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Nonlinear dynamic FE analysis of structures consisting of rigid and deformable parts Part I – Formulation

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Abstract. Some structures under the action of some specific loads can be treated as consisting of rigid and deformable parts. The paper presents a way to include rigid elements into a finite element model accounting for geometrical and material nonlinearities. Lagrange multipliers technique is used to derive equations of motion for the coupled deformable-rigid system. Solution algorithm based on the elimination of the Lagrangian multipliers and dependent kinematic unknowns at the element level is described. A follow-up paper(Rojek and Kleiber 1993) complements the discussion by giving details of the computer implementation and presenting some realistic test examples.

Key words: nonlinear finite element analysis; rigid body dynamics; inelastic structures.

1. Introduction

Depending on the application, structures are usually modelled as either deformable or rigid. However, in some practical situations strain-inducing deformations may occur only in selected parts of the structure, while its remaining parts can be regarded as deforming as a collection of rigid bodies. Although the equations of motion used in structural mechanics dealing with deformable bodies as well as the equations describing rigid body motion originate from the same principles, the theories of structural dynamics and rigid body dynamics have been developing separately and have now different methodologies worked out for the analysis of motion.

The aim of the paper is to combine the nonlinear FE equations with the equations of rigid body dynamics. The Lagrange multipliers technique will be used to derive equations of motion for the coupled deformable-rigid system.

Assuming a model consisting of both deformable finite elements and rigid bodies can often be an optimal approach in view of the effectiveness of the analysis. The problem of combining rigid bodies and deformable finite elements in nonlinear analysis has been studied by several authors, cf. Belytschko, et al. 1977, Benson and Hallquist 1986, Park and Saczalski 1974, Saczalski and Huang 1972, Saczalski and Park 1974. We shall present an alternative for-

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mulation of the equations of motion for the coupled system which has been implemented into a program for nonlinear analysis of frame and shell structures(Kleiber and Rojek 1992, Rojek 1992). The computer implementation together with the results of the analysis of a practical problem in the form of a simulation of a tractor safety cab will be presented in Part II of this paper(Rojek and Kleiber 1993).

2. Incremental equations of motion for the continuum

2.1. Incremental description of motion

We are interested in the motion of a body(Fig. 1) in the time interval $[0, \bar{t}]$. The volume of the body and the surrounding boundary surface are denoted by V^t and S^t, respectively. The body is subjected to the external loading consisting of the body forces $\boldsymbol{b}^t$ and traction $\boldsymbol{p}^t$. We introduce the following reference systems: (a) a fixed Cartesian coordinate system $\{X\ Y\ Z\}$, (b) corotational Cartesian coordinate system $\{x^\tau y^\tau z^\tau\}$, $\tau=0,\ \Delta t,\ 2\Delta t, \ldots$, fixed in the time interval$[\tau,\ \tau+\Delta t]$ and updated step-wise. The approach used is known as the updated Lagrangian description. The two coordinate systems are related by the transformation

$$\boldsymbol{x}^\tau = \boldsymbol{L}^\tau \boldsymbol{X} \tag{1}$$

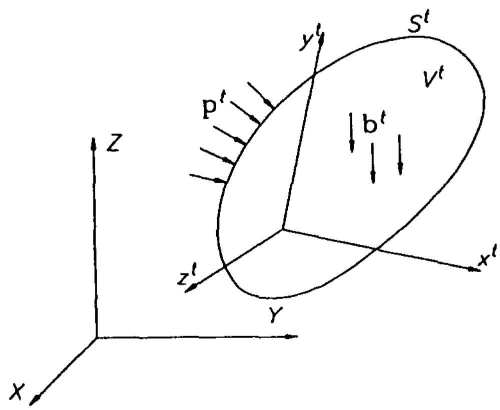

Fig. 1 Motion of a body—configuration at time instant t

where $\boldsymbol{X}=\{X\ Y\ Z\}^T$ are the components of any column vector in the coordinate system $\{X\ Y\ Z\}$, $\boldsymbol{x}^\tau=\{x^\tau\ y^\tau\ z^\tau\}^T$ is the representation of the same vector with respect to the coordinate system $\{x^\tau\ y^\tau\ z^\tau\}$, and $\boldsymbol{L}^\tau$ is the time-dependent matrix of rotation, the index T indicating transposition.

The location of a given particle of the body at time t, $\boldsymbol{x}^t=\{x^t\ y^t\ z^t\}^T$, cf. Fig. 2, can be described by its initial position $\boldsymbol{x}^0=\{x^0\ y^0\ z^0\}^T$ and its displacement $\boldsymbol{u}^t=\{u_x^t\ u_y^t\ u_z^t\}^T$ from time 0 to time t so that

$$\boldsymbol{x}^t = \boldsymbol{x}^0 + \boldsymbol{u}^t \tag{2}$$

In the incremental nonlinear analysis we look for the vector of the particle position change $\Delta\boldsymbol{u}=\{\Delta u_x\ \Delta u_y\ \Delta u_z\}^T$ during the time interval from t to $t+\Delta t$.

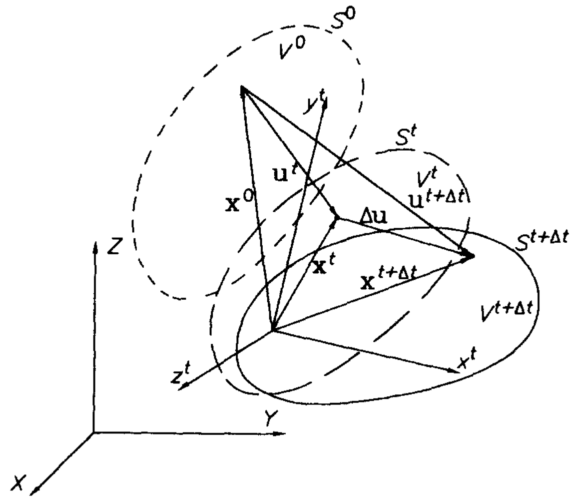

Fig. 2 The location and displacement of a material point

2.2. *The principle of virtual work*

Using the principle of virtual work the condition of dynamic equilibrium can be written in the following form

