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Empirical formulas to estimate cable tension by cable fundamental frequency

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Abstract. The cable tension plays an important role in the construction, assessment and long-term health monitoring of cable structures. The cable vibration equation is nonlinear if cable sag and bending stiffness are included. The engineering implementation of a vibration-based cable tension evaluation is mostly carried out by the simple taut string theory. However, the simple theory may cause unacceptable errors in many applications since the cable sag and bending stiffness are ignored. From the practical point of view, it is necessary to have empirical formulas if they are simple and yet accurate. Based on the solutions by means of energy method and fitting the exact solutions of cable vibration equations where the cable sag and bending stiffness are respectively taken into account, the empirical formulas are proposed in the paper to estimate cable tension based on the cable fundamental frequency only. The applicability of the proposed formulas is verified by comparing the results with those reported in the literatures and with the experimental results carried out on the stay cables in the laboratory. The proposed formulas are straightforward and they are convenient for practical engineers to fast estimate the cable tension by the cable fundamental frequency.

Key words: cable structure; fundamental frequency; cable tension; practical formulas; vibration method; cable sag; cable bending stiffness.

1. Introduction

Cables, which are normally held in position by tensions along their length and end forces at supports, are very efficient structural members used in tension structures such as cable-supported bridges, cable roofs, and guyed towers/masts etc. due to their high strength and light weight (Leonard 1998). The uses of cables may vary from several kilometers long in the case of suspension bridges to less than a meter in the case of musical string instruments. Of particular concerns in cable structures are the cable tension forces and their variations since cable tensions control the internal force distribution as well as structural alignments. As a result, the cable tension plays an important role in the quality control during construction, the health monitoring during service, as well as a cheap and quick maintenance of cable-supported structures. A simple, speedy and reliable

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technique to estimate the cable tension is often required for the practical engineers.

The direct measurement of cable tension forces can be carried out by lift-off tests, which provide the actual stresses in the cable only. However, these tests cannot be repeated too often because the wedges to anchor the strands get damaged and valuable corrosion protection has to be removed. Another way is to pre-install the calibrated load cells under the cable anchorage during construction. But the initial cost of these heavy load cells has frequently limited their use. Considerable shift and drift in their output has also questioned the load cell application for long-term monitoring of tensioned cables. A method involving attachment of strain gages to stranded systems or wires seems neither appropriate nor reliable for cable-supported structures.

Instead of direct measurement of cable forces, vibration-based methods have been most widely used in the estimation of cable tension forces (Casas 1994, Yen *et al.* 1997, Russell and Lardner 1998, Cunha *et al.* 2001). The cable vibrations excited by either ambient or manual sources are first measured and the cable vibration frequencies are identified accordingly. Cable tensions can then be indirectly calculated from the measured cable frequencies using the relationship between cable tensions and their corresponding natural frequencies. The simplest relationship to calculate the cable forces based on cable vibration frequencies is so called chord equation where the cables are idealized as taut strings. This idealization simplifies the analysis but may cause unacceptable errors in many situations since the cable sag and bending stiffness effects are neglected. The taut-string formula should be modified by taking into account cable's sag and bending stiffness effects.

The increasing use of cable structures in modern construction requires a better understanding of cable dynamic behavior. A history of the derivation and dynamic solutions of inclined cables was intensively studied by Irvine (1981, 1974). Many investigators (Starossek 1991, 1994, Mehrabi and Tabatabai 1998, Zheng *et al.* 2001, Ni *et al.* 2002) have addressed the effects of sag extensibility or bending stiffness on cable vibrations where various analytical tools, finite difference (FD) method and nonlinear finite element (FE) method were implemented. The difficulty to include the cable sag and bending stiffness is that there is no closed-form solution between cable tensions and frequencies where either the equations are nonlinear or the solutions are the transcendental equations. Nowadays, with the widespread use of personal computers and good software such as MATLAB, it is not a difficult task to solve these nonlinear equations. Nevertheless, from practical point of view, it may be good to have empirical formulas to calculate the cable tension force through measuring the cable frequencies if they are simple and yet accurate. Practical formulas to estimate the cable forces from measured frequencies were proposed by Zui *et al*. (1996) where the first and second natural frequencies of cable vibration are needed. A higher frequency difference-based method was presented including the effects of cable sag and bending stiffness (Chen 1994).

Considering that the fundamental frequency is easily implemented by practical engineers, the aim of the paper is to derive some empirical formulas to estimate the cable tensions based on the cable fundamental frequency. The energy method is first applied to solve the equations of cable vibration and the relationships between cable tensions and cable fundamental frequency are established. The cable sag and bending stiffness are studied respectively depending on the cable types and behavior. The curve fitting method is then applied to the exact solutions of cable vibration considering the effects of cable sag and cable bending stiffness respectively. Finally, the empirical formulas are given where the cable tensions can be easily estimated by the measured fundamental frequency. The errors between the proposed formulas and the exact solutions are less than 1%. The applicability of the empirical formulas is verified with the comparison of the results reported in the literature.

While the understanding of the dynamic behavior of cables is quite complete, there has been little experimental verification of the predictions for cable tension forces (Russell and Lardner 1998). One of the purposes of this paper is to present the laboratory dynamic tests on stay cables with different tension levels. Of particular interest is to compare the actual cable tensions with those predicted from the proposed empirical formulas by using measured cable fundamental frequency. The practical formulas match well with the experimental data. The verification of this practical tool is addressed in the paper.

2. Sag effect on cable fundamental frequency

Fig. 1 shows the schematic diagram of a vibrating cable. The following assumptions are made in the analysis:

- The cable is horizontal. In the case of inclined cable, the component of self weight in the chord direction is ignored.
- The sag-to-cable span ratio is less than 1:8.
- The bending stiffness of cable is ignored.

