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Green's functions and boundary element method for a magneto-electro-elastic half-plane

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1. Introduction

In recent years, intelligent structures and smart systems have become a hot topic. Piezoelectric and magneto-electro-elastic materials have been widely used to make various transducers and actuators and have important applications in high technology. For piezoelectric materials, Lee and Jiang (1994) obtained the fundamental solutions by employing the double Fourier transform, Sosa and Castro (1994) also gave the solutions for a point force applied vertically on the boundary of a piezoelectric half-plane. Ding *et al.* (1998a,b) derived a general solution in terms of three harmonic functions and studied the Green's functions for a two-phase infinite plane using the method of trial-and-error. Ding *et al.* (1998c) constructed the fundamental solutions following the general solution in terms of harmonic functions, and performed the numerical examples by the corresponding boundary element method.

Recently, more and more attention has been paid to magneto-electro-elastic materials. Pan (2001) derived the exact three-dimensional solution of a simply supported, multi-layered rectangular plate under static load. Pan (2002) presented 3D Green's functions for an-isotropic full space, half space, and bi-materials based on the extended Stroh formalism. Pan *et al.* (2002) extended the analytical method of Pan (2001) to the free vibration of three-dimensional, linear an-isotropic, simply supported, and multi-layered rectangular plates. Wang and Shen (2002) obtained the general solution expressed by five potential functions, which was then applied to find the fundamental solution for a generalized dislocation. Hou *et al.* (2003) analyzed the elliptical Hertzian contact of transversely isotropic magneto-electro-elastic bodies by virtue of the general solution that was expressed in terms of five harmonic functions. Ding and Jiang (2003) obtained the fundamental solution of infinite magneto-electro-elastic solid in five harmonic functions, derived the boundary

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integral formulation and performed two numerical calculations.

In this paper, the basic equations of 2D magneto-electro-elastic media are established on the basis of the constitutive equations and the assumption of plane-strain, and the work of Ding *et al.* (1998a,b,c) is generalized to the magneto-electro-elastic half-plane problem. Firstly, by virtue of the strict differential operator theory, the general solution for the case of distinct material eigenvalues is derived, which is expressed in terms of four harmonic functions. Then, based on the obtained general solution along with the trial-and-error method, the Green's functions for point forces, point charge and point current acting in interior of a magneto-electro-elastic half-plane are systematically derived. Finally, the fundamental solutions and the corresponding boundary integral formulation are given and numerical examples are presented. It is shown that our fundamental solutions are not only easy to understand and convenient to check, but also very simple and uniform that can be widely used.

2. Governing equations and the general solutions

The basic equations of transversely isotropic magneto-electro-elastic bodies can be found in Pan (2001) and Ding *et al.* (2003). If the displacements u_i, the electric potential Φ and the magnetic potential Ψ are independent of y, then we will have the so-called plane-strain problem. If xoy plane is selected to be the isotropic plane, the basic equations for two-dimensional magneto-electro-elastic solid in absence of body forces, free electric charge and free magnetic charge in the (x, z) coordinates can be simplified as follows:

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{zx}}{\partial z}=0,\quad \frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\sigma_z}{\partial z}=0,\quad \frac{\partial D_x}{\partial x}+\frac{\partial D_z}{\partial z}=0,\quad \frac{\partial B_x}{\partial x}+\frac{\partial B_z}{\partial z}=0 \tag{1}$$

$$\sigma_x = c_{11}\frac{\partial u}{\partial x}+c_{13}\frac{\partial w}{\partial z}+e_{31}\frac{\partial\Phi}{\partial z}+d_{31}\frac{\partial\Psi}{\partial z}$$

$$\tau_{zx} = c_{44}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+e_{15}\frac{\partial\Phi}{\partial x}+d_{15}\frac{\partial\Psi}{\partial x},\quad \sigma_z = c_{13}\frac{\partial u}{\partial x}+c_{33}\frac{\partial w}{\partial z}+e_{33}\frac{\partial\Phi}{\partial z}+d_{33}\frac{\partial\Psi}{\partial z}$$

