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Nonlinear dynamic FE analysis of structures consisting of rigid and deformable parts Part II – Computer implementation and test examples

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Abstract. This is the second part of the paper (Rojek and Kleiber 1993) devoted to nonlinear dynamic analysis of structures consisting of rigid and deformable parts. The first part contains a theoretical formulation of nonlinear equations of motion for the coupled system as well as a solution algorithm. The second part presents the computer implementation of the equations derived in the first part with a short review of the capabilities of the computer program used and the library of finite elements. Details of material nonlinearity treatment are also given. The paper is illustrated by discussing a practical problem of a safety cab analysis for an agricultural tractor.

Key words: nonlinear finite element analysis; rigid body dynamics; rigid-deformable systems

1. Introduction

In Part I of our paper (Rojek and Kleiber 1993) the formulation of incremental equations of motion for structures consisting of deformable and rigid parts was presented. The deformable parts were discretized with finite elements while the equations of rigid body dynamics in the form of the Newton-Euler equations were employed for the rigid parts. Lagrangian multiplier technique was used to couple the equations for the whole system.

The algorithm for nonlinear dynamic analysis of structures consisting of rigid and deformable parts has been implemented in our in-house computer code called AKAB, (Rojek 1992, Rojek and Kleiber 1991). In Part II we would like to provide some information on the computer implementation, reviewing the capabilities of nonlinear analysis with AKAB and describing the library of finite elements that can be used in modeling of structures. The library currently includes: rigid element, truss element, beam element, and flat shell element. The updated Lagrangian description has been adopted. Towards the end we shall present the application of the computer program to the analysis of a practical engineering problem, namely the analysis of a safety cab installed on an agricultural tractor and being impacted in order to test its performance in a possible roll-over accident. This is a typical example of a structure that can be modelled as a system consisting of deformable and rigid parts. The purpose of

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the analysis is to illustrate the practical importance of the method of analysis presented in the paper.

2. Description of the finite elements used in the computer code

2.1. Rigid element

A rigid element represents a part of the structure where no deformations are expected. It can be viewed as an arbitrary rigid body with known mass and principal inertia moments with respect to the axes of the local coordinate system passing through the center of mass. The algorithm for the analysis of structures consisting of deformable and rigid parts, implemented in AKAB, is based on the elimination of the undetermined Lagrange multipliers and constrained kinematic unknowns on the element level prior to the assembling of the global matrices and vectors. The set of the equations is symmetrized by shifting the terms introducing nonsymmetry to the right hand side and updating them iteratively.

2.2. Truss element

It is a simple 2 node truss element with constant strain and stress along its length, (Bathe, et al. 1974, Kleiber and Wozniak 1991). Three cases are possible: (i) the element subjected both to compressive and tensile loading, (ii) the element carrying load only when compressed, and (iii) the element carrying load only under stretching. The case (iii) is useful in modeling cables. By selecting the option (ii) we can use the elements to represent springs modelling vehicle tyres, for instance.

2.3. Beam element

2.3.1. General remarks

A simplest straight, prismatic two node beam element based on the Euler-Bernoulli theory neglecting the influence of shear stresses has been used. The equations for the beam element are formulated in the space of generalized strains and generalized stresses (stress resultants). The shape functions are constructed assuming cubic bending displacement variation and a linear variation in the axial and torsional displacements, which gives the element matrices of the form that can be found in Kleiber and Wozniak 1991, for instance.

2.3.2. Elastic-plastic stiffness matrix for the beam element

The elastic-plastic constitutive matrix is denoted as $\boldsymbol{k}^{(ep)}$. When no yielding occurs $\boldsymbol{k}^{(ep)}$ coincides with the elastic constitutive stiffness matrix $\boldsymbol{k}^{(e)}$. The effect of yielding is considered globally on the cross-section level. An immediate plastification of the whole cross-section is assumed when a limit state condition is satisfied. The yield condition, which can be written generally as

$$f=F(\hat{\boldsymbol{\sigma}})-k(\bar{\hat{\varepsilon}}^{(p)})=0 \tag{1}$$

depends on the stress resultants $\hat{\boldsymbol{\sigma}}$, with k being the generalized yield stress, and $\bar{\bar{\varepsilon}}^{(p)}$ being the equivalent generalized plastic strain in the section, playing the role of a hardening parameter. In the vector $\hat{\boldsymbol{\sigma}}$ the axial force F_x, the torsional moment M_x and bending moments with respect to the central principal axes of the cross-section M_y and M_z are included, i.e.

$$\hat{\boldsymbol{\sigma}}=\{F_x\ M_x\ M_y\ M_z\}^T \tag{2}$$

The shear forces are neglected. The limit state surface defined in Eq. (1) can expand due to the strain-hardening of the material without changing its shape. Exact treatment of strain-hardening effects in the elastic-plastic analysis at the cross-section level in a general case is impossible. A simple approach presented here enables one to perform the analysis which should be at least qualitatively better than that based on elastic-ideally plastic model of material, (Toi and Yang 1991, Ueda and Fujikubo 1992). The equivalent generalized plastic strain in the section $\bar{\bar{\varepsilon}}^{(p)}$ is defined by the following relation, (Ueda and Fujikubo 1992)

$$\bar{\bar{\sigma}}\, d\bar{\bar{\varepsilon}}^{(p)}=\hat{\boldsymbol{\sigma}}^{T} d\bar{\bar{\boldsymbol{\varepsilon}}}^{(p)} \tag{3}$$

Here, $\bar{\bar{\sigma}}$ is the equivalent generalized stress associated with the equivalent generalized plastic strain $\bar{\bar{\varepsilon}}^{(p)}$. The vector of incremental generalized strains corresponding to the vector of generalized stresses $\hat{\boldsymbol{\sigma}}$ defined by Eq. (2) has the following components

$$d\hat{\boldsymbol{\varepsilon}}=\{d\hat{\varepsilon}_{xx}\ \frac{d(d\theta_x)}{dx'}\ d\kappa_z\ -d\kappa_y\}^T \tag{4}$$

where $\hat{\varepsilon}_{xx}$ is axial strain of the beam axes, θ_x is the angle of torsion of the beam cross-section, while κ_z,κ_y are the beam axis curvatures.