$$\int_{V^t} \delta(\Delta \boldsymbol{u})^T \rho^t \ddot{\boldsymbol{u}}^{t+\Delta t} dV + \int_{V^t} c^t \dot{\boldsymbol{u}}^{t+\Delta t} \delta(\Delta \boldsymbol{u})\, dV +$$
$$- \int_{V^t} \delta(\Delta \boldsymbol{u})^T \boldsymbol{b}^{t+\Delta t} dV - \int_{S^{(\sigma)}} \delta(\Delta \boldsymbol{u})^T \boldsymbol{p}^{t+\Delta t} dS + \int_{V^t} \delta(\Delta \boldsymbol{\varepsilon})^T \boldsymbol{\sigma}^{t+\Delta t} dV = 0 \tag{3}$$

which expresses the condition of dynamic equilibrium at time $t+\Delta t$, for which the solution is to be found. In Eq. (3) we have used the following notation: ρ^t – material density, c^t – damping coefficient, $\boldsymbol{\sigma}^{t+\Delta t}$ – vector of components of the second Piola-Kirchhoff stress tensor at $t+\Delta t$, $\Delta\boldsymbol{\varepsilon}$ – vector of components of the incremental Green-Lagrange strain tensor. The boundary kinematic conditions are taken into account implicitly by assuming that variations of the incremental displacements $\delta(\Delta\boldsymbol{u})$ satisfy them *a priori*.

Eq. (3) is valid for both deformable and rigid bodies; in the latter case the virtual work of internal stresses is equal to zero.

2.3. *The condition of dynamic equilibrium for the body divided into deformable and rigid parts*

Following our assumptions for the physical model we divide the body volume V^t into disjoint subregions: $N^{(f)}$ deformable parts $V_i^{(f)t}$ and $(i=1,\ldots,N^{(f)})$ and $N^{(r)}$ parts $V_j^{(r)t}$ $(j=1,\ldots, N^{(r)})$ that can be regarded as rigid. The regions taken up by all the deformable and rigid parts at time t are denoted as $V^{(f)t}$ and $V^{(r)t}$, respectively.

In this way we have divided all the quantities into those referred to deformable parts denoted by superscript "f" (flexible) and into those referred to rigid parts denoted by superscript "r" (rigid). We shall further assume that allowable incremental displacements fields $\Delta\boldsymbol{u}^{(f)}$ and

$\Delta\boldsymbol{u}^{(r)}$ satisfy a priori kinematic boundary conditions and that the kinematic condition of continuity across the surface $S^{(f-r)t}$ separating deformable and rigid parts is accounted for by adding the additional constraint equation

$$\chi\,(\Delta\boldsymbol{u}^{(f)},\ \Delta\boldsymbol{u}^{(r)})=\Delta\boldsymbol{u}^{(f)}-\Delta\boldsymbol{u}^{(r)}=\mathbf{0},\ \boldsymbol{x}^t\in S^{(f-r)t} \tag{4}$$

For the coupled system the equation of dynamic equilibrium (3) can be rewritten as

$$\begin{aligned}&\int_{V^{(f)t}}\delta(\Delta\boldsymbol{u}^{(f)})^T\rho\ddot{\boldsymbol{u}}^{(f)t+\Delta t}dV+\int_{V^{(f)t}}\delta(\Delta\varepsilon)^T\boldsymbol{\sigma}^{t+\Delta t}\ dV+\\ &-\int_{V^{(f)t}}\delta(\Delta\boldsymbol{u}^{(f)})^T\boldsymbol{b}^{(f)t+\Delta t}dV-\int_{S^{(f)t}}\delta(\Delta\boldsymbol{u}^{(f)})^T\boldsymbol{p}^{(f)t+\Delta t}dS+\int_{V^{(r)t}}\delta(\Delta\boldsymbol{u}^{(r)})^T\rho\ddot{\boldsymbol{u}}^{(r)t+\Delta t}dS+\\ &-\int_{V^{(r)t}}\delta(\Delta\boldsymbol{u}^{(r)})^T\boldsymbol{b}^{(r)t+\Delta t}dV-\int_{S^{(r)t}}\delta(\Delta\boldsymbol{u}^{(r)})^T\boldsymbol{p}^{(r)t+\Delta t}dS+\\ &+\int_{S^{(f-r)t}}\left[\delta(\Delta\boldsymbol{u}^{(f)})^T\left(\frac{\partial\chi}{\partial(\Delta\boldsymbol{u}^{(f)})}\right)^T+\delta(\Delta\boldsymbol{u}^{(r)})^T\left(\frac{\partial\chi}{\partial(\Delta\boldsymbol{u}^{(r)})}\right)^T\right]\boldsymbol{\lambda}^{*t+\Delta t}dS=0\end{aligned} \tag{5}$$

where the integrals for the rigid and deformable parts are written separately and the virtual work of the constraint reactions is included explicitly by adding the terms with Lagrangian multipliers $\boldsymbol{\lambda}^{*t}$.

3. Discretization of equations of motion for deformable bodies

3.1. Linearization of the incremental equations of motion for deformable bodies

We shall consider a motion of a deformable body(Fig. 2) using the condition of dynamic equilibrium (3). By using the decomposition of the incremental strains $\Delta\boldsymbol{\varepsilon}$ into linear and nonlinear parts, $\Delta\boldsymbol{e}$ and $\Delta\boldsymbol{\eta}$, respectively

$$\Delta\boldsymbol{\varepsilon}=\Delta\boldsymbol{e}+\Delta\boldsymbol{\eta} \tag{6}$$

and the stress decomposition

$$\boldsymbol{\sigma}^{t+\Delta t}=\boldsymbol{\sigma}^t+\Delta\boldsymbol{\sigma} \tag{7}$$

we can transform Eq. (3) into the following linearized form

$$\begin{aligned}&\int_{V^t}\delta(\Delta\boldsymbol{u})^T\rho^t\ddot{\boldsymbol{u}}^{t+\Delta t}dV+\int_{V^t}\delta(\Delta\boldsymbol{u})^Tc^t\dot{\boldsymbol{u}}^{t+\Delta t}\ dV+\\ &+\int_{V^t}\delta(\Delta\boldsymbol{e})^T\Delta\boldsymbol{\sigma}\,dV+\int_{V^t}\delta(\Delta\boldsymbol{\eta})^T\boldsymbol{\sigma}^t dV+\\ &-\int_{V^t}\delta(\Delta\boldsymbol{u})^T\boldsymbol{b}^{t+\Delta t}dV-\int_{S^{(\sigma)t}}\delta(\Delta\boldsymbol{u})^T\boldsymbol{p}^{t+\Delta t}dS+\int_{V^t}\delta(\Delta\boldsymbol{e})^T\boldsymbol{\sigma}^t dV=0\end{aligned} \tag{8}$$

