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The analysis of the plane frames having columns which has principal axes not in the plane of frame

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1. Introduction

The structural analysis of space frames in most cases is not practical because of some complications in data preparation and interpretation of results. Therefore the analysis of reinforced concrete and steel structures realized sometimes by idealization of them as plane frames although they are space frames. This approach seems to be sufficient for the design of structures and can be applied to the idealized plane frames provided that at least one of the principal axes of the elements (beams and columns) is laying in the plane of frame. Although the beams of the frames generally do satisfy this condition the columns may or may not satisfy. For these types of columns the frame plane and bending plane are not coinciding and the columns are actually being under the effect of asymmetric bending. For the structural analysis of such frames the frame must be taken as a space frame (Kuo and Yang 1993, Aristazabal-Ochoa 2003), or a special plane frame analysis technique (Kim and Lee 2000, Nunes and Soriano 2003) which takes into consideration the asymmetric bending of columns must be used. In this work a displacement type of finite element method is proposed for a plane frame having columns which has principal axes not in the plane of frame.

The global axes of the plane frame (X, Y, Z) and the local axes of column 1 $(\hat{x}, \hat{y}, \hat{z})$ are shown in Fig. 1. In the analysis of a frame using the method of Finite Elements it is sufficient to define the stiffness matrix of the column in the X, Y system. The local displacements and the forces can be determined by transformation.

2. Local stiffness matrix

The local displacements and forces of an element of a planar and space frame are shown in Fig. 2. $\underline{\hat{u}}_s$ and $\underline{\hat{S}}_s$ are local displacement and force vectors respectively.
The local equilibrium equations for both types of elements (Przemieniecki 1968):

Key words: finite element; plane frames; principal axes not in the plane; space frames.

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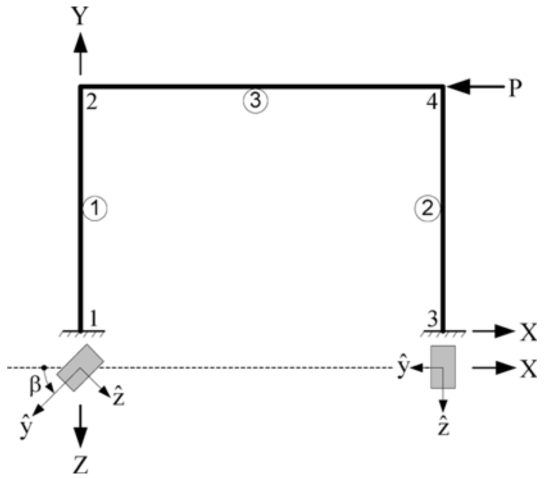