The motion equations of above cables subjected to a small perturbation around the static equilibrium position were derived by Irvine and Caughey (1974)

$$\frac{\partial}{\partial s}\left\{(T+\tau)\left(\frac{dx}{ds}+\frac{\partial u}{\partial s}\right)\right\} = m\frac{\partial^2 u}{\partial t^2} \tag{1a}$$

$$\frac{\partial}{\partial s}\left\{(T+\tau)\left(\frac{dy}{ds}+\frac{\partial v}{\partial s}\right)\right\} = m\frac{\partial^2 v}{\partial t^2} - mg \tag{1b}$$

$$\frac{\partial}{\partial s}\left\{(T+\tau)\frac{\partial w}{\partial s}\right\} = m\frac{\partial^2 w}{\partial t^2} \tag{1c}$$

where s is the arc length co-ordinate; u is the longitudinal in-plane component of motion; v is the vertical in-plane component of motion; w is the out-of-plane component of motion; m is the mass per unit length of the cable; T is cable tension; and τ is the additional cable tension due to the vibration.

For the in-plane vibration, the solutions of cable natural frequencies are as follows (Irvine and Caughey 1974):

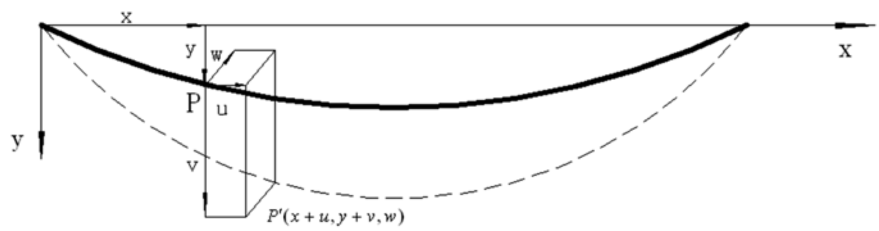

Fig. 1 Schematic of vibrating cable

(i) Anti-symmetric modes

$$\omega_n = \frac{2n\pi}{l}\sqrt{\frac{H}{m}} \qquad n = 1, 2, 3, \dots \tag{2}$$

(ii) Symmetric modes

$$\tan\left(\frac{\beta l}{2}\right) = \left(\frac{\beta l}{2}\right) - \frac{4}{\lambda^2}\left(\frac{\beta l}{2}\right)^3 \tag{3}$$

in which ω_n is the nth order circular frequency of natural vibration. H is the cable force in the chord direction where $H = T\frac{dx}{ds} =$ constant and $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{mgx}{H}\right)^2}$. l is the chord length of cable. Two important characteristic parameters of Eq. (3) are defined by

$$\beta^2 = \frac{m\omega^2}{H}; \qquad \lambda^2 = \left(\frac{mgl}{H}\right)^2\frac{EAl}{HL_e} \tag{4}$$

where E is the modulus of elasticity of cable material; A is the cross-sectional area of cable; and $L_e = \int_0^l \left(\frac{ds}{dx}\right)^3 dx \approx l\left\{1+\frac{1}{8}\left(\frac{mgl}{H}\right)^2\right\}$.

The nature of the roots of Eq. (3) highly depends on the size of the non-dimensional parameter λ^2 where the first term reflects the cable geometry effect (cable sag effect) and the second term reflects the cable elasticity (EA) effect. The eigenvalue problem specified by Eq. (3) is nonlinear with respect to this parameter. The solution of cable in-plane symmetric modes (3) is a transcendental equation where there is no explicit relationship between cable tension force and natural frequencies. The energy method is used here to find out this relationship approximately.

The static equilibrium configuration of a cable under its self weight can be expressed as a parabola from the assumption where the ratio of sag to cable span is less than 1:8 (Irvine 1981)

$$y = \frac{mgl^2}{2H}\left\{\frac{x}{l} - \left(\frac{x}{l}\right)^2\right\} \tag{5}$$

By using Lagrangian strain measure, the cable extensional strain due to vibration around its static equilibrium position is (Ni *et al.* 2002)

$$\varepsilon = \frac{dx}{ds}\frac{\partial u}{\partial s} + \frac{dy}{ds}\frac{\partial v}{\partial s} + \frac{1}{2}\left(\frac{\partial u}{\partial s}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial s}\right)^2 \tag{6}$$

Since the longitudinal displacement is small, its quadratic derivative term can be ignored. (The profile of cable is shallow and the longitudinal displacement is considered unimportant.) Thus the extensional strain becomes

$$\varepsilon = \frac{dy}{ds}\frac{\partial v}{\partial s} + \frac{1}{2}\left(\frac{\partial v}{\partial s}\right)^2 \tag{7}$$

The following formulation is derived from the principle of virtual displacements:

$$\delta W_I = \delta W_E \tag{8}$$

where δW_E is the external virtual work due to the fictitious inertia force f_I acting through the virtual displacements δv:

$$\delta W_E = \int_0^L f_I \delta v ds \tag{9a}$$

where L is the arc length of cable and by D'Alembert's principle

$$f_I = -m\frac{\partial^2 v}{\partial t^2} \tag{9b}$$

The infinitesimal internal virtual work $d(\delta W_I)$ is due to the initially existed cable force T and additional tension force $EA\varepsilon$ acting through the virtual elongation of the infinitesimal section of cable $\delta\varepsilon ds$:

$$d(\delta W_I) = EA\varepsilon \cdot \delta\varepsilon ds + T \cdot \delta\varepsilon ds \tag{10a}$$

Therefore

$$\delta W_I = \int_0^L \delta\varepsilon(EA\varepsilon + T)ds \tag{10b}$$

By implementation of variable separation, $v(x, t)$ becomes

$$v(x, t) = \phi(x)q(t) \tag{11}$$

in which the shape function $\phi(x)$ must satisfy the displacement boundary conditions. Substituting (11) into (7), (9) and (10), after arc length integral, the final expressions for δW_E and δW_I can be obtained. Then the principle of virtual displacements (9) yields

$$\delta q[M\ddot{q}(t) + K_1 q(t) + K_2 q^2(t) + K_3 q^3(t) - P] = 0 \tag{12}$$

From Eq. (11), the governing equation of cable motion can be obtained as

$$M\ddot{q}(t) + K_1 q(t) + K_2 q^2(t) + K_3 q^3(t) = P \tag{13a}$$

where

$$M = \int_0^l m\phi^2(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx \tag{13b}$$

$$K_1 = \int_0^l \left[H\left(\frac{d\phi}{dx}\right)^2 + EA\left(\frac{dy}{dx}\frac{d\phi}{dx}\right)^2\left(\frac{dx}{ds}\right)^3\right]dx \tag{13c}$$

$$K_2 = \int_0^l \left[\frac{3}{2}EA\frac{dy}{dx}\left(\frac{d\phi}{dx}\right)^3\left(\frac{dx}{ds}\right)^3\right]dx \tag{13d}$$

$$K_3 = \int_0^l \left[\frac{1}{2}EA\left(\frac{d\phi}{dx}\right)^4\left(\frac{dx}{ds}\right)^3\right]dx \tag{13e}$$

$$P = \int_0^l \left[H\left(\frac{dy}{dx}\frac{d\phi}{dx}\right) \right] dx \tag{13f}$$