For elastic-plastic processes described in terms of generalized strains and stresses we assume the additive decomposition of the generalized strain increments into the elastic $\Delta\hat{\boldsymbol{\varepsilon}}^{(e)}$ and plastic $\Delta\hat{\boldsymbol{\varepsilon}}^{(p)}$ parts. The associative flow rule in terms of generalized strains and stresses, Olszak, et al. 1965 reads

$$\Delta\hat{\boldsymbol{\varepsilon}}^{(p)}=\Delta\lambda\left(\frac{\partial f}{\partial\hat{\boldsymbol{\sigma}}}\right)^T \tag{5}$$

During yielding we should have

$$\dot{f}=0 \tag{6}$$

which, upon introducing f from Eq. (1) and considering a small time increment, can be written in the incremental form as

$$\frac{\partial F}{\partial\hat{\boldsymbol{\sigma}}}\Delta\hat{\boldsymbol{\sigma}}-\hat{H}\Delta\bar{\bar{\varepsilon}}^{(p)}=0 \tag{7}$$

where $\hat{H}=\dfrac{dk}{d\bar{\bar{\varepsilon}}^{(p)}}$ can be called a cross-section equivalent strain-hardening modulus. A possible approximate procedure for expressing $\hat{H}$ as a function of material strain-hardening modulus and geometric properties of the section can be found in Ueda and Fujikubo 1992.

Eqs. (1)-(7) refer to the yielded beam section without considering the distribution of plastic regions along the beam axes. In many practical examples of frame structures it is justified to assume that plastic deformations are concentrated in very small regions. These regions can be idealized as the so-called plastic hinges with plastic strains limited only to one section of the beam. Since generalized strain velocities in the plastic hinges tend to infinity, it is necessary to define now the generalized strains in the following way, (Olszak, et al. 1965):

$$\Delta u_x^{(p)} = \lim_{\Delta\bar{x}\to 0} \int_x^{x+\Delta\bar{x}} \Delta\varepsilon_{xx}^{(p)} \, dx \qquad \Delta\theta_x^{(p)} = \lim_{\Delta\bar{x}\to 0} \int_x^{x+\Delta\bar{x}} \frac{d(\Delta\theta_x^{(p)})}{dx'} \, dx$$
$$\Delta\theta_y^{(p)} = \lim_{\Delta\bar{x}\to 0} \int_x^{x+\Delta\bar{x}} \Delta\kappa_y^{(p)} \, dx \qquad \Delta\theta_z^{(p)} = \lim_{\Delta\bar{x}\to 0} \int_x^{x+\Delta\bar{x}} \Delta\kappa_z^{(p)} \, dx \tag{8}$$

where $\Delta\bar{x}$ is the width of the region where plastic strains are concentrated. The generalized strains defined by formulas (8) have the physical character of incremental displacement and rotation angles.

In the finite element analysis, when we use the beam element with linear variation of the bending moments and the constant axial force and torsional moment, plastic hinges can be located at the element nodes only. Incremental generalized plastic strains defined by Eq. (8) correspond then to the appropriate components of the vector of the incremental generalized displacements $\Delta\boldsymbol{q}^{(\alpha)}$ defined at the α-th node ($\alpha=1, 2$) and are equal to their plastic parts $\Delta\boldsymbol{q}^{(\alpha)(p)}$, the following additive decomposition being assumed

$$\Delta\boldsymbol{q} = \Delta\boldsymbol{q}^{(e)} + \Delta\boldsymbol{q}^{(p)} \tag{9}$$

$\Delta\boldsymbol{q} = \{\Delta\boldsymbol{q}^{(1)\,T} \ \Delta\boldsymbol{q}^{(2)\,T}\}$, etc. The generalized stresses $\hat{\boldsymbol{\sigma}}$ at the α-th node are the appropriate components $\boldsymbol{P}^{(\alpha)}$ of the vector of generalized nodal forces at this node. The limit state condition (1) for the α-th node is written in the following form

$$f^{(\alpha)}(\boldsymbol{P}^{(\alpha)}, \bar{\bar{\varepsilon}}^{(\alpha)(p)}) = F^{(\alpha)}(\boldsymbol{P}^{(\alpha)}) - k^{(\alpha)}(\bar{\bar{\varepsilon}}^{(\alpha)(p)}) = 0 \tag{10}$$

The associative flow rule for the generalized displacements $\Delta\boldsymbol{q}^{(p)}$ has the form

$$\Delta\boldsymbol{q}^{(\alpha)(p)} = \Delta\lambda_\alpha^* \boldsymbol{n}^{(\alpha)} \tag{11}$$

where

$$\boldsymbol{n}^{(\alpha)} = \left(\frac{\partial F^{(\alpha)}}{\partial \boldsymbol{P}^{(\alpha)}} \right)^T \tag{12}$$

If we take into account the possibility of yielding at both nodes of the element and combine the two equations into one we have

$$\Delta\boldsymbol{q}^{(p)} = \boldsymbol{n}\Delta\boldsymbol{\lambda}^* \tag{13}$$

where

$$\boldsymbol{n} = \begin{bmatrix} \boldsymbol{n}^{(1)} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{n}^{(2)} \end{bmatrix}, \quad \Delta\boldsymbol{\lambda}^* = \{\Delta\lambda_1^* \ \Delta\lambda_2^*\}^T \tag{14}$$

The increments of the generalized nodal forces are equal to the increments which result from the elastic deformations in the element

$$\Delta\boldsymbol{P} = \boldsymbol{k}^{(e)} \Delta\boldsymbol{q}^{(e)} \tag{15}$$

where $\boldsymbol{k}^{(e)}$ is the elastic constitutive stiffness matrix. On the other hand, the elastic-plastic constitutive stiffness matrix enters the equation

$$\Delta \boldsymbol{P} = \boldsymbol{k}^{(ep)} \Delta \boldsymbol{q} \tag{16}$$

The condition (7) can be written for the α-th node as

$$\boldsymbol{n}^{(\alpha)\,T} \Delta \boldsymbol{P}^{(\alpha)} - \hat{H}^{(\alpha)} \Delta \bar{\bar{\varepsilon}}^{(\alpha)(p)} = 0 \tag{17}$$

Inserting the relationship

$$\hat{H}^{(\alpha)*} = \frac{1}{\Delta \lambda_\alpha^*} \hat{H}^{(\alpha)} \Delta \bar{\bar{\varepsilon}}^{(\alpha)(p)} \tag{18}$$

into Eq. (17) we get

$$\boldsymbol{n}^{(\alpha)\,T} \Delta \boldsymbol{P}^{(\alpha)} - \hat{H}^{(\alpha)*} \Delta \lambda_\alpha^* = 0 \tag{19}$$

The parameter $\hat{H}^{(\alpha)*}$ is called the nodal 'hardening' modulus. The formulae which make it possible to calculate the value of this modulus will be derived later in this paper.