$$D_x = e_{15}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-\varepsilon_{11}\frac{\partial\Phi}{\partial x}-g_{11}\frac{\partial\Psi}{\partial x},\quad D_z = e_{31}\frac{\partial u}{\partial x}+e_{33}\frac{\partial w}{\partial z}-\varepsilon_{33}\frac{\partial\Phi}{\partial z}-g_{33}\frac{\partial\Psi}{\partial z}$$

$$B_x = d_{15}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-g_{11}\frac{\partial\Phi}{\partial x}-\mu_{11}\frac{\partial\Psi}{\partial x},\quad B_z = d_{31}\frac{\partial u}{\partial x}+d_{33}\frac{\partial w}{\partial z}-g_{33}\frac{\partial\Phi}{\partial z}-\mu_{33}\frac{\partial\Psi}{\partial z} \tag{2}$$

With the method and the strict differential operator theorem presented by Ding *et al.* (1998a,b), the general solutions for Eqs. (1), (2) in the case of distinct eigenvalues can be derived as follows:

$$u=\sum_{j=1}^{4}\frac{\partial\psi_j}{\partial x},\quad w_m=\sum_{j=1}^{4}s_jk_{mj}\frac{\partial\psi_j}{\partial z_j},\quad \sigma_x=\sum_{j=1}^{4}\omega_{4j}\frac{\partial^2\psi_j}{\partial z_j^2}$$

$$\sigma_m=\sum_{j=1}^{4}\omega_{mj}\frac{\partial^2\psi_j}{\partial z_j^2},\quad \tau_m=\sum_{j=1}^{4}s_j\omega_{mj}\frac{\partial^2\psi_j}{\partial x\,\partial z_j}\quad (m=1\sim 3) \tag{3}$$

where $\omega_{4j} = -\omega_1 s_j^2$, and the generalized displacements and stresses are defined as follows:

$$w_1 = w,\quad w_2 = \Phi,\quad w_3 = \Psi,\quad \sigma_1 = \sigma_z,\quad \sigma_2 = D_z,\quad \sigma_3 = B_z,\quad \tau_1 = \tau_{xz},\quad \tau_2 = D_x, \tau_3 = B_x \tag{4}$$

The functions ψ_j satisfy the following equations:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2}\right)\psi_j = 0 \quad (j = 1\sim4) \tag{5}$$

where $z_j = s_j z\ (j = 1\sim4)$ and s_j^2 are the four roots of the equation (we take $\mathrm{Re}(s_j) > 0$)

$$a_1 s^8 - a_2 s^6 + a_3 s^4 - a_4 s^2 + a_5 = 0 \tag{6}$$

where k_{mj} and ω_{mj} in Eq. (3) and a_n in Eq. (6) are the same as those derived by Hou *et al.* (2003).

3. Green's functions for magneto-electro-elastic half-plane

Taking the origin of the coordinates at the straight linear boundary of the half-plane with x axis is along with the straight line and z axis pointed inwards. Point forces, point charge and point current are assumed be applied at the point $(0, h)$, the boundary conditions $(z = 0)$ can be known as

$$\sigma_z = D_z = B_z = 0, \quad \tau_{xz} = 0 \tag{7}$$

For the sake of convenience, we introduce a series of denotations:

$$h_j = s_j h,\quad z_{jk} = z_j + h_k,\quad R_{jk} = \sqrt{x^2 + z_{jk}^2},\quad \bar{z}_{jk} = z_j - h_k,\quad \bar{R}_{jk} = \sqrt{x^2 + \bar{z}_{jk}^2} \quad (j, k = 1\sim4) \tag{8}$$

3.1 Green's functions to the problem of combination of point force P_z in z direction, point charge Q and point current J