Material nonlinearities are taken into account in Eq. (8) by using an appropriate constitutive equation to express $\Delta\boldsymbol{\sigma}$ in terms of $\Delta\boldsymbol{\varepsilon}$. On the basis of Eq. (8) discretized equations of the so-called tangent stiffness method can be obtained. Another way of accounting for nonlinear effects, which is particularly effective for material nonlinearities, is based on introducing pseudo-forces on the right-hand side of the equilibrium equations. The approach is called the initial load method. In our program both these iterative methods can alternatively be used.

3.2. Incremental finite element equations

Using displacement expansions typical of FEM, performing appropriate local-to-global transformations and introducing the result into Eq. (8) we get the condition of dynamic equilibrium in the implicit form

$$\boldsymbol{M}\ddot{\boldsymbol{r}}^{t+\Delta t}+\boldsymbol{C}^{t}\dot{\boldsymbol{r}}^{t+\Delta t}+\boldsymbol{K}^{t+\Delta t}\Delta\boldsymbol{r}=\boldsymbol{R}^{t+\Delta t}-\boldsymbol{F}^{t} \tag{9}$$

where

$$\boldsymbol{K}=\boldsymbol{K}^{(con)}+\boldsymbol{K}^{(\sigma)} \tag{10}$$

$\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}^{(con)}$ and $\boldsymbol{K}^{(\sigma)}$ are the global mass, damping, constitutive stiffness and initial stress matrices, $\boldsymbol{r}$ is the global vector of generalized nodal displacements, $\boldsymbol{R}$ is the global vector of external load and $\boldsymbol{F}$ is the global vector of internal nodal forces. The matrix $\boldsymbol{K}^{(con)}$ depends on material properties. The elastic and elastic-plastic constitutive stiffness matrices will be denoted by $\boldsymbol{K}^{(e)}$ and $\boldsymbol{K}^{(ep)}$, respectively. For the initial load method we similarly have

$$\boldsymbol{M}\ddot{\boldsymbol{r}}^{t+\Delta t}+\boldsymbol{C}\dot{\boldsymbol{r}}^{t+\Delta t}+\boldsymbol{K}^{t+\Delta t}\Delta\boldsymbol{r}=\boldsymbol{R}^{t+\Delta t}-\boldsymbol{F}^{t}+\Delta\boldsymbol{\Psi} \tag{11}$$

where now

$$\boldsymbol{K}=\boldsymbol{K}^{(e)}+\boldsymbol{K}^{(\sigma)} \tag{12}$$

and $\Delta\boldsymbol{\Psi}$ is the global vector of fictitious forces due to material nonlinearities within the increment considered.

3.3. Nonlinear material models in FE equations

3.3.1. Elastic-plastic material model

The flow theory of plasticity is used below and the additive decomposition of the strain rate $\dot{\boldsymbol{\varepsilon}}$ into the elastic $\dot{\boldsymbol{\varepsilon}}^{(e)}$ and plastic $\dot{\boldsymbol{\varepsilon}}^{(p)}$ parts is postulated

$$\dot{\boldsymbol{\varepsilon}}=\dot{\boldsymbol{\varepsilon}}^{(e)}+\dot{\boldsymbol{\varepsilon}}^{(p)} \tag{13}$$

Assuming the associative flow rule and the yield condition with isotropic hardening we can obtain for the elastic-plastic states of the material behaviour the constitutive matrix in the following form(Owen and Hinton 1980)

$$\boldsymbol{C}^{(ep)}=\boldsymbol{C}^{(e)}-\frac{\boldsymbol{C}^{(e)}\left(\dfrac{\partial f}{\partial\sigma}\right)^{T}\dfrac{\partial f}{\partial\sigma}\boldsymbol{C}^{(e)}}{H+\dfrac{\partial f}{\partial\sigma}\boldsymbol{C}^{(e)}\left(\dfrac{\partial f}{\partial\sigma}\right)^{T}}=\boldsymbol{C}^{(e)}-\boldsymbol{C}^{(p)} \tag{14}$$

where H is the so-called hardening modulus, which can be determined from the uniaxial tension test as

$$H = \frac{d\sigma^{(Y)}}{d\varepsilon^{(P)}} \tag{15}$$

with $\sigma^{(Y)}$ being the yield stress.

3.3.2. Elastic-viscoplastic material model

The influence of the strain-rate on the yield stress is accounted for in the elastic-viscoplastic model. In analogy to Eq. (13) we assume the decomposition

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{(e)} + \dot{\boldsymbol{\varepsilon}}^{(vp)} \tag{16}$$

and the viscoplastic flow rule in the associative form suggested first in Perzyna 1966 as

$$\dot{\boldsymbol{\varepsilon}}^{(vp)} = \gamma \, \langle \Phi(f) \rangle \, \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{17}$$

with γ being the viscosity parameter, f the yield function and the symbol $\langle \Phi(f) \rangle$ defined as

$$\langle \Phi(f) \rangle = \begin{cases} \Phi(f) & \text{if } f > 0 \\ 0 & \text{if } f \leq 0 \end{cases} \tag{18}$$

The so-called overstress function $\Phi(f)$ should be taken in a form which is in best agreement with experimental results available.

4. Equations of multibody dynamics

4.1. Rigid body kinematics

We shall now consider a single rigid body which can be treated as the d-th rigid part of the system consisting of $N^{(r)}$ rigid and $N^{(f)}$ deformable parts defined in Sec. 2.3. The forces acting from the rest of the system on the part selected will be treated as included in the external loading.