Fig. 1 The locations of the plane frame and columns on the plan

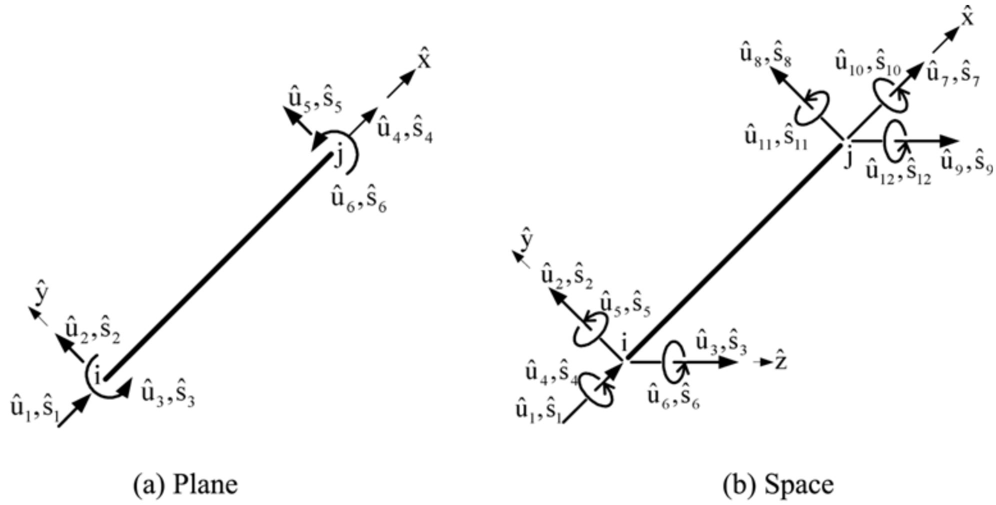

Fig. 2 The local components of frame element

$$\hat{\underline{S}}_s = \hat{\underline{k}}_s \hat{\underline{u}}_s \tag{1}$$

$$\hat{\underline{S}}_p = \hat{\underline{k}}_p \hat{\underline{u}}_p \tag{2}$$

where the indices *s* and *p* represents the space and plane respectively.

Here, $\hat{\underline{k}}_s$ [12 by 12] and $\hat{\underline{k}}_p$ [6 by 6] are local stiffness matrices. These matrices which can be found in any FEM book (Przemieniecki 1968, Balfour 1992, Smith 1998) and rigid body motions, are given below to accomplish the consistency in notation.

$$\underline{\hat{k}}_s = \begin{bmatrix} k_{11} & & & & & & & & & & & \\ 0 & k_{22} & & & & & & & & & & \\ 0 & 0 & k_{33} & & & & & & & & & \\ 0 & 0 & 0 & k_{44} & & & & \text{Symmetric} & & & & \\ 0 & 0 & k_{53} & 0 & k_{55} & & & & & & & \\ 0 & k_{62} & 0 & 0 & 0 & k_{66} & & & & & & \\ -k_{11} & 0 & 0 & 0 & 0 & 0 & k_{11} & & & & & \\ 0 & -k_{22} & 0 & 0 & 0 & -k_{62} & 0 & k_{22} & & & & \\ 0 & 0 & -k_{33} & 0 & -k_{53} & 0 & 0 & 0 & k_{33} & & & \\ 0 & 0 & 0 & -k_{44} & 0 & 0 & 0 & 0 & 0 & k_{44} & & \\ 0 & 0 & k_{53} & 0 & k_{115} & 0 & 0 & 0 & -k_{53} & 0 & k_{55} & \\ 0 & k_{62} & 0 & 0 & 0 & k_{126} & 0 & -k_{62} & 0 & 0 & 0 & k_{66} \end{bmatrix} \tag{3}$$

$$\underline{\hat{k}}_p = \begin{bmatrix} k_{11} & & & & & \\ 0 & k_{22} & & & & \\ 0 & k_{62} & k_{66} & & \text{Symmetric} & \\ -k_{11} & 0 & 0 & k_{11} & & \\ 0 & -k_{22} & -k_{62} & 0 & k_{22} & \\ 0 & k_{62} & k_{126} & 0 & -k_{62} & k_{66} \end{bmatrix} \tag{4}$$

$$\left.\begin{aligned} k_{11} &= EA/L & k_{53} &= -6EI_y/[L^2(1+\phi_z)] \\ k_{22} &= 12EI_z/[L^3(1+\phi_y)] & k_{62} &= 6EI_z/[L^2(1+\phi_y)] \\ k_{33} &= 12EI_y/[L^3(1+\phi_z)] & k_{115} &= (2-\phi_z)EI_y/[L(1+\phi_z)] \\ k_{44} &= GJ/L & k_{126} &= (2-\phi_y)EI_z/[L(1+\phi_y)] \\ k_{55} &= (4+\phi_z)EI_y/[L(1+\phi_z)] & & \\ k_{66} &= (4+\phi_y)EI_z/[L(1+\phi_y)] & & \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} \phi_y &= 12EI_z/(GA_y L^2) \\ \phi_z &= 12EI_y/(GA_z L^2) \end{aligned}\right\} \tag{6}$$

where E and G are the modulus of elasticity and shear modulus, A and L are the cross-sectional area and length, A_y and A_z are the areas which resists against the shear forces in directions $\hat{y}$ and $\hat{z}$, I_y and I_z are the moment of Inertia of the cross-section due to $\hat{y}$ and $\hat{z}$ axes respectively and J is the torsional moment of Inertia of the cross-section.