Assuming the displacement functions ψ_j to be in the following form:

$$\psi_j = A_j\left(\bar{z}_{jj}\ln\bar{R}_{jj} - x\arctan\frac{x}{\bar{z}_{jj}} - \bar{z}_{jj}\right) + \sum_{k=1}^{4} A_{jk}\left(z_{jk}\ln R_{jk} - x\arctan\frac{x}{z_{jk}} - z_{jk}\right) \quad (j = 1\sim4) \tag{9}$$

where $A_j, A_{jk}(j, k = 1\sim4)$ are arbitrary constants. Substituting Eq. (9) into Eq. (3), we have

$$u = -\sum_{j=1}^{4}\left(A_j\arctan\frac{x}{\bar{z}_{jj}} + \sum_{k=1}^{4} A_{jk}\arctan\frac{x}{z_{jk}}\right),\quad w_m = \sum_{j=1}^{4} s_j k_{mj}\left(A_j\ln\bar{R}_{jj} + \sum_{k=1}^{4} A_{jk}\ln R_{jk}\right)$$

$$\sigma_m = \sum_{j=1}^{4}\omega_{mj}\left(\frac{A_j\bar{z}_{jj}}{\bar{R}_{jj}^2} + \sum_{k=1}^{4}\frac{A_{jk}z_{jk}}{R_{jk}^2}\right),\quad \tau_m = \sum_{j=1}^{4} s_j\omega_{mj}\left(\frac{A_j x}{\bar{R}_{jj}^2} + \sum_{k=1}^{4}\frac{A_{jk}x}{R_{jk}^2}\right) \quad (m = 1\sim3) \tag{10}$$

Considering the continuity condition of u at $z = h$ yields

$$\sum_{j=1}^{4} A_j = 0 \tag{11}$$

Considering the equilibrium of the elastic strip cut by two lines of $z = h \pm \varepsilon$, where ε can be arbitrary value, gives three additional equations:

$$2\pi \sum_{j=1}^{4} \omega_{mj} A_j = -P_m \quad (m = 1\sim3) \tag{12}$$

where, $P_1 = P_z, P_2 = -Q, P_3 = -J$.

It is shown that the combination of Eqs. (11) and (12) will give constants A_j. Substituting Eq. (10) into the boundary condition (7) gives the equations to determine the constants A_{jk} as follows

$$\omega_{mj} A_j - \sum_{k=1}^{4} \omega_{mk} A_{kj} = 0 \qquad (m = 1\sim3, j = 1\sim4) \tag{13}$$

$$s_j \omega_{1j} A_j + \sum_{k=1}^{4} s_k \omega_{1k} A_{kj} = 0 \qquad (j = 1\sim4) \tag{14}$$

3.2 Green's functions to the problem of point force P_x in x direction

Assuming the displacement functions ψ_j are:

$$\psi_j = E_j\left(x \ln \bar{R}_{jj} + \bar{z}_{jj} \arctan\frac{x}{\bar{z}_{jj}} - x\right) + \sum_{k=1}^{4} E_{jk}\left(x \ln R_{jk} + z_{jk} \arctan\frac{x}{z_{jk}} - x\right) \quad (j = 1\sim4) \tag{15}$$

where E_j and E_{jk} are arbitrary constants to be determined. Substituting Eq. (15) into (3) yields

$$u = \sum_{j=1}^{4}\left(E_j \ln \bar{R}_{jj} + \sum_{k=1}^{4} E_{jk} \ln R_{jk}\right), \quad w_m = \sum_{j=1}^{4} s_j k_{mj}\left[E_j \arctan\frac{x}{\bar{z}_{jj}} + \sum_{k=1}^{4} E_{jk} \arctan\frac{x}{z_{jk}}\right]$$

$$\sigma_m = -\sum_{j=1}^{4} \omega_{mj}\left(\frac{E_j x}{\bar{R}_{jj}^2} + \sum_{k=1}^{4} \frac{E_{jk} x}{R_{jk}^2}\right), \quad \tau_m = \sum_{j=1}^{4} s_j \omega_{mj}\left(\frac{E_j \bar{z}_{jj}}{\bar{R}_{jj}^2} + \sum_{k=1}^{4} \frac{E_{jk} z_{jk}}{R_{jk}^2}\right) \quad (m = 1\sim3) \tag{16}$$

