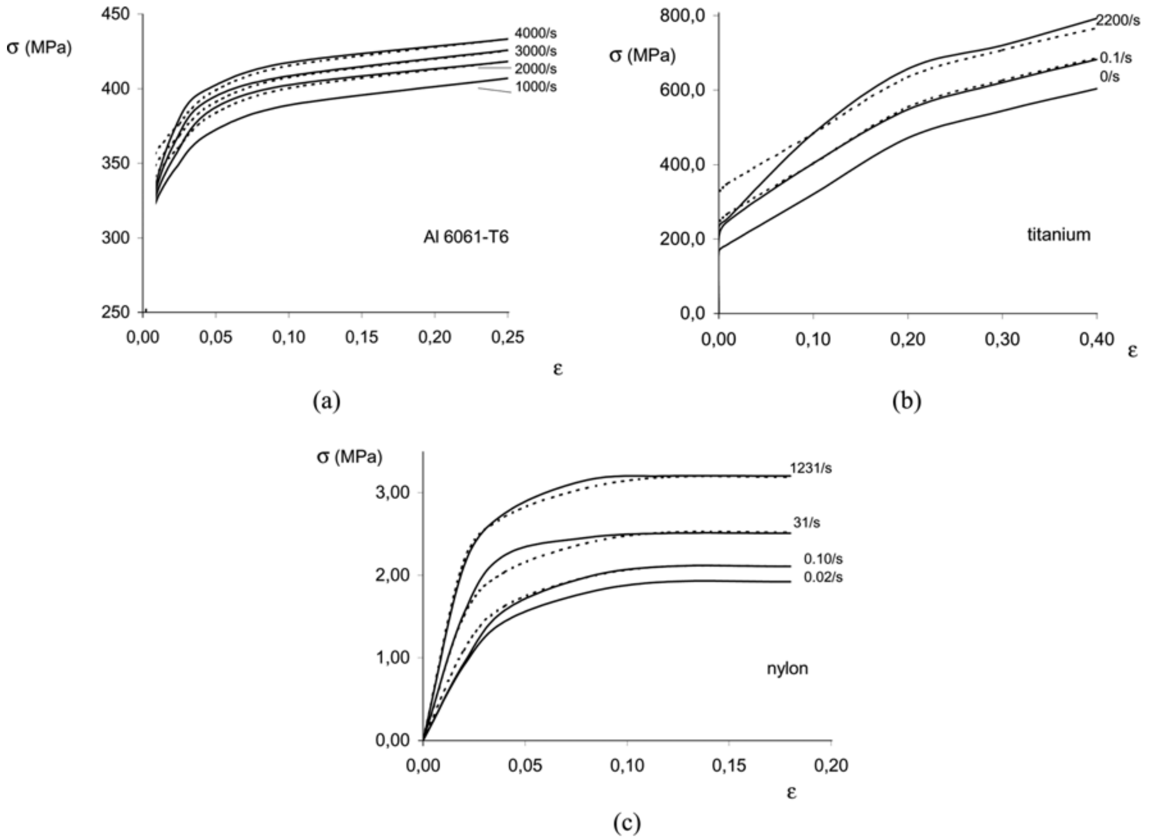

Fig. 5 Experimental results from references in Table 2 given by thick lines. Dashed lines are the predictions according to Eq. (3) and the material constants in Table 2. (a) Structural 6061-T6 aluminium, (b) titanium, (c) nylon

4. Cowper-Symonds equation versus finite-element simulation

The Cowper-Symonds relationship is used in many commercially available finite element analysis packages, including Abaqus. The purpose of this section is to illustrate that errors occur in the stress predictions if the material modelling is done using the normal form of the Cowper-Symonds equation.

The structure used to investigate this issue is a beam impacted centrally. The beam was modelled using 15 second order beam elements (Abaqus element library B22). Full advantage was taken of the symmetry that a central impact offers. A bias placed on the elements position allowed a greater concentration close to both the impactor and the fully clamped end. The dimensions of the beam were taken from beam 5 in Alves and Jones (2002b), i.e., $B = 8.0$ mm, $H = 9.0$ mm and $2L = 102.2$ mm.

The wedge-shaped impactor was modelled using an analytical rigid surface governed by a reference node to which a mass was attached. The initial velocity prescibed to the mass provided the input energy. A series of tests were run in which the mass and velocity were altered but keeping constant the input energy to 250J. This energy was arrived at by conducting a basic static analysis

Table 3 Values of drop mass and initial velocity used in the simulation of a beam under central impact

Case	Mass (kg)	Velocity (m/s)
1	500	1
5	20	5
10	5	10
15	2.22	15
20	1.25	20

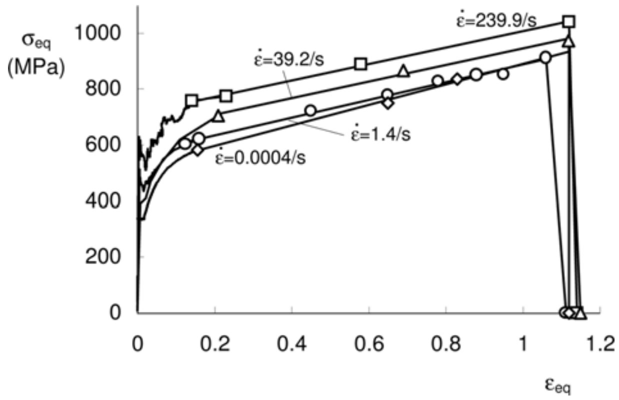

Fig. 6 Equivalent stress-strain curve for steel BS EN 10025 FE430A at different strain rates

of the beam which incorporated the formation of plastic hinges and membrane effects (Alves and Jones 2002a). Table 3 shows the mass and velocity used in each case.