Eq. (19) can be combined for $\alpha=1$ and $\alpha=2$ to give

$$\boldsymbol{n}^T \Delta \boldsymbol{P} - \hat{\boldsymbol{H}}^* \Delta \boldsymbol{\lambda}^* = 0 \tag{20}$$

where

$$\hat{\boldsymbol{H}}^* = \begin{bmatrix} \hat{H}^{(1)*} & 0 \\ 0 & \hat{H}^{(2)*} \end{bmatrix} \tag{21}$$

On introducing the vector $\Delta \boldsymbol{q}^{(e)}$ defined by Eq. (9) into Eq. (15) we arrive at

$$\Delta \boldsymbol{P} = \boldsymbol{k}^{(e)} (\Delta \boldsymbol{q} - \Delta \boldsymbol{q}^{(p)}) \tag{22}$$

By using Eqs. (22) and (13) in Eq. (20) we get the following formula

$$\Delta \boldsymbol{\lambda}^* = (\boldsymbol{n}^T \boldsymbol{k}^{(e)} \boldsymbol{n} + \hat{\boldsymbol{H}}^*)^{-1} \boldsymbol{n}^T \boldsymbol{k}^{(e)} \Delta \boldsymbol{q} \tag{23}$$

By means of Eqs(23), (13) and (22) we obtain the expression for the elastic-plastic constitutive stiffness matrix of the beam element in the form

$$\boldsymbol{k}^{(ep)} = \boldsymbol{k}^{(e)} - \boldsymbol{k}^{(e)} \boldsymbol{n} (\boldsymbol{n}^T \boldsymbol{k}^{(e)} \boldsymbol{n} + \hat{\boldsymbol{H}}^*)^{-1} \boldsymbol{n}^T \boldsymbol{k}^{(e)} \tag{24}$$

We shall now return to the problem of finding nodal 'hardening' moduli $\hat{H}^{(\alpha)*}$, defined by Eq. (18), which appear on the diagonal in the matrix $\hat{\boldsymbol{H}}^*$ given by Eq. (21). The modulus $\hat{H}^{(\alpha)*}$ can be obtained from (18), provided we know $\Delta \bar{\bar{\varepsilon}}^{(\alpha)(p)}$. In order to obtain $\Delta \bar{\bar{\varepsilon}}^{(\alpha)(p)}$ we can assume (Ueda and Fujikubo 1992) that plastic work in the plastic hinge is equal to the plastic work in the real yielded zone spreading in the element along a certain length $l_{(\alpha)}^{(p)}$. On expressing the distribution of the equivalent plastic strain increment $\Delta \bar{\bar{\varepsilon}}^{(\alpha)}$ in the yielded zone as a function of the equivalent plastic strain at the α-th node $\Delta \bar{\bar{\varepsilon}}^{(\alpha)(p)}$ by means of a certain distribution function $g(x)$

$$\Delta \bar{\bar{\varepsilon}}^{(p)}(x) = g(x) \Delta \bar{\bar{\varepsilon}}^{(\alpha)(p)} \tag{25}$$

we can write plastic work equality in the following form

$$\left(\int_{l_{(\alpha)}^{(p)}} \bar{\bar{\sigma}}^{(\alpha)(p)} g(x)dx\right)\Delta\bar{\bar{\varepsilon}}^{(\alpha)(p)} = \Delta\lambda_{\alpha}^{*} \boldsymbol{n}^{(\alpha)} \boldsymbol{P}^{(\alpha)} \tag{26}$$

The value of $\Delta\bar{\bar{\varepsilon}}^{(\alpha)(p)}$ can be obtained from Eq. (26). The integral in this equation can be evaluated numerically. The length of the yielded zone is determined by checking the yield condition along the beam length. The values of the distribution function $g(x)$ can be computed from the expression obtained from (25) after introducing $\Delta\bar{\bar{\varepsilon}}^{(p)}(x)$ and $\Delta\bar{\bar{\varepsilon}}^{(\alpha)(p)}$ defined by (7) and (17), respectively.

2.3.3. Yield conditions for beams under complex loading

Exact derivation of the yield conditions (1) in their explicit form for beams with arbitrary cross-sections under complex loading is a very difficult task, cf. Zyczkowski 1981. Therefore approximate equations for the yield surfaces are usually assumed, cf. Egger and Kröplin 1978, Kleiber and Woźniak 1991, Sosnowski 1989, Tong and Rossetos 1977, Ueda and Fujikubo 1992, Ueda and Yao 1982. In this work we have used an approximate form of the yield condition, written for the α-th node as

$$f^{(\alpha)} = \left(\frac{M_x^{(\alpha)}}{M_x^{(0)}}\right)^2 + \left(\frac{M_y^{(\alpha)}}{M_y^{(0)}}\right)^2 + \left(\frac{M_z^{(\alpha)}}{M_z^{(0)}}\right)^2 + \left(\frac{F_x^{(\alpha)}}{F_x^{(0)}}\right)^2 - \left[k^{(\alpha)}(\bar{\bar{\varepsilon}}^{(\alpha)(p)})\right]^2 = 0 \tag{27}$$

Eq. (27) is based on the known values of the limit axial force $F_x^{(0)}$, limit torsional moment $\mathbf{M}_x^{(0)}$, and limit bending moments $\mathbf{M}_y^{(0)}$ and $\mathbf{M}_z^{(0)}$. It is also assumed that $k^{(\alpha)}$ (the generalized normalized yield stress) is a linear function of the hardening parameter previously defined

$$k^{(\alpha)} = 1 + \hat{H}\bar{\bar{\varepsilon}}^{(\alpha)(p)} \tag{28}$$

Expression (27) is similar to those used in other formulations, cf. Sosnowski 1989, Tong and Rossetos 1977.