The motion of the rigid body can be described by means of three translational and three rotational coordinates. In the rigid body dynamics the motion of a body is often investigated in body-fixed coordinates with Euler and Bryant angles, Euler parameters or direction cosines used as generalized coordinates, (Nikravesh 1982, Wittenburg 1977) defining the angular location of the moving coordinates with respect to the stationary coordinates.

In our formulation the system $\{x'y'z'\}$, kept constant within the time step Δt, is considered as the reference coordinate system for the description of the rotational motion of rigid bodies. The equations of motion are transformed to the global coordinates by means of the rotation matrix which is updated at every time step. This description is fully consistent with that adopted in the deformable finite element analysis.

On the basis of rigid body kinematics (Euler's and Chasle's theorems, Lurie 1961, Meirowithch 1970, Wittenburg 1977) any displacement of the rigid body can be represented by a sum of the translation of any point and a rotation about an axis passing through this point. As a reference point for every rigid element we choose their centers of mass since the rotational equations have then a simpler form. Accordingly, the velocity of any point belonging to the

d-th rigid body $\dot{\boldsymbol{u}}$ can be expressed by using the velocity of the reference point $\dot{\boldsymbol{u}}^{(d)}$ and the angular velocity $\boldsymbol{\omega}^{(d)}$ in the following form, (Meirowitch 1970)

$$\dot{\boldsymbol{u}}=\dot{\boldsymbol{u}}^{(d)}+\dot{\boldsymbol{s}}=\dot{\boldsymbol{u}}^{(d)}+\tilde{\boldsymbol{\omega}}^{(d)}\boldsymbol{s} \tag{19}$$

where $\boldsymbol{s}$ is the vector connecting the given point with the reference point(the center of mass) and $\tilde{\boldsymbol{\omega}}^{(d)}$ is the skew-symmetric matrix associated with the vector $\boldsymbol{\omega}^{(d)}$

$$\tilde{\boldsymbol{\omega}}^{(d)}=\begin{bmatrix} 0 & -\omega_z^{(d)} & \omega_y^{(d)} \\ \omega_z^{(d)} & 0 & -\omega_x^{(d)} \\ -\omega_y^{(d)} & \omega_x^{(d)} & 0 \end{bmatrix} \tag{20}$$

which is used to express the vector product in algebraic notation. The geometrical interpretation of the equation (19) is given in Fig. 3. Using the properties of product we can rewrite the relation (19) as

$$\dot{\boldsymbol{u}}=\dot{\boldsymbol{u}}^{(d)}+(-\tilde{\boldsymbol{s}})\boldsymbol{\omega}^{(d)} \tag{21}$$

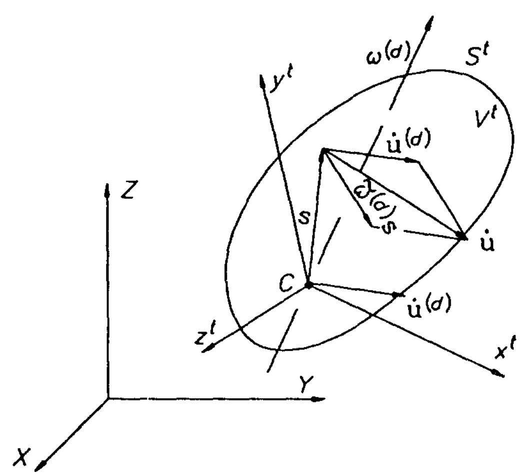

Fig. 3 Velocity of a particle of a rigid body

where $\tilde{s}$ has the structure analogous to that defined in Eq. (20). Differentiating the expression (19) with respect to time yields (Meirowitch 1970)

$$\ddot{\boldsymbol{u}}=\ddot{\boldsymbol{u}}^{(d)}+\ddot{\boldsymbol{s}}=\ddot{\boldsymbol{u}}^{(d)}+(-\tilde{\boldsymbol{s}})\dot{\boldsymbol{\omega}}^{(d)}+\tilde{\boldsymbol{\omega}}^{(d)}\tilde{\boldsymbol{\omega}}^{(d)}\boldsymbol{s} \tag{22}$$

which defines the acceleration $\ddot{u}$ of the given point. Relation (19) can be written in the incremental form as

$$\Delta\boldsymbol{u}=\Delta\boldsymbol{u}^{(d)}+(-\tilde{\boldsymbol{s}})\Delta\boldsymbol{\theta}^{(d)}=\begin{bmatrix}\boldsymbol{I}-\tilde{\boldsymbol{s}}\end{bmatrix}\begin{Bmatrix}\Delta\boldsymbol{u}^{(d)} \\ \Delta\boldsymbol{\theta}^{(d)}\end{Bmatrix}=\boldsymbol{\Phi}\Delta\boldsymbol{q}^{(d)} \tag{23}$$

The relation (23) plays the same role for the rigid parts as the FEM displacement expansion for the deformable parts since it expresses incremental displacements of any point as a function of generalized incremental displacements.

4.2. Equations of motion for single rigid body

Similarly to the derivation of the FE equations we shall now make use of the principle of virtual work to obtain equations of motion for the single rigid body. The condition of dynamic equilibrium can be written in the following form

$$\int_{V(d)} \delta(\Delta \boldsymbol{u})^T \rho \ddot{\boldsymbol{u}}' dV - \int_{V(d)} \delta(\Delta \boldsymbol{u})^T \boldsymbol{b}' dV - \int_{S(d)} \delta(\Delta \boldsymbol{u})^T \boldsymbol{p}' dS = 0 \tag{24}$$