Consideration of the continuity of w_m at $z = h$ yields

$$\sum_{j=1}^{4} s_j k_{mj} E_j = 0 \qquad (m = 1\sim3) \tag{17}$$

Just like that in the previous section, the whole balance condition of the elastic strip cut by two lines of $z = h \pm \varepsilon$ gives

$$2\pi \sum_{j=1}^{4} s_j \omega_{1j} E_j = -P_x \tag{18}$$

Then we can solve constants E_j by combining Eqs. (17) and (18). Further, substituting Eq. (16) into the boundary condition (7), we get the equations to determine that constants E_{jk} as follows:

$$\omega_{mj}E_j + \sum_{k=1}^{4} \omega_{mk}E_{kj} = 0 \quad (m = 1\sim3, j = 1\sim4) \tag{19}$$

$$s_j\omega_{1j}E_j - \sum_{k=1}^{4} s_k\omega_{1k}E_{kj} = 0 \quad (j = 1\sim4) \tag{20}$$

Taking $P_x = 1$, $P_z = Q = J = 0$ or $P_z = 1$, $P_x = Q = J = 0$ or $Q = 1$, $P_x = P_z = J = 0$ or $J = 1$, $P_x = 0$, $P_z = Q = 0$, we can derive the fundamental solutions for a magneto-electro-elastic half-plane.

4. Boundary integral formulation

Let S be the boundary of the domain Ω of a magneto-electro-elastic half-plane. The boundary conditions are expressed by

$$S_t{:}\sigma_{ij}n_j = \bar{t}_i;\quad S_u{:}u_i = \bar{u}_i;\quad S_\omega{:}D_in_i = -\bar{\omega};\quad S_\Phi{:}\Phi = \bar{\Phi};\quad S_\eta{:}B_in_i = -\bar{\eta};\quad S_\Psi{:}\Psi = \bar{\Psi} \tag{21}$$

where t_i represent the surface tractions, ω is the surface charge, η is the surface magnetic induction.

With the extended Somigliana equation, the boundary integral formulation is obtained:

$$\mathbf{C}(\xi)\mathbf{u}(\xi) = \int_S \mathbf{U}^*(\xi, X)\mathbf{t}(X)ds - \int_S \mathbf{T}^*(\xi, X)\mathbf{u}(X)ds + \int_\Omega \mathbf{U}^*(\xi, X)\mathbf{b}(X)d\Omega \tag{22}$$

where $\mathbf{C}$ is the coefficient matrix, and the generalized displacement vector $\mathbf{u}$, surface traction vector $\mathbf{t}$ as well as body force $\mathbf{b}$ are

$$\mathbf{u} = [u \;\; w \;\; -\Phi \;\; -\Psi]^T, \qquad \mathbf{t} = [t_x \;\; t_z \;\; -\omega \;\; -\eta]^T, \qquad \mathbf{b} = [f_x \;\; f_z \;\; -f_e \;\; -f_m]^T \tag{23}$$

The two matrices $\mathbf{U}^*$ and $\mathbf{T}^*$ in Eq. (22) are composed of fundamental solutions

$$\mathbf{U}^* = \begin{bmatrix} u_{11}^* & u_{12}^* & \Phi_1^* & \Psi_1^* \\ u_{21}^* & u_{22}^* & \Phi_2^* & \Psi_2^* \\ u_{31}^* & u_{32}^* & \Phi_3^* & \Psi_3^* \\ u_{41}^* & u_{42}^* & \Phi_4^* & \Psi_4^* \end{bmatrix}, \qquad \mathbf{T}^* = \begin{bmatrix} t_{11}^* & t_{12}^* & \omega_1^* & \eta_1^* \\ t_{21}^* & t_{22}^* & \omega_2^* & \eta_2^* \\ t_{31}^* & t_{32}^* & \omega_3^* & \eta_3^* \\ t_{41}^* & t_{42}^* & \omega_4^* & \eta_4^* \end{bmatrix} \tag{24}$$