The input equivalent stress-strain curve is the quasi-static one shown in Fig. 6, i.e., $\dot{\varepsilon} = 4 \times 10^{-4}$/s. In Abaqus, the material strain rate sensitivity can be taken into account by the Cowper-Symonds, so that the coefficients $C = 550.43\text{s}^{-1}$ and $q = 3.439$ were input in the code, according to Table 1.

In order to increase the amount of information available in the region surrounding yield, a two step analysis was conducted. The first step used a high sampling frequency to ensure sufficient data was available during the analysis of small strains in the system. The second step permitted the structure to be analysed while undergoing large strains.

The stress, strain and strain rate at the middle bottom of the beam were monitored. Figs. 7(a) and (b) show the true equivalent stress-strain curves and the strain versus time for the various cases in Table 3, respectively.

The finite element simulation using the standard Cowper-Symonds equation with constant coefficients yields wrong results for stress levels associated with different strains and strain rates. This can be concluded because material data is available, Fig. 6, which indicates, for a given strain and strain rate, stress levels significantly different from the ones found in the FE simulation, Fig. 7(a).

Indeed, the stress levels in Fig. 6 should be compared with the ones plotted in Fig. 7(a). It is evident that for similar strain rates the stress values in Fig. 6 are significantly lower than the ones in Fig. 7(a). For instance, test number 15 gives a strain rate of 230.7/s and a peak stress of 1100 MPa in Fig. 7(a). The actual experimental material response for such a strain rate level is around 800 MPa in Fig. 6.

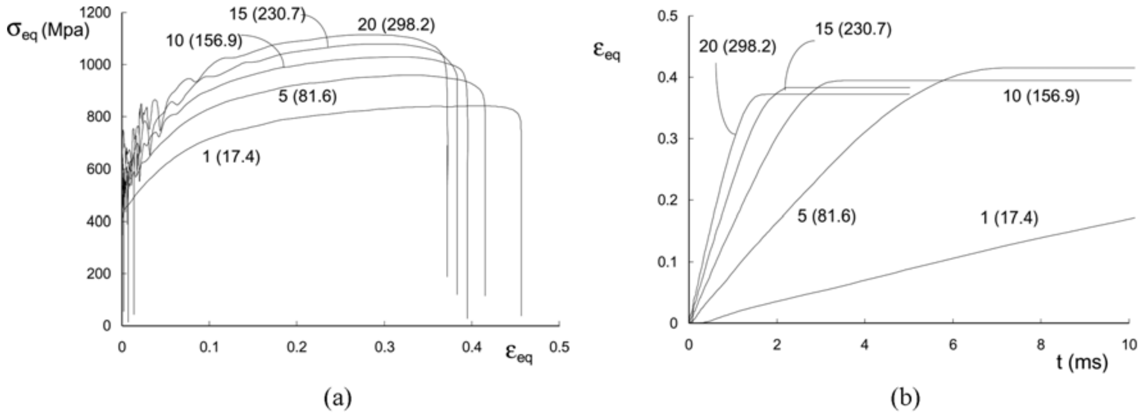

Fig. 7 (a) Equivalent stress-strain curve and (b) equivalent strain versus time at the middle bottom of various impacted beams as calculated by Abaqus. The labels refer to the numbers listed in Table 3, followed by the strain rate (in s^{-1}) calculated using the initial nearly straight line of the respective curve in (b)

The only explanation for the above discrepancy concerning the stress levels is that the coefficients in the Cowper-Symonds equation are kept constant in the simulation, regardless of the strain level. This warns the user of FE codes that the Cowper-Symonds equation cannot be used straightforwardly for the prediction of the dynamic overflow stresses.

5. Conclusions

It is imperative for structural impact analysis to know the material behaviour at high strain rates. Herein, it is proposed to infer the material behaviour using Eq. (3) where its coefficients are evaluated from very elementary test data, e.g. the load at the fracture region and the total test time.

Such a procedure is very sound and has evident advantages over the traditional technique based on accurate load and strain measurements at yielding using head gauges and body gauges, respectively. The approach works fairly well for other materials whose strain hardening behaviour is not very much strain rate sensitive.

Eq. (3) might be seen as an improvement in the Cowper-Symonds equation for it describes better the experimental data. Numerical results using the standard Cowper-Symonds equation show that an error occurs on the prediction of the stress levels for a given strain rate. This is due to its inaccuracy of representing the material behaviour at large strains when its coefficients are kept constants. Hence, it is suggested the use of Eq. (3) in theoretical and numerical models for materials whose strain hardening behaviour is not strain rate sensitive.

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