3. Global stiffness matrix

The transformations of the local stiffness matrix to the global stiffness matrix will be given here. The global displacement vector of the column can be defined as $\underline{u}_s = [u_1, u_2, u_3, \ldots, u_{11}, u_{12}]^T$.

The force vector, which creates a work because of these displacements, is $\underline{S}_s = [S_1, S_2, S_3, \ldots, S_{11}, S_{12}]^T$.

The relations between the local $\hat{\underline{u}}_s$ and global $\underline{u}_s$ displacement vectors and local $\hat{\underline{k}}_s$ and global $\underline{k}_s$ can be given as below (Przemieniecki 1968, Felton 1997).

$$\hat{\underline{u}}_s = \underline{T}\,\underline{u}_s \tag{7}$$

$$\underline{k}_s = \underline{T}^T \hat{\underline{k}}_s \underline{T} \tag{8}$$

$$\underline{T} = \begin{bmatrix} \underline{t} & & & \\ & \underline{t} & & \\ & & \underline{t} & \\ & & & \underline{t} \end{bmatrix} \quad \text{where} \quad \underline{t} = \begin{bmatrix} 0 & 1 & 0 \\ -\mathrm{Cos}\beta & 0 & \mathrm{Sin}\beta \\ \mathrm{Sin}\beta & 0 & \mathrm{Cos}\beta \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -c & 0 & s \\ s & 0 & c \end{bmatrix} \tag{9}$$

The submatrix $\underline{t}$ transforms the global quantities to local ones.

The global stiffness matrix $\underline{k}_s$ can be obtained using Eq. (8):

$$\underline{k}_s = \left[\begin{array}{cccccccccccc}
\tilde{k}_{11} & & & & & & & & & & & \\
0 & \tilde{k}_{22} & & & & & & & & & & \\
\tilde{k}_{31} & 0 & \tilde{k}_{33} & & & & & & & & & \\
\tilde{k}_{41} & 0 & \tilde{k}_{43} & \tilde{k}_{44} & & & & & \text{Symmetric} & & & \\
0 & 0 & 0 & 0 & \tilde{k}_{55} & & & & & & & \\
\tilde{k}_{61} & 0 & -\tilde{k}_{41} & \tilde{k}_{64} & 0 & \tilde{k}_{66} & & & & & & \\
-\tilde{k}_{11} & 0 & \tilde{k}_{73} & -\tilde{k}_{41} & 0 & -\tilde{k}_{61} & \tilde{k}_{11} & & & & & \\
0 & -\tilde{k}_{22} & 0 & 0 & 0 & 0 & 0 & \tilde{k}_{22} & & & & \\
\tilde{k}_{73} & 0 & -\tilde{k}_{33} & -\tilde{k}_{43} & 0 & \tilde{k}_{41} & -\tilde{k}_{73} & 0 & \tilde{k}_{33} & & & \\
\tilde{k}_{41} & 0 & \tilde{k}_{43} & -k_{104} & 0 & \tilde{k}_{106} & -\tilde{k}_{41} & 0 & -\tilde{k}_{43} & \tilde{k}_{44} & & \\
0 & 0 & 0 & 0 & -\tilde{k}_{55} & 0 & 0 & 0 & 0 & 0 & \tilde{k}_{55} & \\
\tilde{k}_{61} & 0 & -\tilde{k}_{41} & \tilde{k}_{106} & 0 & \tilde{k}_{126} & -\tilde{k}_{61} & 0 & \tilde{k}_{41} & \tilde{k}_{64} & 0 & \tilde{k}_{66}
\end{array}\right] \tag{10}$$