2.3.4. Initial load method

In the initial load formulation material nonlinear effects are taken into account in the vector of fictitious forces $\Delta\boldsymbol{\psi}$ on the right-hand side of the equilibrium equations. The elemental vector of initial load for the beam element can be evaluated from the following equation

$$\Delta\boldsymbol{\psi} = \boldsymbol{k}^{(p)}\Delta\boldsymbol{q} \tag{29}$$

where $\boldsymbol{k}^{(p)}$ is given by

$$\boldsymbol{k}^{(p)} = \boldsymbol{k}^{(e)}\boldsymbol{n}(\boldsymbol{n}^T\boldsymbol{k}^{(e)}\boldsymbol{n} + \hat{\boldsymbol{H}}^{*})^{-1}\boldsymbol{n}^T\boldsymbol{k}^{(e)} \tag{30}$$

Similarly as in the elastic-plastic analysis, in the elastic-viscoplastic analysis of frames we assume that inelastic strains are concentrated at the element nodes and yielding occurs when the yield condition given by Eq. (27) is satisfied. The vector of the initial load (fictitious forces on the right-hand side) can be written as

$$\Delta\boldsymbol{\psi} = \boldsymbol{k}^{(e)}\Delta\boldsymbol{q}^{(vp)} \tag{31}$$

The viscoplastic flow law is used in the form

$$\Delta \boldsymbol{q}^{(\alpha)(vp)} = \Delta t \gamma \left\langle \Phi \left(f^{(\alpha)}\right)\right\rangle \frac{\partial f}{\partial \boldsymbol{P}^{(\alpha)}} \tag{32}$$

where $\alpha=1, 2$ numbers the beam nodes, γ is a certain fluidity parameter and $\Phi(f^{(\alpha)})$ is the so-called overstress function assumed in the simplest form as

$$\Phi(f^{(\alpha)}) = f^{(\alpha)} \tag{33}$$

In the computer implementation expressions (29) and (31) are used in the iterative form

Both elastic-plastic and elastic-viscoplastic algorithms presented above have been implemented for both static and dynamic analysis. In the dynamic analysis inertial properties of the beam element are represented by the lumped mass matrix.

2.4. Shell element

2.4.1. General comments

Shells are approximated by flat triangular elements. With well-known disadvantages these elements have one superior feature over all other shell elements — they are very effective numerically. Therefore they are still often used in non-linear analysis, including practical applications (Haug, et al. 1983, Hayduk, et al. 1983). The shell element employed is obtained by combining the constant strain triangle — the CST membrane element (see, for instance, Cook, et al. 1989, Gallagher 1975) and the linear curvature compatible triangle — the LCCT-9 plate element, known also as the HCT element (Clough and Felippa 1968).

The vector of generalized stresses for this element $\hat{\boldsymbol{\sigma}}$ has as its components the membrane forces $\hat{\boldsymbol{\sigma}}^{(m)}$ and bending moments $\hat{\boldsymbol{\sigma}}^{(b)}$

$$\hat{\boldsymbol{\sigma}} = \{\hat{\boldsymbol{\sigma}}^{(m)}\ \hat{\boldsymbol{\sigma}}^{(b)}\}^T \tag{34}$$

$$\hat{\boldsymbol{\sigma}}^{(m)} = \{N_{xx} N_{yy} N_{xy}\}^T \tag{35}$$

$$\hat{\boldsymbol{\sigma}}^{(b)} = \{M_{xx} M_{yy} M_{xy}\}^T \tag{36}$$

The incremental constitutive relationship for the elastic shell is written as

$$\Delta\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{C}}^{(e)} \Delta\hat{\boldsymbol{\varepsilon}} \tag{37}$$

where $\hat{\boldsymbol{C}}^{(e)}$ is the constitutive elastic matrix for the shell and $\Delta\hat{\boldsymbol{\varepsilon}}$ is the vector of respective incremental generalized strains, consisting of membrane $\Delta\hat{\boldsymbol{\varepsilon}}^{(m)}$ and bending $\Delta\hat{\boldsymbol{\varepsilon}}^{(b)}$ components

$$\Delta\hat{\boldsymbol{\varepsilon}} = \{\Delta\hat{\boldsymbol{\varepsilon}}^{(m)}\ \Delta\hat{\boldsymbol{\varepsilon}}^{(b)}\}^T \tag{38}$$

$$\Delta\hat{\boldsymbol{\varepsilon}}^{(m)} = \{\Delta\hat{\varepsilon}_{xx}\ \Delta\hat{\varepsilon}_{yy}\ 2\Delta\hat{\varepsilon}_{xy}\}^T \tag{39}$$

$$\Delta\hat{\boldsymbol{\varepsilon}}^{(b)} = \{\Delta\kappa_{xx}\ \Delta\kappa_{yy}\ 2\Delta\kappa_{xy}\}^T \tag{40}$$

The constitutive elastic matrix for the shell $\hat{\boldsymbol{C}}^{(e)}$ is written in the following form

$$\hat{\boldsymbol{C}}^{(e)} = \begin{bmatrix} \hat{\boldsymbol{C}}^{(m)(e)} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{C}}^{(b)(e)} \end{bmatrix} \tag{41}$$

where $\hat{\boldsymbol{C}}^{(m)(e)}$ and $\hat{\boldsymbol{C}}^{(b)(e)}$ are the elastic constitutive matrices for the membrane and plate, respectively.

The elastic constitutive stiffness matrix for the shell element $\boldsymbol{k}^{(con)}$ is written as

$$\boldsymbol{k}^{(con)} = \boldsymbol{k}^{(e)} = \begin{bmatrix} \boldsymbol{k}^{(m)(e)} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{k}^{(b)(e)} \end{bmatrix} \tag{42}$$

where $\boldsymbol{k}^{(m)(e)}$ and $\boldsymbol{k}^{(b)(e)}$ are membrane and bending elastic constitutive stiffness matrices, respectively. Since in the constitutive elastic matrix $\boldsymbol{C}^{(e)}$ given by Eq. (41) there is no interaction between membrane and bending states, in the elastic constitutive stiffness matrix $\boldsymbol{k}^{(con)}$ the coupling submatrices for the membrane bending stiffness have only zero elements. The tangent stiffness matrix $\boldsymbol{k}$ used in the nonlinear analysis in the UL formulation is composed of the constitutive stiffness matrix $\boldsymbol{k}^{(con)}$ and the initial stress matrix $\boldsymbol{k}^{(\sigma)}$ as

$$\boldsymbol{k} = \boldsymbol{k}^{(con)} + \boldsymbol{k}^{(\sigma)} \tag{43}$$

The initial stress matrix $\boldsymbol{k}^{(\sigma)}$ used in this work has non-zero elements only in the submatrix corresponding to the bending degrees of freedom

$$\boldsymbol{k}^{(\sigma)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{k}^{(b)(\sigma)} \end{bmatrix} \tag{44}$$