It can be found that the cable stiffness includes the first order stiffness K_1, quadratic stiffness K_2, and cubical stiffness K_3. In this paper, only the first order stiffness term is included and the higher order stiffness terms are ignored. Therefore, the fundamental frequency of the symmetric in-plane mode which takes into account the sag effect can be expressed as

$$\omega^2 = \frac{K_1}{M} = \frac{\int_0^l \left[H\left(\frac{d\phi}{dx}\right)^2 + EA\left(\frac{dy}{dx}\frac{d\phi}{dx}\right)^2 \left(\frac{dx}{ds}\right)^3 \right] dx}{\int_0^l m\phi^2(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx} \tag{14}$$

The first order vibration shape function $\phi(x)$ of a cable is approximately selected as a parabola that is the cable defection shape under self weight:

$$\phi(x) = a(lx - x^2) \tag{15}$$

which satisfies the boundary condition of a vibrating string: $\phi(0) = \phi(l) = 0$. Substituting it into Eq. (14), the fundamental frequency of a cable yields

$$\omega^2 = 10\frac{H}{ml^2} + \frac{3}{2}\frac{EA}{m}\left(\frac{mg}{H}\right)^2 \tag{16}$$

Eq. (16) is in fact the explicit relationship between the fundamental frequency and the cable tension, which is the approximate solution of transcendental Eq. (3) for the first order natural frequency. To compare the approximate solution (16) with the exact solution specified by Eq. (3), both are plotted in Fig. 2 and the effects of sag-extensibility on the cable fundamental frequency can

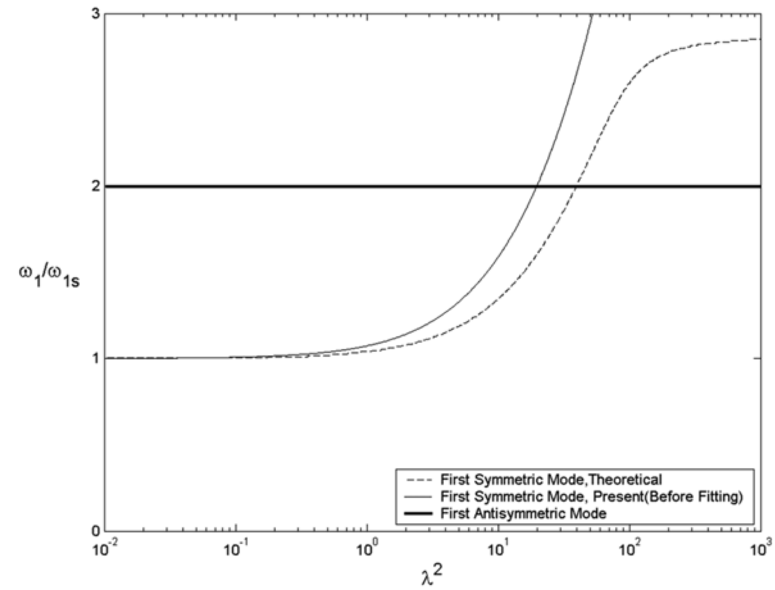

Fig. 2 Fundamental frequency ratio versus parameter λ^2 before fitting

be demonstrated. The curves in Fig. 2 are the relationship between the non-dimensional frequency $\overline{\omega}$ and non-dimensional characteristic parameter λ^2 for the first symmetric or the first anti-symmetric in-plane modes. The non-dimensional frequency $\overline{\omega} = \omega/\omega_{1s}$ is defined as the frequency ratio where ω_{1s} is the fundamental frequency of the taut string theory

$$\omega_{1s} = \frac{\pi}{l}\sqrt{\frac{H}{m}} \tag{17}$$

Fig. 2 clearly demonstrates the "modal crossover" phenomenon (Irvine 1981, Irvine and Caughey 1974). Before the crossover point ($\lambda^2 < 4\pi^2$), the cable fundamental frequency is the frequency of the first symmetric mode, whereas it jumps to the frequency of the first anti-symmetric mode after the crossover point ($\lambda^2 > 4\pi^2$). It can be found that the approximate solution coincides well with the exact solution only in the range of small values of λ^2. The reason is that the selected approximate parabolic mode shape function (15) in current energy method just matches the actual first order mode shape of a cable well in the range of small λ^2. Inspecting the approximate fundamental frequency expression (16), the first term is approximately equal to the fundamental frequency of the taut string theory which is $\pi^2 H/ml^2$, whereas the second term is actually the additional contribution due to the cable sag effect. It is therefore assumed that

$$\omega^2 = \pi^2\frac{H}{ml^2} + a\frac{EA}{m}\left(\frac{mg}{H}\right)^2 \tag{18}$$

Fitting fundamental frequency (18) with the exact solution by the least squares technique, the value of coefficient a is found out to be 0.777. Since the fundamental frequency of the cable is in fact the frequency of the first anti-symmetric mode after the modal crossover point ($\lambda^2 = 4\pi$) and can be obtained easily from Eq. (2), the curve fitting is carried out only within the range of $0 < \lambda^2 < 4\pi^2$. Fig. 3 shows the comparison of $\overline{\omega}_1 \sim \lambda^2$ curves after fitting with that of the exact