After introducing relations (22) and (23) into Eq. (24) we obtain

$$\int_{V(d)} \left[\delta(\Delta \boldsymbol{u}^{(d)})^T + \delta\left(-\tilde{\boldsymbol{s}}\Delta\boldsymbol{\theta}^{(d)}\right)^T\right]\left[\rho\left(\ddot{\boldsymbol{u}}^{(d)\prime} - \tilde{\boldsymbol{s}}\dot{\boldsymbol{\omega}}^{(d)\prime} + \tilde{\boldsymbol{\omega}}^{(d)\prime}\tilde{\boldsymbol{\omega}}^{(d)\prime}\boldsymbol{s}\right) - \boldsymbol{b}'\right] dV +$$
$$- \int_{S(d)} \left[\delta(\Delta \boldsymbol{u}^{(d)})^T + \delta\left(-\tilde{\boldsymbol{s}}\Delta\boldsymbol{\theta}^{(d)}\right)^T\right] \boldsymbol{p}' dS = 0 \tag{25}$$

while Eq. (25) can be transformed to the following form, (Nikravesh 1982, Wittenburg 1977):

$$\delta\,(\Delta \boldsymbol{u}^{(d)})^T\left[m^{(d)}\ddot{\boldsymbol{u}}^{(d)\prime} - \bar{\boldsymbol{f}}^{(d)\prime}\right] +$$
$$+\delta\left(\Delta\boldsymbol{\theta}^{(d)}\right)^T\left[\boldsymbol{j}^{(d)}\dot{\boldsymbol{\omega}}^{(d)\prime} + \tilde{\boldsymbol{\omega}}^{(d)\prime}\boldsymbol{j}^{(d)}\boldsymbol{\omega}^{(d)\prime} - \bar{\boldsymbol{n}}^{(d)\prime}\right] = 0 \tag{26}$$

with $m^{(d)}$ being the mass of the body, $\bar{\boldsymbol{f}}^{(d)}$ the resultant external force, $\bar{\boldsymbol{n}}^{(d)}$ the resultant external moment with respect to the reference point and $\boldsymbol{j}^{(d)}$ being the matrix of inertia containing components of the inertia tensor with respect to the origin of the coordinate system $\{x'y'z'\}$. Since all the variations in $\delta(\Delta\boldsymbol{u}^{(d)})$ and $\delta(\Delta\boldsymbol{\theta}^{(d)})$ are independent the condition (26) yields two equations

$$\begin{aligned} m^{(d)}\ddot{\boldsymbol{u}}^{(d)\prime} &= \bar{\boldsymbol{f}}^{(d)\prime} \\ \boldsymbol{j}^{(d)}\dot{\boldsymbol{\omega}}^{(d)\prime} + \tilde{\boldsymbol{\omega}}^{(d)\prime}\boldsymbol{j}^{(d)}\boldsymbol{\omega}^{(d)\prime} &= \bar{\boldsymbol{n}}^{(d)\prime} \end{aligned} \tag{27}$$

The first of the above equations describes translational motion of the rigid body and is called the Newton's equation. Rotational motion is governed by the second equation called the Euler's equation. Both Eqs (27) are referred as the Newton-Euler's equations.

Using the matrix of rotation $\boldsymbol{L}^{(d)}$ we can transform all the quantities in Eq. (27) from the coordinates $\{x'y'z'\}$ to the coordinates $\{XYZ\}$ as

$$\boldsymbol{J}^{(d)\prime} = \boldsymbol{L}^{(d)\prime\,T}\boldsymbol{j}^{(d)}\boldsymbol{L}^{(d)\prime} \tag{28}$$

$$\ddot{\boldsymbol{U}}^{(d)\prime} = \boldsymbol{L}^{(d)\prime\,T}\ddot{\boldsymbol{u}}^{(d)\prime} \tag{29}$$

$$\boldsymbol{\Omega}^{(d)\prime} = \boldsymbol{L}^{(d)\prime\,T}\boldsymbol{\omega}^{(d)\prime} \tag{30}$$

$$\bar{\boldsymbol{F}}^{(d)\prime} = \boldsymbol{L}^{(d)\prime\,T}\bar{\boldsymbol{f}}^{(d)\prime} \tag{31}$$

$$\bar{\boldsymbol{N}}^{(d)\prime} = \boldsymbol{L}^{(d)\prime\,T}\bar{\boldsymbol{n}}^{(d)\prime} \tag{32}$$

We can write Eq. (27) in terms of the global quantities defined by Eqs (28)-(32) as

$$m^{(d)}\ddot{\boldsymbol{U}}^{(d)\prime} = \bar{\boldsymbol{F}}^{(d)\prime}$$

$$J^{(d)t}\dot{\Omega}^{(d)t}+\tilde{\Omega}^{(d)t}J^{(d)t}\Omega^{(d)t}=\bar{N}^{(d)t} \tag{33}$$

Eqs (33) can be combined in one matrix equation as

$$M^{(d)t}\ddot{r}^{(d)t}+C^{(d)t}\dot{r}^{(d)t}=R^{(d)t} \tag{34}$$

where

$$M^{(d)t}=\begin{bmatrix} m^{(d)} & \mathbf{0} \\ \mathbf{0} & J^{(d)t} \end{bmatrix} \tag{35}$$

$$m^{(d)}=\begin{bmatrix} m^{(d)} & 0 & 0 \\ 0 & m^{(d)} & 0 \\ 0 & 0 & m^{(d)} \end{bmatrix} \tag{36}$$

$$C^{(d)t}=\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\Omega}^{(d)t}J^{(d)t} \end{bmatrix} \tag{37}$$

$$\ddot{r}^{(d)t}=\{\ddot{U}^{(d)t}\ \dot{\Omega}^{(d)t}\}^T \tag{38}$$

$$\dot{r}^{(d)t}=\{\dot{U}^{(d)t}\ \Omega^{(d)t}\}^T \tag{39}$$

$$R^{(d)t}=\{\bar{F}^{(d)t}\ \bar{N}^{(d)t}\}^T \tag{40}$$

4.3. Equations of motion for system of unconstrained rigid bodies

Now we shall consider a system of rigid bodies interconnected by springs and damping elements. The links are treated as sources of the internal loading which is added to the external loading. The principle of virtual work for such a system leads to

$$M^t\ddot{r}^t + C^t\dot{r}^t=R^t \tag{41}$$

where we have introduced global quantities for the whole system assembled from the vectors and matrices defined for a single rigid body by Eqs (35)-(40), (Nikravesh 1982, Wittenburg 1977).