4.1 Examples

A magneto-electro-elastic half-plane with a circular hole of radius r (r = 1), which is centered at point (0, h). Assume there is uniform radial load all around the hole:

$$t_r = t_0(=1), \qquad t_\theta = 0 \tag{25}$$

Table 1 Numerical results by BEM

h	u	σ_z	D_z	B_z	σ_x
10	0.90165E-11	0.82548E+00	−0.45736E-11	−0.61201E-08	−0.99432E+00
20	0.89269E-11	0.76866E+00	−0.10048E-10	−0.61683E-08	−0.99531E+00
30	0.89155E-11	0.75950E+00	−0.11055E-10	−0.61738E-08	−0.99547E+00
∞	0.89073E-11	0.75885E+00	−0.11064E-10	−0.61768E-08	−0.99559E+00

As we know, when $h \to \infty$, the problem above becomes as an infinite magneto-electro-elastic plane with a circular hole with uniform radial load all around, so it can be used to verify our derived fundamental solutions and boundary element method. The displacement, stresses, electric displacement and magnetic induction of the point (r, h) (when $h \to \infty$, it corresponds to the center (r, 0) of an infinite plane) are tabulated in Table 1 for case of the same material constants in Ding (2003). It can be found that the numerical results become closer to those of infinite plane problem as h get larger.

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References

Ding, H.J., Wang, G.Q. and Chen, W.Q. (1998a), "Green's functions for a piezoelectric half plane", *Science in China (Series E)*, **41**(1), 70-75.

Ding, H.J. and Jiang, A.M. (2003), "Fundamental solutions for transversely isotropic magneto-electro-elastic media", *Science in China (Series E)*, **46**(6), 607-619.

Ding, H.J., Wang, G.Q. and Chen, W.Q. (1998b), "Green's functions for a two-phase infinite piezoelectric plane", *Proc. R. Soc. Lond.*, **453A**, 2241-2257.

Ding, H.J., Wang, G.Q. and Chen, W.Q. (1998c), "A boundary integral formulation and 2D fundamental solutions for piezoelectric media", *Comput. Meth. Appl. Mech. Engng.*, **158**, 65-80.

Hou, P.F., Leung Andrew, Y.T. and Ding, H.J. (2003), "The elliptical Hertizan contact of transversely isotropic magneto-electro-elastic bodies", *Int. J. Solids Struct.*, **40**, 2833~2850.

Lee, J.S. and Jiang, L.Z. (1994), "A boundary integral formulation and 2D fundamental solution for piezoelectric media", *Mech. Res. Comm.*, **22**, 47-54.

Pan, E. (2001), "Exact solution for simply supported and multilayered magneto-electro-elastic plates", *J. Appl. Mech.*, **68**, 608-618.

Pan, E. (2002), "Three-dimensional Green's functions in an-isotropic magneto-electro-elastic bi-materials", *Z. Angrew. Math. Phys.*, **53**, 815-838.

Pan, E. and Heylinger, P.R. (2002), "Free vibration of simply supported and multi-layered magneto-electro-elastic plates", *J. Sound Vib.*, **253**(3), 429-442.

Sosa, H.A. and Castro, M.A. (1994), "On concentrated load at boundary of a piezoelectric half-plane", *J. Mech. Phys. Solids*, **42**, 1105-1122.

Wang, X. and Shen, Y.P. (2002), "The general solution of three-dimensional problems in magneto-electro-elastic media", *Int. J. Engng. Sci.*, **40**, 1069-1080.