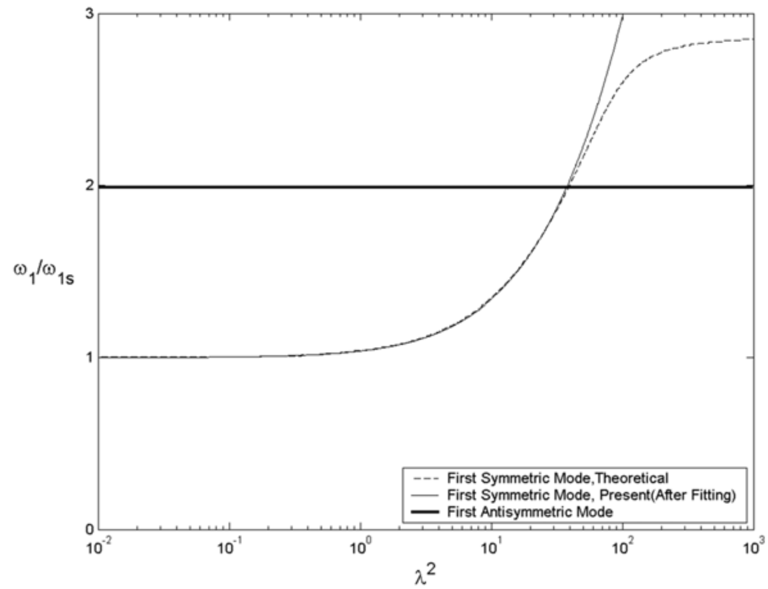

Fig. 3 Fundamental frequency ratio versus parameterλ^2 after fitting

solution. The maximum difference between the two curves is less than 1% and it occurs at the modal crossover point $\lambda^2 = 4\pi$. By inspecting the curve of the theoretical solution, it is demonstrated that the fundamental frequency considering the cable sag effect is almost same as that of the taut string theory when $\lambda < 0.174$. Therefore, the fundamental frequency of the cable can be calculated by the simple taut string theory within this range. The explicit relationship between the cable tension and the cable fundamental frequency including the sag effect can be summarized as

$$\omega = \frac{\pi}{l}\sqrt{\frac{H}{m}} \qquad (\lambda^2 \le 0.17) \tag{19a}$$

$$\omega^2 = \pi^2 \frac{H}{ml^2} + 0.777\frac{EA}{m}\left(\frac{mg}{H}\right)^2 \qquad (0.17 < \lambda^2 < 4\pi^2) \tag{19b}$$

$$\omega = \frac{2\pi}{l}\sqrt{\frac{H}{m}} \qquad (4\pi^2 \le \lambda^2) \tag{19c}$$

3. Bending stiffness effect on cable fundamental frequency

By ignoring the cable sag and taking the cable bending stiffness into account, the cable can be treated as a beam subjected to an axial tension force. The equation of motion is

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} - T\frac{\partial^2 v(x,t)}{\partial x^2} + m\frac{\partial^2 v(x,t)}{\partial t^2} = 0 \tag{20}$$

where EI is the bending stiffness of cable. The natural frequencies solutions of above equation depend on the boundary conditions of the cable (Humar 1990).

- Simply supported at both ends

$$\omega_n^2 = \left(\frac{n\pi}{l}\right)^2\frac{T}{m} + \left(\frac{n\pi}{l}\right)^4\frac{EI}{m} \tag{21}$$

- Fixed at both ends

$$2(\alpha l)(\beta l)[1 - \cos(\alpha l)\cosh(\beta l)] + [(\beta l)^2 - (\alpha l)^2]\sin(\alpha l)\sinh(\beta l) = 0 \tag{22}$$

where $\alpha l = l\sqrt{\sqrt{\zeta^4 + \gamma^4} - \zeta^2}$, $\beta l = l\sqrt{\sqrt{\zeta^4 + \gamma^4} + \zeta^2}$, $\zeta^2 = T/2EI$, $\gamma^4 = m\omega^2/EI$. Two non-dimensional parameters are introduced

$$\xi = \sqrt{\frac{T}{EI}}l \quad \text{and} \quad \eta_n = \frac{f_n}{f_n^s} \tag{23}$$

where, $f = \omega/2\pi$, f_n is the nth order cable natural frequency considering the cable bending stiffness, and $f_n^s = (n/2l)\sqrt{T/m}$ is the nth order natural frequency of a taut string. Eq. (22) can be rewritten in a more compact form (Zui *et al.* 1996)

$$2n\pi\eta_n(1 - \cos\alpha l\cosh\beta l) + \xi\sin\alpha l\sinh\beta l = 0 \tag{24}$$

It is clearly shown that the non-dimensional parameter ξ (23) reflects the effect of cable bending stiffness on the natural frequencies of the cable vibration. The nature of the roots of the Eq. (24) highly depends on the size of parameter ξ. Eq. (22) or (24) is the transcendental equations where the solution of η_n for a given value of ξ can be only obtained by an iterative procedure such as Newton-Raphson method. The energy method and least square fitting are used again to establish the explicit relationship between cable tension and fundamental frequency where the cable bending stiffness is taken into account. The following two cases are investigated respectively depending on the size of non-dimensional parameter ξ.

3.1 ξ is relatively small ($0 \le \xi \le 18$)

When ξ is relatively small, the cable with relatively large bending stiffness is subjected to relatively small tension. In such a case, the cable behaviors are much like an axially tensioned beam. By implementation of variable separation and determination of cable kinetic energy, strain energy and the work done by external force, the governing equation of cable motion can be obtained according to Hamilton's principle (8)

$$M\ddot{q}(t) + Kq(t) = 0 \tag{25}$$

where

$$M = \int_0^l m\phi^2(x)dx \tag{26a}$$

$$K = \int_0^l \left[EI\left(\frac{d^2\phi}{dx^2}\right)^2 + T\left(\frac{d\phi}{dx}\right)^2 \right] dx \tag{26b}$$

Therefore, the fundamental frequency taking the bending stiffness effect into account can be determined from

$$\omega^2 = \frac{K}{M} = \frac{\int_0^l \left[EI\left(\frac{d^2\phi}{dx^2}\right)^2 + T\left(\frac{d\phi}{dx}\right)^2 \right] dx}{\int_0^l m\phi^2(x)dx} \tag{27}$$