5. Discretization of equations of motion for system consisting of deformable and rigid parts

5.1. Discrete model of system consisting of deformable and rigid parts

Now we shall synthesize our results obtained so far in order to get discretized equations for the coupled system consisting of deformable and rigid parts. We consider discrete model of the system defined in Sec. 2.3. Deformable parts are discretized with finite elements and rigid parts are represented by rigid bodies with known locations of the centers of mass and inertial properties. The deformable and rigid parts are connected with each other by some of the nodes of the finite element mesh lying on the boundary surfaces of the rigid parts.

The incremental generalized displacements Δr can be divided into those referred to the nodes defining finite element mesh $\Delta\boldsymbol{r}^{(f)}$ and those referred to the nodes being the centers of mass of rigid parts $\Delta\boldsymbol{r}^{(r)}$, which is written as

$$\Delta\boldsymbol{r}=\{\Delta\boldsymbol{r}^{(f)}\ \Delta\boldsymbol{r}^{(r)}\}^T \tag{42}$$

In the vector $\Delta\boldsymbol{r}^{(f)}$ we further distinguish the components referred to the nodes lying on the boundary surfaces of rigid bodies $\Delta\boldsymbol{r}^{(c)}$ ("c"=constrained) and the components referred to the nodes not connected to the rigid parts $\Delta\boldsymbol{r}^{(u)}$ ("u"=unconstrained), i.e.

$$\Delta\boldsymbol{r}^{(f)}=\{\Delta\boldsymbol{r}^{(u)}\ \Delta\boldsymbol{r}^{(c)}\}^T \tag{43}$$

Now we can write the vector $\Delta\boldsymbol{r}$ as consisting of the three subvectors

$$\Delta\boldsymbol{r}=\{\Delta\boldsymbol{r}^{(u)}\ \Delta\boldsymbol{r}^{(c)}\ \Delta\boldsymbol{r}^{(r)}\}^T \tag{44}$$

5.2. Discretized equations of constraints

The problem of getting discretized equations of constraints is simplified by considering the condition of continuity only at the nodes of finite element mesh lying on the boundary surface of rigid bodies, cf. Gallagher 1975. When we apply the kinematic relations (19), (21), (22) and (23) to the constrained nodes of the FE mesh (to the b-th node connected to the d-th rigid part, for instance), we can obtain the discretized equations for the velocities, accelerations and incremental displacements as

$$\dot{\boldsymbol{r}}^{(b)(c)t}=\boldsymbol{D}^{(b)t}\dot{\boldsymbol{r}}^{(d)(r)t} \tag{45}$$

$$\ddot{\boldsymbol{r}}^{(b)(c)t}=\boldsymbol{D}^{(b)t}\ddot{\boldsymbol{r}}^{(d)(r)t}+\boldsymbol{g}^{(b)t} \tag{46}$$

$$\Delta\boldsymbol{r}^{(b)(c)}=\boldsymbol{D}^{(b)t}\Delta\boldsymbol{r}^{(d)(r)} \tag{47}$$

where

$$\boldsymbol{D}^{(b)t}=\begin{bmatrix}\boldsymbol{I} & -\tilde{\boldsymbol{S}}^{(b)t}\\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix} \tag{48}$$

$$\boldsymbol{g}^{(b)t}=\{\tilde{\boldsymbol{\Omega}}^{(d)(r)t}\tilde{\boldsymbol{\Omega}}^{(d)(r)t}\boldsymbol{S}^{(b)t}\ \boldsymbol{0}\}^T \tag{49}$$

with $\boldsymbol{S}^{(b)t}$ being the vector connecting the b-th constrained node with the center of mass of the d-th rigid body and $\boldsymbol{\Omega}^{(d)(r)t}$ being the angular velocity of the d-th rigid body. It should be noted that the constraint equations (45)-(47) have been written in the global coordinates $\{XYZ\}$ and that the condition of equality of the translational nodal kinematic quantities have been supplemented with the condition of equality of the rotational kinematic quantities, which is important if structural elements with rotational degrees of freedom(beams or shells) are used in modeling the structure. If some of the six degrees of freedom of the b-th node are not constrained then the rows corresponding to them should be removed from the equations (45)-(47).

By combining the relations (45)-(47) for all $W^{(c)}$ constrained nodes we can obtain joint equations in the form

$$\dot{\boldsymbol{r}}^{(c)t}-\boldsymbol{D}^t\dot{\boldsymbol{r}}^{(r)t}=\boldsymbol{0} \tag{50}$$

$$\ddot{\boldsymbol{r}}^{(c)t}-\boldsymbol{D}^t\ddot{\boldsymbol{r}}^{(r)t}-\boldsymbol{g}^t=\boldsymbol{0} \tag{51}$$

$$\Delta r^{(c)} - D' \Delta r^{(r)} = 0 \tag{52}$$

where

$$D' = \sum_{b=1}^{b=W^{(c)}} \Lambda^{(b)(c)\,T} D^{(b)} \Lambda^{(d)(r)} \tag{53}$$

$$g' = \sum_{b=1}^{b=W^{(c)}} \Lambda^{(b)(c)\,T} g^{(b)} \tag{54}$$

with $\Lambda^{(b)(c)}$ and $\Lambda^{(d)(r)}$ being appropriate boolean matrices relating the components of the vectors $\Delta r^{(b)(c)}$ and $\Delta r^{(d)(r)}$ with components of the vectors $\Delta r^{(c)}$ and $\Delta r^{(r)}$, respectively.