The matrix $\boldsymbol{k}^{(b)(\sigma)}$ used is the so-called inconsistent matrix evaluated using simplified shape functions. As it was proved in many tests, (Hien, et al. 1989), by using this matrix one can often obtain very good results. The membrane element used in the formulation has no stiffness for rotation about the normal to its surface. Therefore, to avoid numerical problems when coplanar elements are assembled we have added an artificial stiffness term for the rotation about the normal in accordance with the suggestion made by Bathe (1982). In the dynamic analysis of shells we have used the lumped matrix with non-zero rotational terms according to Surana 1978.

2.4.2. Elastic-plastic analysis of shells

By considering the yielding of the shell in the space of the stress resultants given by Eq. (34), there is no need to perform a numerical integration through the thickness. The limit state condition is taken in the Ilyushin's form, (Crisfield 1981, Eidsheim and Larsen 1981) as

$$f_L = \frac{(\bar{N})^2}{(N^{(0)})^2} + \frac{(\bar{M})^2}{(M^{(0)})^2} + \frac{1}{\sqrt{3}}\frac{|\overline{MN}|}{M^{(0)}N^{(0)}} - 1 = 0 \tag{45}$$

where

$$\begin{aligned} N^{(0)} &= \sigma^{(Y)} h \\ M^{(0)} &= \frac{\sigma^{(Y)} h^2}{4} \\ (\bar{N})^2 &= (N_{xx})^2 + (N_{yy})^2 - N_{xx}N_{yy} + 3(N_{xy})^2 \\ (\bar{M})^2 &= (M_{xx})^2 + (M_{yy})^2 - M_{xx}M_{yy} + 3(M_{xy})^2 \\ \overline{MN} &= M_{xx}N_{xx} + M_{yy}N_{yy} - \frac{1}{2}(M_{xx}N_{yy} + M_{yy}N_{xx}) + 3M_{xy}N_{xy} \end{aligned} \tag{46}$$

Another effective concept which preserves all the advantages of the formulation in the stress resultant space and at the same time accounts for the gradual development of yielding through the shell thickness was presented in Crisfield 1981, Eidsheim and Larsen 1981. There, it was suggested to introduce a control parameter a to enable the limit state condition expanding from the initial yield condition given by the following formula

$$f_o=\frac{(\overline{N})^2}{(N^{(0)})^2}+\frac{9}{4}\frac{(\overline{M})^2}{(M^{(0)})^2}+\frac{3\,|\,\overline{MN}\,|}{M^{(0)}N^{(0)}}-1=0 \tag{47}$$

to the limit yield condition given by Eq. (45). The form of the yield condition which covers this range was suggested as

$$f=\frac{(\overline{N})^2}{(N^{(0)})^2}+\frac{(\overline{M})^2}{a^2(M^{(0)})^2}+\frac{1}{\sqrt{3}}\frac{|\,\overline{MN}\,|}{M^{(0)}N^{(0)}b}-1=0 \tag{48}$$

The coefficients a and b in Eq. (48) are given in terms of the effective plastic curvature $\bar{\kappa}^{(p)}$ as

$$a(\bar{\kappa}^{(p)})=1-\frac{1}{3}\exp\left(\frac{8}{3}\;\frac{\kappa^{(p)}}{\kappa^{(0)}}\right) \tag{49}$$

$$b(\bar{\kappa}^{(p)})=1-\left(3-\frac{1}{\sqrt{3}}\right)\left(1-a(\bar{\kappa}^{(p)})\right) \tag{50}$$

$$\kappa^{(0)}\;=\;\frac{2\sigma^{(Y)}}{Eh} \tag{51}$$

$$\bar{\kappa}^{(p)}=\int_0^{\bar{\kappa}^{(p)}}d\bar{\kappa}^{(p)}=\int_0^{\bar{\kappa}^{(p)}}\frac{2}{\sqrt{3}}\left[\left(d\bar{\kappa}_{xx}^{(p)}\right)^2+\left(d\kappa_{xx}^{(p)}\right)^2+d\kappa_{xx}^{(p)}\,d\kappa_{yy}^{(p)}+\frac{\left(d\kappa_{xy}^{(p)}\right)^2}{4}\right]^{\frac{1}{2}} \tag{52}$$

For $\bar{\kappa}^{(p)}=0$ Eq. (48) is equivalent to the condition of the initial yielding (47), while for $\bar{\kappa}^{(p)}\rightarrow\infty$ Eq. (48) coincides with the limit state condition (45).

After assuming the yield condition in the form (48) and the associative flow law

$$\Delta\hat{\varepsilon}^{(p)}=\Delta\lambda\left(\frac{\partial f}{\partial\hat{\sigma}}\right)^T \tag{53}$$

we can obtain the elastic-plastic constitutive matrix in the form

$$\hat{C}^{(ep)}=\hat{C}^{(e)}-\frac{\hat{C}^{(e)}\left(\dfrac{\partial f}{\partial\hat{\sigma}}\right)^T\dfrac{\partial f}{\partial\hat{\sigma}}\,\hat{C}^{(e)}}{\hat{H}+\dfrac{\partial f}{\partial\hat{\sigma}}\,\hat{C}^{(e)}\left(\dfrac{\partial f}{\partial\hat{\sigma}}\right)^T}=\hat{C}^{(e)}-\hat{C}^{(p)} \tag{54}$$

where

$$\hat{H}=-\frac{\partial f}{\partial a}\,\frac{\partial a}{\partial\bar{\kappa}^{(p)}}\,\frac{\partial\bar{\kappa}^{(p)}}{\partial\lambda} \tag{55}$$

and

$$\frac{\partial \bar{\kappa}^{(p)}}{\partial \lambda} = \frac{2}{\sqrt{3}} \left[\left(\frac{\partial f}{\partial M_{xx}} \right)^2 + \left(\frac{\partial f}{\partial M_{yy}} \right)^2 + \frac{\partial f}{\partial M_{xx}} \frac{\partial f}{\partial M_{yy}} + \frac{1}{4} \left(\frac{\partial f}{\partial M_{xy}} \right)^2 \right]^{\frac{1}{2}} \tag{56}$$

which can be introduced into Eq. (55).