The first order vibration shape function $\phi(x)$ is chosen as the first order mode shape of a beam fixed at both ends

$$\begin{aligned} \phi(x) &= (\sin 4.731 + \sinh 4.731)\left(\sin\frac{4.731}{l}x - \sinh\frac{4.731}{l}x\right) \\ &+ (\cos 4.731 - \cosh 4.731)\left(\cos\frac{4.731}{l}x - \cosh\frac{4.731}{l}x\right) \end{aligned} \tag{28}$$

Eq. (27) yields

$$\omega^2 = 12.32\frac{T}{ml^2} + 501.404\frac{EI}{ml^4} \tag{29}$$

It is actually the explicit relationship of Eq. (22) or (24) between the cable fundamental frequency and the cable tension where the cable bending stiffness is taken into account. To inspect the difference between the approximate solution and the exact solution, both solutions are compared in Fig. 4 where two curves are plotted in the non-dimensional form with a range of $0 \le \xi \le 18$. The fundamental frequency ratio is defined as the cable fundamental frequency ω_1 taking the bending stiffness effect into account over corresponding frequency ω_{1b} of a beam fixed at both ends. It can

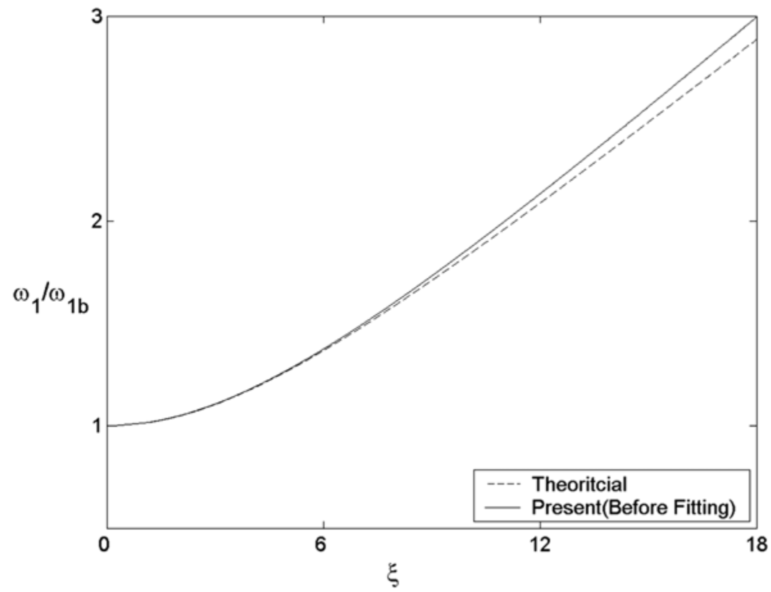

Fig. 4 Fundamental frequency ratio versus parameter ξ before fitting ($\xi \le 18$)

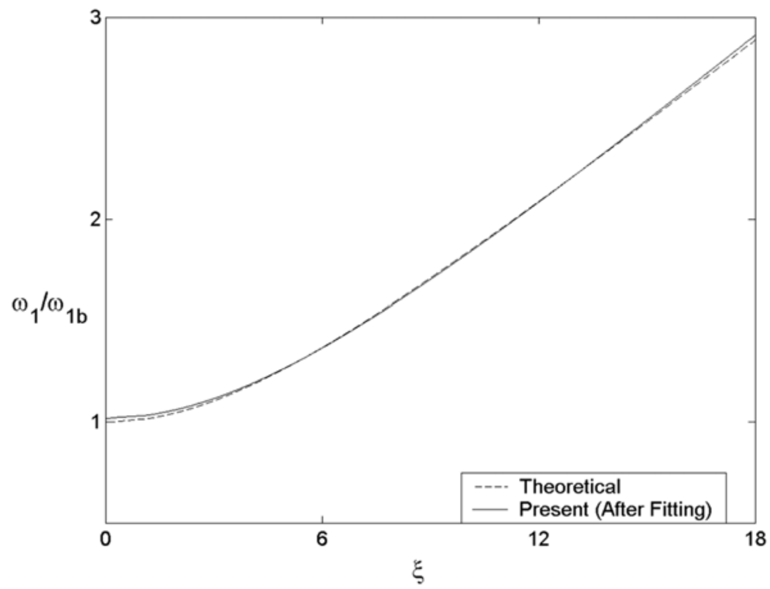

Fig. 5 Fundamental frequency ratio versus parameter ξ after fitting ($\xi \le 18$)

be seen from Fig. 4 that two curves coincide well in the range of small values of $\xi(\xi \leq 8)$ where the discrepancy between two curves is less than 1%. However, the discrepancy becomes larger when ξ increases. When $\xi = 18$, for instance, the difference goes to 3.8%. The reason is that the effect of bending stiffness on the cable vibration frequencies decreases with the increase in non-dimensional parameter ξ. In such a case, the cable is over stiffened by using the first order mode shape (28) of a beam fixed at both ends as the actual mode shape of a cable. To fit the exact solution well, similarly, it is assumed that

$$\omega^2 = a\frac{T}{ml^2} + b\frac{EI}{ml^4} \tag{30}$$

Fitting Eq. (30) with the exact fundamental frequency solution in the range of $0 \leq \xi \leq 18$ by the least square fitting method, two coefficients are found to be $a = 11.49$, $b = 519.24$. Fig. 5 shows the comparison of the fundamental frequency between the explicit solution and theoretical solution. The maximum discrepancy between two curves goes down to 0.86% at $\xi = 18$.

3.2 ξ is relatively large ($\xi > 18$)

With the increase in the size of ξ, the effect of bending stiffness on the cable vibration frequencies decreases since the cable has relatively small bending stiffness subjected to relatively large tension. In such a case, the cable fundamental frequency tends to that of a taut string. It is assumed within the range of $18 < \xi \leq 500$ that

$$\omega = \frac{\pi}{l}\sqrt{\frac{T}{m}} + a\frac{1}{l^2}\sqrt{\frac{EI}{m}} \tag{31}$$

where the first term of above equation is the fundamental frequency of the taut string theory, and