Here,

$$\left.\begin{array}{lll}
\tilde{k}_{11} = k_{22}c^2 + k_{33}s^2 & \tilde{k}_{22} = k_{11} & \tilde{k}_{31} = -(k_{22} + k_{33})c^2s^2 \\
\tilde{k}_{33} = k_{22}s^2 + k_{33}c^2 & \tilde{k}_{41} = -(k_{62} + k_{53})c^2s^2 & \tilde{k}_{43} = k_{62}s^2 + k_{53}c^2 \\
\tilde{k}_{44} = k_{55}c^2 + k_{66}s^2 & \tilde{k}_{55} = k_{44} & \tilde{k}_{61} = -k_{62}c^2 + k_{53}s^2 \\
\tilde{k}_{64} = (-k_{55} + k_{66})c^2s^2 & \tilde{k}_{66} = k_{55}s^2 + k_{66}c^2 & \tilde{k}_{73} = (k_{22} - k_{33})c^2s^2 \\
\tilde{k}_{104} = k_{115}c^2 + k_{126}s^2 & \tilde{k}_{106} = (-k_{115} + k_{126})c^2s^2 & \tilde{k}_{126} = k_{115}s^2 + k_{126}c^2
\end{array}\right\} \tag{11}$$

4. Global plane stiffness matrix

The global stiffness matrix given in Eq. (10) must be reduced to the plane of frame X-Y in order to use it for the analysis of a frame, which is given in Fig. 1. For a plane frame the displacement components normal to the frame plane must be zero, $u_3 = u_4 = u_5 = u_9 = u_{10} = u_{11} = 0$. In order to be in equilibrium, the total potential of the column must be minimum.

$$\pi_{\text{column}} = \frac{1}{2}\underline{u}_s^T \underline{k}_s \underline{u}_s - \underline{u}_s^T \underline{S}_s \tag{12}$$

After the analysis of Eq. (12), we conclude that there won't be any contribution from the zero terms of Eq. (12) to the total potential of the column. Therefore, the total potential that is calculated by using $[6 \times 6]$ matrix of $\underline{k}_p$ obtained by discarding the rows and columns corresponding the zero displacement components u_3, u_4, u_5, u_9, u_{10} and u_{11} in Eq. (12) will not be changed.

$$\pi_{\text{column}} = \frac{1}{2}\underline{u}_p^T \underline{k}_p \underline{u}_p - \underline{u}_p^T \underline{S}_p \tag{13}$$

Here $\underline{u}_p$ and $\underline{S}_p$ are the displacement vectors obtained by discarding the terms corresponding the zero displacements from $\underline{u}_s$ and $\underline{S}_s$. The expression, which makes the Eq. (13) minimum, $(\partial\pi_{\text{column}} / \partial\underline{u}_p = 0)$

$$\underline{k}_p \underline{u}_p = \underline{S}_p \tag{14}$$

is the equilibrium equation of the column in global X-Y system, where;

$$\underline{k}_p = \begin{bmatrix} \tilde{k}_{11} & & & & & \\ 0 & \tilde{k}_{22} & & & \text{Symmetric} & \\ \tilde{k}_{61} & 0 & \tilde{k}_{66} & & & \\ -\tilde{k}_{11} & 0 & -\tilde{k}_{61} & \tilde{k}_{11} & & \\ 0 & -\tilde{k}_{22} & 0 & 0 & \tilde{k}_{22} & \\ \tilde{k}_{61} & 0 & \tilde{k}_{126} & -\tilde{k}_{61} & 0 & \tilde{k}_{66} \end{bmatrix} \tag{15}$$

5. Local displacements and forces

The equilibrium equations of the frame can be established by using the global stiffness matrix of element 1 which is given in Eq. (15).

The displacements of the frame at points i and j which are found, must be equivalent to the global displacements of the columns joined at joints i and j. The following expression is obtained using the transformation in Eq. (7).

$$\underline{\hat{u}}_s = [U_{i2}\ \ -cU_{i1}\ \ sU_{i1}\ \ 0\ \ sU_{i3}\ \ cU_{i3}\ \ U_{j2}\ \ -cU_{j1}\ \ sU_{j1}\ \ 0\ \ sU_{j3}\ \ cU_{j3}]^T \tag{16}$$

The local forces of column now can be determined using Eq. (1).