The elastic-plastic constitutive matrix can be divided into the following submatrices

$$\hat{C}^{(ep)} = \begin{bmatrix} \hat{C}^{(m)(ep)} & \hat{C}^{(mb)(ep)} \\ \hat{C}^{(mb)(ep)\,T} & \hat{C}^{(b)(ep)} \end{bmatrix} \tag{57}$$

where $\hat{C}^{(m)(ep)}$ is the membrane elastic-plastic constitutive matrix, $\hat{C}^{(b)(ep)}$ is the bending elastic-plastic constitutive matrix and $\hat{C}^{(mb)(ep)}$ is the elastic-plastic constitutive matrix coupling me mbrane and bending modes (which is no longer zero in contrast to Eq. (41), cf. Rojek, et al. 1991). The elastic-plastic constitutive matrix given by Eqs. (54) and (57) can be used to get the elastic-plastic constitutive stiffness matrix for the shell element.

The constitutive relationships obtained above can also be used to evaluate the initial load vector accounting for the plastic deformations. The elemental vector of initial loads can be obtained from the following equation

$$\Delta\Psi = \int_A B^{(L)\,T} \Delta\hat{\sigma}^{(p)} dA \tag{58}$$

where $B^{(L)}$ is the linear strain operator matrix, A is the area of an element and

$$\Delta\hat{\sigma}^{(p)} = \hat{C}^{(p)} \Delta\hat{\varepsilon} \tag{59}$$

matrix $\hat{C}^{(p)}$ being defined in Eq. (57). In the numerical algorithm Eqs. (58)-(59) are used in the iterative form as the relationships for consecutive approximations. Similarly, relationships for the tangential stiffness method in the numerical algorithm are used in the form adequate to the iterative method employed, (Rojek 1992).

2.4.3. Elastic-viscoplastic analysis of shells

The elastic-viscoplastic analysis has been implemented for the shell element using the initial load technique. The constitutive law for viscoplastic flow is taken as

$$\Delta\hat{\varepsilon}^{(vp)} = \Delta t \gamma \, \langle \Phi(f) \rangle \left(\frac{\partial f}{\partial \hat{\sigma}} \right)^T \tag{60}$$

where γ is a viscosity parameter and $\Phi(f)$ is the ovestress function assumed in the simplest form

$$\Phi(f) = f \tag{61}$$

The vector of initial load is obtained from the integral

$$\Delta\Psi = \int_A B^{(L)\,T} \hat{C}^{(e)} \Delta\hat{\varepsilon}^{(vp)} dA \tag{62}$$

which in the computer implementation is used iteratively as consecutive approximations to the exact value.

3. Computer simulation of dynamic tests for safety cabs of agricultural tractors

3.1. Requirements for cab design and technical conditions of laboratory tests

The computer program AKAB has been verified on a great number of test examples of beams, frames, plates and shells for different cases of nonlinear analysis, both under static and dynamic loading (Kleiber and Rojek 1992, Rojek 1992, Rojek and Kleiber 1991). Test examples of structures consisting of deformable and rigid parts (Kleiber and Rojek 1992, Rojek 1992, Rojek and Kleiber 1991) have also been analyzed. The results have generally shown good agreement with theoretical solutions and numerical results presented in literature. The program was then applied to the analysis of a practical problem in the form of a computer simulation of the dynamic tests conducted to verify the strength of the roll-over protective structures (ROPS), either frames or cabs, installed on agricultural tractors.

National and international standards, such as ASAE Standard, ISO 3463/3, 5700, establish test procedures to verify the effectiveness of ROPS. Regulations require the structures to pass a specific set of laboratory quasi-static or impact tests simulating rollover. The dynamic test procedure according to the ISO standards (ISO 3463/3) consists of the following tests: impact at rear, static crush test with loading applied at rear, impact at front, impact at side and static crush test with loading applied at front. The standard requirements for stiffness are met if a certain space around the place of the driver, the so-called zone of clearance, is preserved free in all the tests. On the other hand small deformations prove the structure is overdesigned.

The laboratory tests are costly and time consuming. Their number can be greatly reduced if the computer simulation is performed during the design phase, although a number of tests must still be conducted to validate the design concepts basing on the numerical analysis.

Our purpose was to perform the computer simulation of impact tests conducted for ROPS of agricultural tractors. For the impact tests the tractor with a cab standing on the wheels is restrained from moving by means of restraining cables and beams positioned against the wheels. The dynamic loading is generated by use of a 2000 kg pendulum. The standard determines the height of the center of gravity of the pendulum pulled back before the test, which ensures the required energy of impact.

3.2. Mechanical and numerical model of tractor with cab subjected to impact

In the numerical analysis of roll-over protective structures for agricultural tractors, presented in Kearns 1987, Kecman 1983, Rusinski 1983, 1984, Tidbury 1984, Yeh, et al. for instance, models of elastic-plastic frames subjected to quasi-static loading have mainly been used. Formulation of equations for the systems consisting of rigid and deformable parts, presented in our paper, enables us to examine the behavior of the whole vehicle (and not only the frame) under the dynamic loading. According to our knowledge the structures on hand have not been treated in this way in the available literature. On the other hand, the problem of stiffness evaluation of the protective structures for agricultural structures belongs to a wider class of problems considered in the so-called crashworthiness analysis. At present the power of today's mainframe and supercomputers makes it possible to use very large and complicated models of structures with hundreds of thousands unknowns, and such models are also employed in the crashworthiness analysis, cf. Belytschko 1992, Haug, et al. 1989, Schweizerkof, et al. 1991.

However, even when using the contemporary mesh generators preparing data for a large model of a structure like a car body is a time-consuming process. Therefore the number of possible design solutions we can analyze in this way is limited. Thus, the use of medium size models similar to that used in our work has some advantages. A great number of design alternatives can be analyzed at early stages of the design process.