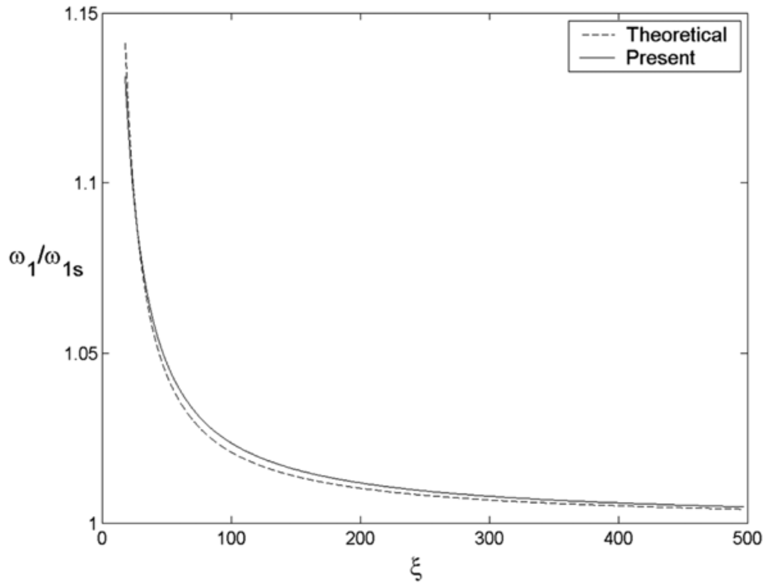

Fig. 6 Frequency comparison of present formulas with theoretical solution ($18 < \xi \leq 500$)

the second term is the additional contribution due to cable bending stiffness effect. Fitting Eq. (31) with the exact solution of a cable within the range of $18 < \xi \le 500$ by the least square fitting, the coefficient can be obtained as $a = 7.42$.

Fig. 6 shows the comparison between the approximate solution and the exact solution in the range of $18 < \xi \le 500$. Both curves are plotted in the non-dimensional form. The fundamental frequency ratio is the ratio of cable fundamental frequency ω_1 taking the bending stiffness effect into account to the corresponding frequency ω_{1s} of a taut string. It can be seen from Fig. 6 that two curves coincide with each other well. The maximum difference is about 0.95% which occurs at $\xi = 18$.

By further analyzing the curve of the theoretical solution, it can be found that, the cable fundamental frequency taking the bending stiffness effect into account goes to the fundamental frequency ω_{1s} of a taut string when $\xi > 208.1$. Namely, the fundamental frequency of a cable can be calculated by the simple taut string theory where the cable bending stiffness can be neglected. Considering the bending stiffness effect, as a result, the explicit relationship between the cable fundamental frequency and the cable tension can be summarized as follows

$$\omega^2 = 11.49\frac{T}{ml^2} + 519.24\frac{EI}{ml^4} \qquad (0 \le \xi \le 18) \tag{32a}$$

$$\omega = \frac{\pi}{l}\sqrt{\frac{T}{m}} + \frac{7.42}{l^2}\sqrt{\frac{EI}{m}} \qquad (18 < \xi \le 210) \tag{32b}$$

$$\omega = \frac{\pi}{l}\sqrt{\frac{T}{m}} \qquad (210 < \xi) \tag{32c}$$

4. Empirical formulas to estimate cable tensions

Eqs. (19) and (32) represent the explicit relationships between the fundamental frequency and cable tension where the cable sag and bending stiffness effects are taken into account respectively. Therefore, the empirical formulas to determine the cable tensions by using cable fundamental frequency can be derived. Note that the relationship $\omega = 2\pi f$ is applied so that the frequency f in the formulas is in Hz.

- Empirical formulas to estimate cable tension considering cable sag effect

$$T = 4ml^2f^2 \qquad (\lambda^2 \le 0.17) \tag{33a}$$

$$T = \sqrt[3]{ml^2(4f^2T^2 - 7.569mEA)} \qquad (0.17 < \lambda^2 < 4\pi^2) \tag{33b}$$

$$T = ml^2f^2 \qquad (4\pi^2 \le \lambda^2) \tag{33c}$$

- Empirical formulas to estimate cable tensions considering cable bending stiffness effect

$$T = 3.432ml^2f^2 - 45.191\frac{EI}{l^2} \qquad (0 \le \xi \le 18) \tag{34a}$$

$$T = m\left(2lf - \frac{2.363}{l}\sqrt{\frac{EI}{m}}\right)^2 \qquad 18 < \xi \leq 210 \tag{34b}$$

$$T = 4ml^2f^2 \qquad (210 < \xi) \tag{34c}$$

It is noticed in Eq. (19) that H is the cable tension component in the chord direction where $T = H\frac{ds}{dx}$. In the case of small ratio of sag to span (less than 1:8), $ds \approx dx$, thus $T \approx H$. The applicability of above practical formulas highly depends on the sizes of two non-dimensional parameters λ^2 and ξ that characterize the cable sag and bending stiffness effects respectively. The parameters λ^2 and ξ can be calculated from cable structural and physical parameters as well as the designed cable tension force.

Eq. (33) is applicable to the cables where the effect of bending stiffness is negligible but the cable sag effect has to be taken into account. In more than 95% of the stay cables in cable-stayed bridges, for example, the ranges of parameters λ^2 and ξ are within $\lambda^2 < 3.1$ and $\xi > 50$ (Mehrabi and Tabatabai 1998). When $\lambda^2 < 3.1$, the maximum difference between the cable fundamental frequency and the fundamental frequency predicted by a taut string will reach 10.7% according the exact solution of Eq. (3), which indicates that the cable sag effect cannot be neglected. Considering the cable bending stiffness effect only within $\xi > 50$, the maximum difference between the cable fundamental frequency and corresponding frequency of a taut string is found to be 4.4% according to the exact solution of Eq. (24) where the cable is fixed at both ends. When the stay cables in cable-stayed bridges are assumed to be simply supported at both ends, it is demonstrated that the effect of bending stiffness on the fundamental frequency becomes even smaller according to Eq. (21). Thus, the practical formulas as shown in Eq. (33) can be useful for engineers to estimate the cable tension by the fundamental frequency only.

Eq. (34) is applicable to the cables where the effect of sag is negligible but the cable bending stiffness effect has to be taken into account. One example is the pre-tensioned suspenders and tied bars in arch bridges or beams subjected to axial pre-tensions. In such a case, the value of characteristic parameter ξ is relatively large due to relatively short length and large tension.