5.3. Discrete equations for system consisting of deformable and rigid parts

We shall now derive the discrete equations for the system consisting of deformable and rigid parts by using the condition of dynamic equilibrium for the coupled system given by Eq. (5) and performing the discretization typical of the finite element method for the deformable parts and employing the discrete equations of rigid body dynamics. By making use of previously obtained equations — Eq. (9) for the deformable parts and Eq. (41) written in an implicit form for the rigid parts — we can rewrite Eq. (5) in the discrete form as

$$\begin{aligned}
&\delta(\Delta r^{(f)})^T \left(M^{(f)} \ddot{r}^{(f)t+\Delta t} + C^{(f)} \dot{r}^{(f)t+\Delta t} + K^{(f)} \Delta r^{(f)} - R^{(f)t+\Delta t} + F^{(f)}\right) \\
&+ \delta(\Delta r^{(r)})^T \left(M^{(r)} \ddot{r}^{(r)t+\Delta t} + C^{(r)} \dot{r}^{(r)t+\Delta t} - R^{(r)t+\Delta t}\right) \\
&+ \delta(\Delta r^{(f)})^T \left[\frac{\partial \chi^t}{\partial(\Delta r^{(f)})}\right]^T \lambda^{t+\Delta t} + \delta(\Delta r^{(r)})^T \left[\frac{\partial \chi^t}{\partial(\Delta r^{(r)})}\right]^T \lambda^{t+\Delta t} = 0
\end{aligned} \tag{55}$$

If we take into account the explicit form of the equations of constraints (47) and divide the matrices and vectors referring to the deformable parts in accordance with the split of the vector $\Delta r^{(f)}$ expressed by Eq. (43) the following form of Eq. (55) is obtained

$$\begin{aligned}
&\delta \begin{Bmatrix} \Delta r^{(u)} \\ \Delta r^{(c)} \end{Bmatrix}^T \left(\begin{bmatrix} M^{(uu)} & M^{(uc)} \\ M^{(cu)} & M^{(cc)} \end{bmatrix} \begin{Bmatrix} \ddot{r}^{(u)t+\Delta t} \\ \ddot{r}^{(c)t+\Delta t} \end{Bmatrix} + \begin{bmatrix} C^{(uu)} & C^{(uc)} \\ C^{(cu)} & C^{(cc)} \end{bmatrix} \begin{Bmatrix} \dot{r}^{(u)t+\Delta t} \\ \dot{r}^{(c)t+\Delta t} \end{Bmatrix} \right. \\
&\left. + \begin{bmatrix} K^{(uu)} & K^{(uc)} \\ K^{(cu)} & K^{(cc)} \end{bmatrix} \begin{Bmatrix} \Delta r^{(u)} \\ \Delta r^{(c)} \end{Bmatrix} - \begin{Bmatrix} R^{(u)t+\Delta t} \\ R^{(c)t+\Delta t} \end{Bmatrix} + \begin{Bmatrix} F^{(u)} \\ F^{(c)} \end{Bmatrix} \right) \\
&+ \delta \left\{\Delta r^{(r)}\right\}^T \left(\left[M^{(r)}\right] \left\{\ddot{r}^{(r)t+\Delta t}\right\} + \left[C^{(r)}\right] \left\{\dot{r}^{(r)t+\Delta t}\right\} - \left\{R^{(r)t+\Delta t}\right\} \right) \\
&+ \delta \left\{\Delta r^{(c)}\right\}^T \left\{\lambda^{t+\Delta t}\right\} - \delta \left\{\Delta r^{(r)}\right\}^T \left[D'\right]^T \left\{\lambda^{t+\Delta t}\right\} = 0
\end{aligned} \tag{56}$$

By using the condition that Eq. (56) must be satisfied for any variation of the incremental displacements and adding the equation of constraints expressed by accelerations (51) we arrive at the following system of equations

$$
\begin{bmatrix} \boldsymbol{M}^{(uu)} & \boldsymbol{M}^{(uc)} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{M}^{(cu)} & \boldsymbol{M}^{(cc)} & \mathbf{0} & \boldsymbol{I} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{M}^{(r)t} & -\boldsymbol{D}^{t\,T} \\ \mathbf{0} & \boldsymbol{I} & -\boldsymbol{D}^{t} & \mathbf{0} \end{bmatrix}
\begin{Bmatrix} \ddot{\boldsymbol{r}}^{(u)t+\Delta t} \\ \ddot{\boldsymbol{r}}^{(c)t+\Delta t} \\ \ddot{\boldsymbol{r}}^{(r)t+\Delta t} \\ \boldsymbol{\lambda}^{t+\Delta t} \end{Bmatrix}
+
\begin{bmatrix} \boldsymbol{C}^{(uu)} & \boldsymbol{C}^{(uc)} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{C}^{(cu)} & \boldsymbol{C}^{(cc)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{C}^{(r)t} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
\begin{Bmatrix} \dot{\boldsymbol{r}}^{(u)t+\Delta t} \\ \dot{\boldsymbol{r}}^{(c)t+\Delta t} \\ \dot{\boldsymbol{r}}^{(r)t+\Delta t} \\ \mathbf{0} \end{Bmatrix}
$$

$$
+\begin{bmatrix} \boldsymbol{K}^{(uu)t} & \boldsymbol{K}^{(uc)t} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{K}^{(cu)t} & \boldsymbol{K}^{(cc)t} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
\begin{Bmatrix} \Delta\boldsymbol{r}^{(u)} \\ \Delta\boldsymbol{r}^{(c)} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}
=
\begin{Bmatrix} \boldsymbol{R}^{(u)t+\Delta t} \\ \boldsymbol{R}^{(c)t+\Delta t} \\ \boldsymbol{R}^{(r)t+\Delta t} \\ \boldsymbol{g}^{t} \end{Bmatrix}
-
\begin{Bmatrix} \boldsymbol{F}^{(u)t} \\ \boldsymbol{F}^{(c)t} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}
\tag{57}
$$