Example (Fig. 1)

The loading, the dimensions and the material properties

$E = 2.10^7$ kN/m^2, $G = 10^7$ kN/m^2, Section : 0.25 × 0.50 mxm, Beam span = 5 m, Height = 3 m, P = 10 kN, $\beta = 45^\circ$

Table 1 Properties of the members in example

Member	A (m^2)	I_z (m^4)	I_y (m^4)	J (m^4)	β (degree)
1	0.125	0.002604	0.000651	0.001789	45
2	0.125	0.000651	0.002604	0.001789	0
3	0.125	0.002604	0.000651	0.001789	-

This frame is analyzed using SAP 2000 and the proposed method. The results for the selected node and member are outlined below. Values without parenthesis belong to SAP 2000 and the ones in parenthesis belong to this work. The shear strains are neglected in both of the methods and the sign convention of SAP 2000 is used for both of the analysis in order to make a comparison (Habibullah 1986, Wilson 1999).

Table 2 Results of global displacements at Node 2

Node	X Direction	Y Direction	Rotation $-Z$
2	−0.6863 E-3 (−0.68633 E-3)	0.3123 E-5 (0.312 E-5)	0.1647 E-3 (0.16474 E-3)

Table 3 Results of local forces for Member 1

Member	Node	N_x	Q_y	Q_z	M_x	M_y	M_z
1	1	−2.60 (−2.60)	−7.19 (−7.19)	1.80 (1.80)	0 (0)	−3.20 (−3.20)	12.81 (12.81)
	2	−2.60 (−2.60)	−7.19 (−7.19)	1.80 (1.80)	0 (0)	2.19 (2.19)	−8.76 (−8.76)

6. Conclusions

In the analysis of a plane frame having columns which have principal axes not coinciding with the plane of the frame, the global stiffness matrix in Eq. (15) the local displacement vector in Eq. (16) and the force vector in Eq. (1) must be used.

The analysis of the plane frame in Fig. 1 is done for $\beta = 0^\circ, 45^\circ, 90^\circ, ..., 360^\circ$ and only the results for $\beta = 45^\circ$, is given in Example. As is seen in the Tables 2 and 3 in comparison with the other solutions the results are exactly the same.

References

Aristizabal-Ochoa, J.D. (2003), “Elastic stability and second order analysis of three dimensional frames: Effects of column orientation”, *J. Eng. Mech.*, ASCE, **129**(11), 1254-1267.

Balfour, J.A.D. (1992), *Computer Analysis of Structural Frameworks*, Oxford University Press, New York.

Felton, P.L. and Nelson, R.B. (1997), *Matrix Structural Analysis*, John Wiley & Sons, Inc., New York.

Habibullah, A. (1986), “Extended three dimensional analysis of building systems”, *Comput. Struct.*, Inc., Berkeley, California USA.

Kim, J.K. and Lee, S.S. (2000), “The behavior of reinforced concrete columns subjected to axial forces and biaxial bending”, *Eng. Struct.*, **22**(11), 1518-1528.

Kuo, S.-R., Yang, Y.-B. and Chou, J.-H. (1993), “Nonlinear analysis of space frames with finite rotations”, *J. Struct. Eng.*, **119**, 1-15.

Nunes, C.C., Soriano, H.L. and Venancio filho, F. (2003), “Geometric non-linear analysis of space frame with rotation greater than 90°, with Euler angles and quasi-fixed local axes system”, *Int. J. of Non-Linear Mech.*, **38**(8), 1195-1204

Przemieniecki, J.S. (1968), *Theory of Matrix Structural Analysis*, McGraw-Hill, New York.

Smith, I.M. and Griffiths, D.V. (1998), *Programming the Finite Element Method*, John Wiley & Sons, West Sussex, England.

Wilson, E.L. (1999), *SAP 2000 Three Dimensional Static and Dynamic Analysis of Structures - Analysis Reference,* Third Edition, Computers and Structures, Inc., Berkeley, California USA.