In the case of main cables of suspension bridges during the freely hanged stage where only the tower-cable system is erected but none of deck segments has been hoisted into position, the value of parameter λ^2 is far greater than $4\pi^2$ (mode crossover point). In such a case, the cable fundamental frequency is actually the frequency of the first anti-symmetric mode. The boundary condition of such main cables can be assumed to be simply supported at both ends, and the cable tension force can be derived from Eq. (21)

$$T = ml^2f^2 - \frac{4\pi^2}{l^2}EI \tag{35}$$

The cable tension forces can be determined straightforwardly through proposed practical formulas (33) and (34) by using the cable fundamental frequency except for Eq. (33b) where the iteration calculation is required. The initial value of T can be given as that calculated from the taut string theory. The convergence is fast and only several iterations are often needed. A typical stay cable of a cable-stayed bridge is taken as an example where cable length density $m = 400$ kg/m, length $l = 100.0$ m, elasticity modulus $E = 1.5988 \times 10^{10}$ Pa, and cross-sectional area $A = 7.8507 \times 10^{-3}$ m^2.

Table 1 Iterative calculation of cable tension force (kN)

T_0	T_1	T_2	T_3	T_4
3097.6	3043.9	3006.6	2980.7	2980.7

If the measured cable fundamental frequency is $f = 0.44$ Hz and the convergence criterion is defined as that the difference between two consecutive solutions is less than 1%, it is demonstrated that only 4 iterations are needed for the convergent solution. Table 1 shows the details of the iterative calculation of cable tension force.

5. Verification of the empirical formulas

5.1 The effect of cable sag

To verify the applicability of proposed empirical formulas to include the cable sag effect, two cables with different sag parameters λ^2 are analyzed and the results are compared with those reported in literatures. The physical and geometric parameters of two cables are listed in Table 2. Since the values of the parameter ξ of two cables are relatively large, the influence of bending stiffness on the cables' fundamental frequency can be neglected. The difference between the cable fundamental frequency and that of a taut string is 0.67% when $\xi = 303$ according to the exact solution (24) where the cable bending stiffness is taken into account, whereas the difference is only 0.33% when $\xi = 605$. The first cable ($\lambda^2 = 0.79$) has a moderate sag and the second cable ($\lambda^2 = 50.70$) has a large sag. Table 3 shows the comparison of the fundamental frequencies computed by the proposed formula (19) with those obtained from the finite difference method (Mehrabi and Tabatabai 1998), nonlinear finite element method (Ni *et al.* 2002) and the taut string theory. It is observed that the proposed empirical formulas coincide well with the results of complicated numerical methods reported in the literatures. The simple taut string theory ignoring the sag effect results in the unacceptable errors for the analyzed cables.

Table 2 Parameters of two analyzed cables

Cable	λ^2	ξ	m (kg/m)	L (m)	H (kN)	E (Pa)	A (m^2)	I (m^4)
1	0.79	605.5	400.0	100.0	2903.60	1.5988E10	7.8507E-03	4.9535E-6
2	50.70	302.7	400.0	100.0	725.90	1.7186E10	7.6110E-03	4.6097E-6

Table 3 Comparison of computed cable fundamental frequencies (Hz)

Cable no.	λ^2	ξ	String theory	Finite difference method (Mehrabi and Tabatabai 1998)	Finite element method (Ni *et al.* 2002)	Present method
1	0.79	605.5	0.426	0.440	0.441	0.439
2	50.70	302.7	0.213	0.428	0.421	0.426

Table 4 Comparison of computed cable tensions (kN)

Fundamental frequency (Hz)	String theory	Zui *et al.* (1996)	Present formulas
6	171.66	143.05	142.82
8	305.17	267.20	266.27
10	476.83	429.50	427.87
12	686.63	629.94	627.62

5.2 The effect of cable bending stiffness

By approximately solving cable characteristic Eq. (24), Zui *et al.* (1996) proposed the practical formulas to calculate the cable tensions considering cable bending stiffness as well as cable sag by using the first and second-order frequency. Experiments were conducted to verify their practical formulas. The same cable is herein studied to see the applicability of the proposed empirical formulas in the paper. The cable parameters include that $l = 9.98$ m, $m = 12.04$ kg/m and $EI = 23.5\ \mathrm{kN \cdot m^2}$. Table 4 shows the comparison of the cable tension forces calculated by the present formulas (34) with those obtained from Zui's formulas (1996) as well as the taut string theory. It is demonstrated that the cable tension estimations by the present formulas agree well with those by Zui *et al*. The simple taut string theory ignoring the bending stiffness effect results in the unacceptable error for the analyzed cables.

6. Laboratory tests on stay cables

To evaluate the cable tension by using measured cable fundamental frequency, dynamic tests on the stay cables with different levels of tensions were carried out in the laboratory. A pair of steel stands consisting of 7 ϕ5 high strength wires were used as two parallel stay cables. One end of the stay cables is fixed to the resisting wall (Fig. 7a) and another end is tensioned by a hydraulic jack

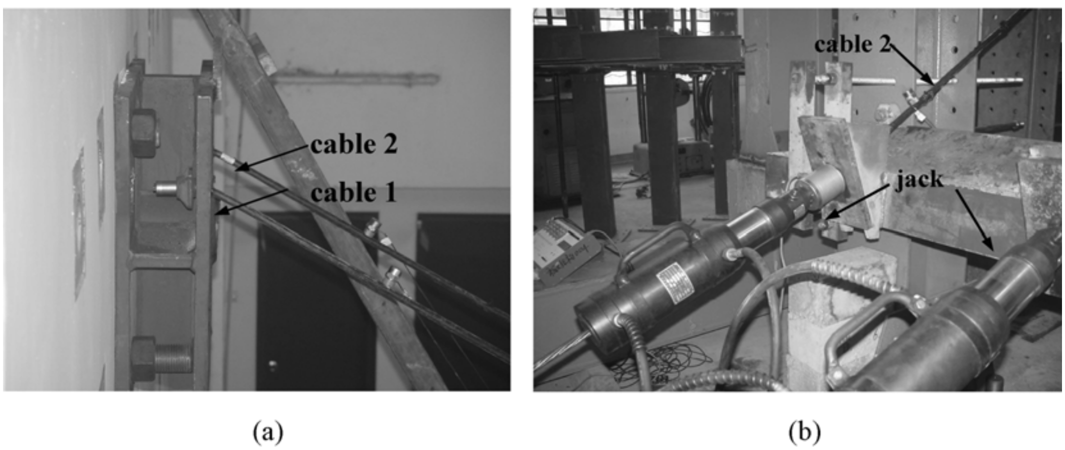

Fig. 7 Stay cables tested in the Lab. (a) Cable ends fixed to the wall, (b) Cable ends tensioned by jack

Table 5 Parameters of tested stay cables

Cable length	Mass/unit length	Sectional area	Inertia moment	Young's modulus
13.6 m	1.2031 kg/m	1237.44 mm^2	6.4×10^{-8} m^4	1.8×10^5 MPa

(Fig. 7b) at different levels from 50 kN to 120 kN. The applied cable tension forces are measured by a load cell. The tested cable parameters are listed in Table 5.