In the analysis we have used a model of the tractor in which the cab with fenders, the mounting adaptors of the cab, the tyres, and the restraining cables were treated as deformable parts, while the rest of the structure (the tractor chassis, engine, gear box, front and rear axles) was replaced by one rigid body. We neglected the existence of possible gaps and mechanisms in the structure. The mass of the wheels was added to the rigid body, and at the same time the rotational motion of the wheels was neglected, as was the friction between the wheels and the ground. Tyres were replaced by nonlinear elastic springs subjected only to compression in the direction perpendicular to the ground. The cab with fenders is treated as a frame and shell structure that can undergo large displacements and inelastic deformations, either plastic or viscoplastic ones, depending on the material model assumed. The deformable frame is connected to the rigid body by means of two nonlinear elastic springs at the rear bottom corners and two hinges at the front bottom part of the structure. Structural damping is considered in an approximate way by assuming it to be of the Rayleigh's type. Restraining cables are represented in the model by spring elements that can carry only extensional loads. Gravitational forces are taken into account by the constant resultant force applied at the center of mass of the rigid element. At the initial moment this force is in equilibrium with initial prestressing forces given in the springs representing the tyres. Initial tension should also be introduced into the restraining cables; however, these forces were neglected in the present analysis, since it was difficult to evaluate them accurately enough from available experimental data.

The impact loading is modelled by assuming an initial velocity at the place of impact. We assume that the pendulum is in permanent contact with the structure after the impact, and its mass is added to the structure as distributed over the place of contact. As we are interested in the global collapse mode, we neglect local effects at the place of impact as well as the wave propagation effects after impact.

3.3. Results of computer simulation

The analysis was performed for the safety cab of the type M87U mounted on the tractor Ursus 1634. The FE model has 160 beam and 204 shell elements, 16 springs and one rigid element. The model was defined with the use of about 200 nodes, which resulted in about 1200 unknowns.

We present the results of the computer simulations for two load cases — impact at rear and impact at side. Both elastic-plastic and elastic-viscoplastic material models are considered, and different values of damping coefficients are assumed. For impact at rear the pendulum has the velocity 4.01 m/s at the moment of impact, while for the impact at side the velocity of the pendulum is 4.2 m/s. The analysis was continued until the maximum deflection at the point of impact was achieved as this is an essential fact for the validation of the structural stiffness. Time integration was performed with a constant time step of 0.1 ms. One case required about 1500 time steps. The CPU time for one step on a SPARC 2 workstation was about 40 s.

Fig. 1 presents the deformed configuration after the impact at rear together with the initial configuration. Deformed structure after the impact at side is given in Fig. 2 which proves that theoretical deformations are outside the zone of clearance. Incremental analysis made it also possible to observe the sequence of plastic hinge development in the frame. The location of plastic hinges at different time instants is shown in Fig. 3.

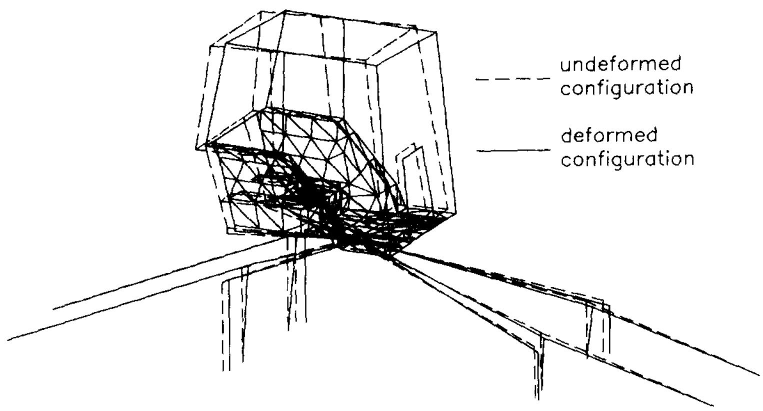

Fig. 1 Deformation of the cab after the impact at rear

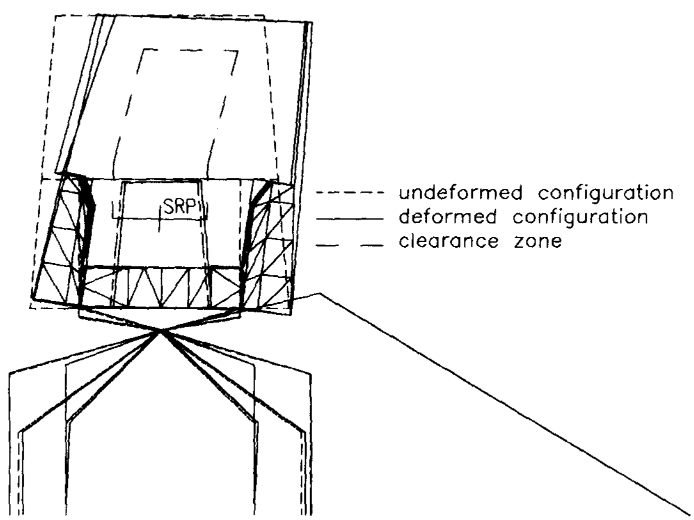

Fig. 2 Deformation of the cab after the impact at side

In Fig. 4 we see the results of the dynamic analysis in the form of the relationship between the horizontal displacement at the point of impact and the time after impact. Response histories are presented for elastic and elastic-plastic material models with different values of damping.

Similarly, we present the response histories for the elastic-viscoplastic model for the impact at rear (Fig. 5). The elastic-viscoplastic analysis was performed for a fixed value of damping and for a set of different viscosity parameters. The elastic-viscoplastic solutions are compared with the results of the elastic and elastic-plastic solutions. For small values of viscosity para-