A total five piezoelectric accelerometers were attached to each cable using the magnetic bases. They were arranged in a symmetrical manner as shown in Fig. 8. All the accelerometers were oriented perpendicular to the cable to measure in-plane vibration. A 16-channel data acquisition system was used and corresponding accelerometer signals were recorded in a computer.

The excitation was provided by slightly tapping the cable with a hammer. The signals were recorded for 5 minutes with sampling frequency of 2,000 Hz. The natural frequencies of the cables

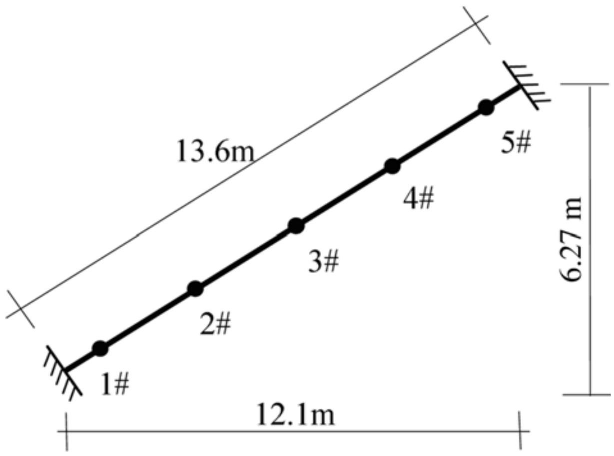

Fig. 8 Accelerometer arrangements on the tested cables

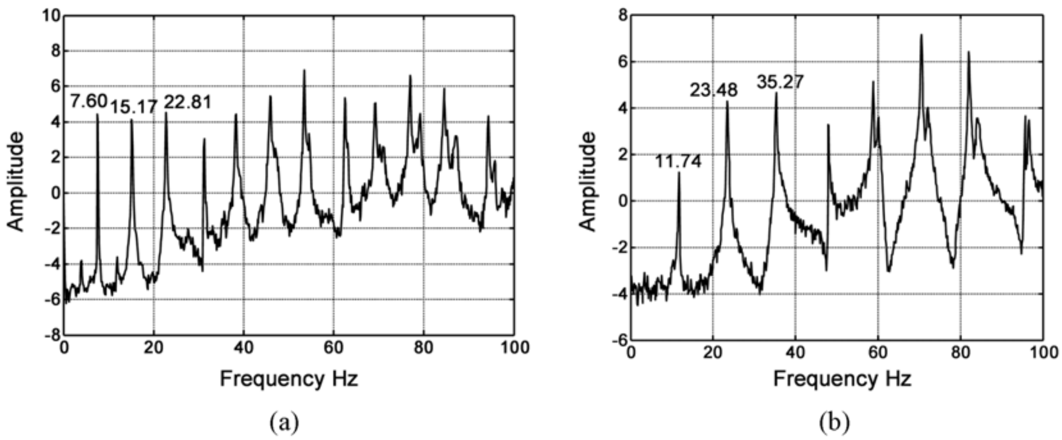

Fig. 9 Power spectral densities of cable acceleration measurements. (a) Cable tension level at 50 kN, (b) Cable tension level at 120 kN

Table 6 Comparison of tested and calculated cable tensions

Measured fundamental frequency (Hz)	7.60	8.32	8.96	9.59	10.16	10.74	11.23	11.74
Calculated cable tension (kN)	51.41	61.62	71.46	81.86	91.88	102.67	112.25	122.68
Applied cable tension (kN)	50.00	60.00	70.00	80.00	90.00	100.00	110.00	120.00
Error (%)	2.82	2.70	2.09	2.33	2.09	2.67	2.06	2.23

were determined by identifying the peaks from the Power Spectral Density (PSD) of recorded acceleration time histories. The spectral parameters include Hanning window, 50% overlap, and 81,920 data point window length, which results in the frequency resolution of 0.024 Hz. The PSDs of the signals collected by the accelerometer (2#) mounted on one-quarter of the cable are shown in Fig. 9 at the cable tension levels of 50 kN and 120 kN respectively. The peaks are clearly shown and the cable fundamental frequency that corresponds to the first peak can be obtained.

For the tested stay cables, the values of two non-dimensional parameters are in the range of $\xi \geq 210$ and $\lambda^2 \leq 0.01$. Eq. (34c) is therefore employed to estimate the cable tension forces by using identified cable fundamental frequencies. The calculated cable tensions are compared with applied cable tensions in Table 6. It is clearly shown that a good agreement has been achieved between the applied cable tensions and those calculated from proposed empirical formula with an error of less than 3%.

7. Conclusions

The empirical formulas to estimate cable tension forces by using cable fundamental frequency are proposed in the paper where the cable sag and bending stiffness are taken into account respectively. To avoid solving the transcendental equations, the empirical formulas are established by the approximate solutions with the help of energy method and fitting the exact solutions of cable vibration. The applicability of proposed empirical formulas depends on the sizes of two non-dimensional characteristic cable parameters that are λ^2 characterizing the cable sag effect and ξ reflecting the cable bending stiffness effect. The proposed empirical formulas are verified by the comparison of the results with those reported in the literatures and with the experimental results carried out on the stay cables in the laboratory. The formulas are simple in form and they are very convenient for practical engineers to fast evaluate the cable tensions.

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