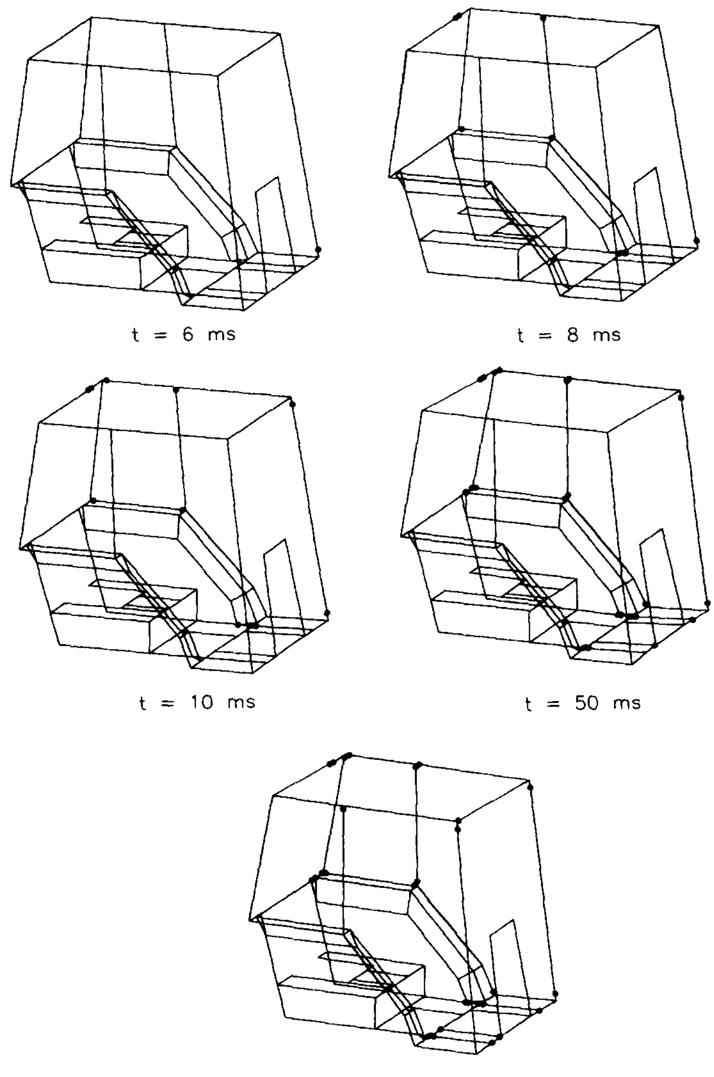

Fig. 3 Distribution of plastic hinges at different time instants after the impact at rear

meter the elastic-viscoplastic solution is close to the elastic solution, and as the value of the viscosity parameter increases the elastic-viscoplastic solution approaches the elastic-plastic response.

The theoretical maximum deformation has been compared with available experimental results. The maximum deflections at the point of impact at rear, as predicted theoretically, are between 140 and 170 mm with respect to the reference frame fixed at the rigid body modeling tractor chassis, with the values varying depending on the damping assumed. The results taken from several tests were ranging from 75 to 145 mm. Similarly we can compare the results for the impact at side. Theoretical horizontal displacements vary from 340 to 440 mm depending on the value of damping, while the experimental results from various tests yield

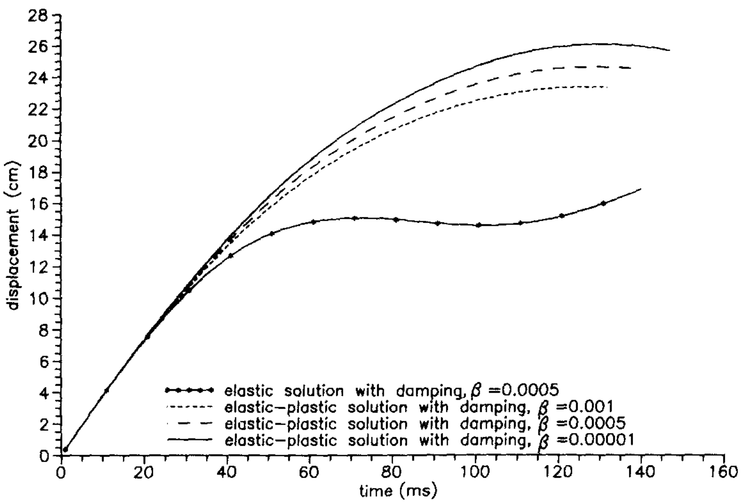

Fig. 4 Horizontal displacement of the point of impact during the impact at rear

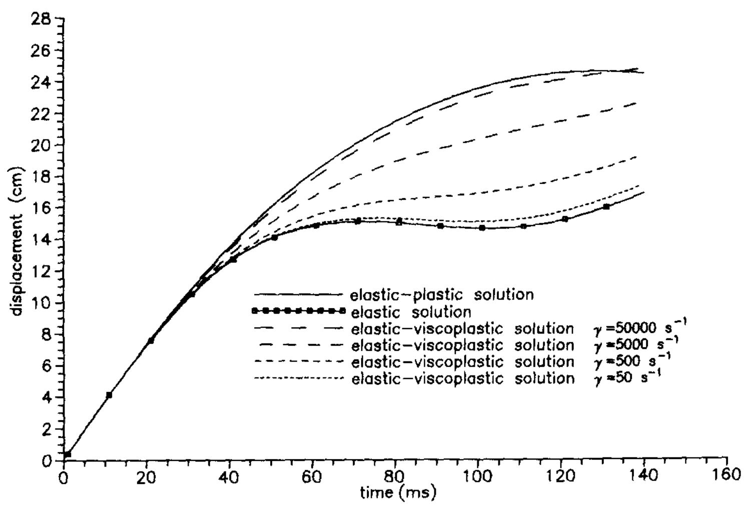

Fig. 5 Horizontal displacement of the point of impact during the impact at rear—results of elastic-viscoplastic analysis compared with elastic and elastic-plastic solutions

values of 315 to 440 mm. As we can see the theoretical results show a qualitative agreement with the experimental results. These, and other results allow us to regard the program AKAB as a useful tool of assessment the stiffness of the structure during the design process. The possibility of using deformable and rigid elements to model the structure enables us to consider the whole structure by means of the medium size model.

4. Concluding remarks

The formulation of equations of motion for structures consisting of rigid and deformable parts, presented in Part I, has been implemented in the computer code called AKAB. The program has wide capabilities of nonlinear analysis which have been briefly described in this paper. Finite elements available in the program have also been described with special attention paid to the way material nonlinearities are accounted for. The capabilities of the computer code have been illustrated with the results of the analysis of an agricultural tractor with a protective structure subjected to the impact loading according to the standard requirements. The structure was modelled as a system of deformable components and a rigid body. The model proved the efficiency of the method discussed. The theoretical results have shown an acceptable agreement with the experimental results; however, we are aware that further validation of the computer program is necessary while many improvements in it are feasible.

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