5.4. Elimination of Lagrange multipliers and constrained displacements

The equation of motion for the structure consisting of deformable and rigid parts (57) can be solved directly for the kinematic unknowns and Lagrange multipliers using any implicit method of step-by-step integration. However, it seems to be more effective to eliminate the Lagrange multipliers and kinematic unknowns referred to the constrained nodes prior to the solution. Instead of Eq. (57) we then have

$$
\begin{bmatrix} \boldsymbol{M}^{(uu)} & \boldsymbol{M}^{(uc)}\boldsymbol{D}^{t} \\ \boldsymbol{D}^{t\,T}\boldsymbol{M}^{(cu)} & \boldsymbol{D}^{t\,T}\boldsymbol{M}^{(cc)}\boldsymbol{D}^{t}+\boldsymbol{M}^{(r)t} \end{bmatrix}
\begin{Bmatrix} \ddot{\boldsymbol{r}}^{(u)t+\Delta t} \\ \ddot{\boldsymbol{r}}^{(r)t+\Delta t} \end{Bmatrix}
$$

$$
+\begin{bmatrix} \boldsymbol{C}^{(uu)} & \boldsymbol{C}^{(uc)}\boldsymbol{D}^{t} \\ \boldsymbol{D}^{t\,T}\boldsymbol{C}^{(cu)} & \boldsymbol{D}^{t\,T}\boldsymbol{C}^{(cc)}\boldsymbol{D}^{t}+\boldsymbol{C}^{(r)t} \end{bmatrix}
\begin{Bmatrix} \dot{\boldsymbol{r}}^{(u)t+\Delta t} \\ \dot{\boldsymbol{r}}^{(r)t+\Delta t} \end{Bmatrix}
+\begin{bmatrix} \boldsymbol{K}^{(uu)t} & \boldsymbol{K}^{(uc)t}\boldsymbol{D}^{t} \\ \boldsymbol{D}^{t\,T}\boldsymbol{K}^{(cu)t} & \boldsymbol{D}^{t\,T}\boldsymbol{K}^{(cc)t}\boldsymbol{D}^{t} \end{bmatrix}
\begin{Bmatrix} \Delta\boldsymbol{r}^{(u)} \\ \Delta\boldsymbol{r}^{(r)} \end{Bmatrix}
$$

$$
=\begin{Bmatrix} \boldsymbol{R}^{(u)t+\Delta t}-\boldsymbol{F}^{(u)t}-\boldsymbol{M}^{(uc)}\boldsymbol{g}^{t} \\ \boldsymbol{R}^{(r)t+\Delta t}+\boldsymbol{D}^{t\,T}(\boldsymbol{R}^{(c)t+\Delta t}-\boldsymbol{F}^{(c)t}-\boldsymbol{M}^{(cc)}\boldsymbol{g}^{t}) \end{Bmatrix}
\tag{58}
$$

In practical examples the equations of motion for the structures consisting of deformable and rigid parts can be considerably reduced by making various simplifying assumptions, such as:
(1) taking diagonal mass and damping matrices,
(2) neglecting damping,
(3) ignoring mass of deformable parts concentrated at the constrained nodes ($\boldsymbol{M}^{(cc)}=\mathbf{0}$),
(4) completely neglecting mass of the deformable parts ($\boldsymbol{M}^{(cc)}=\boldsymbol{M}^{(uu)}=\boldsymbol{M}^{(cu)}=\boldsymbol{M}^{(uc)}=\mathbf{0}$)

Eqs (57) and (58) are based on the split-up of the specially ordered degrees of freedom. This helps to understand the structure of the equations; in the computer implementation, however, there are no requirements as to the ordering of the degrees of freedom so that the nodes of different types can be mixed up. The reduction of Eq. (57) leading to Eq. (58) was shown to be performed by operations on global matrices. In the computer implementation elimination of the Lagrange multipliers and dependent unknowns is accomplished on the element level before assembling global matrices. In our paper we have presented equations of motion for structures consisting of deformable and rigid parts only in the implicit form; explicit equations can easily be derived as well. Any implicit time integration procedure can be applied to solve Eqs. (57) and (58). Our implementation is based on the Wilson θ scheme with modified Newton-Raphson iteration (Kleiber 1989).

We should observe that the left side of Eq. (57) and (58) can, in general, be unsymmetric due to the matrix $C^{(r)t}$ which is assembled from the submatrices defined by Eq. (37). The problem can be solved in its unsymmetric form or it can be symmetrized by moving the unsymmetric term $C^{(r)t}$ over to the right-hand side.

The vector of incremental generalized displacements $\Delta \boldsymbol{r}$ enables us to obtain the new configuration at time $t+\Delta t$. The new configuration can be described by the displacements of all the nodes

$$U^{(n)t+\Delta t} = U^{(n)t} + \Delta U^{(n)} \tag{59}$$

and their angular location characterized by the matrix of rotation $L^{(n)t+\Delta t}$, which can be determined from the following relation

$$L^{(n)t+\Delta t} = L^{(n)t+\Delta t,\,t} L^{(n)t} \tag{60}$$

where $L^{(n)t}$ is the matrix of rotation from the global coordinates $\{XYZ\}$ to the local coordinates $\{x'y'z'\}$, and $L^{(n)t+\Delta t,\,t}$ is the matrix of rotation of the local coordinates in the time interval from t to $t+\Delta t$. The matrix $L^{(n)t+\Delta t,\,t}$ can be defined using the components of the vector of incremental rotations $\Delta\Theta^{(n)}$ as the first-order approximation

$$L^{(n)t+\Delta t,\,t} = I + \widetilde{\Delta\Theta}^{(n)} \tag{61}$$

or the second-order approximation to the matrix of finite rotation (Argyris 1982).

$$L^{(n)t+\Delta t,\,t} = I + \widetilde{\Delta\Theta}^{(n)} + \frac{1}{2}\widetilde{\Delta\Theta}^{(n)}\widetilde{\Delta\Theta}^{(n)} \tag{62}$$

where I is the identity matrix and $\widetilde{\Delta\Theta}^{(n)}$ is the skew-symmetric matrix similar to that given by Eq. (20).

6. Concluding remarks

In this paper we presented the formulation of the equations of motion for structures with deformable and rigid parts. For both deformable and rigid parts large displacements are taken into account; material nonlinearities are considered in the case of deformable parts, using elastic-plastic or elastic-viscoplastic model of the material. The equations of motion for deformable and rigid parts are coupled by means of Lagrangian multipliers. Discretized equations for the coupled system are presented in the implicit form for the tangent stiffness and initial load methods. Explicit equations can be obtained in the same way. We present the algorithm of solution used in the computer implementation which is based on the elimination of the Lagrangian multipliers and dependent kinematic unknowns during the assembling of elemental matrices and vectors. The presented method of modeling have obvious advantages when applied to the analysis of practical problems. It allows us to setup a model of a complex structure with a reduced number of unknowns. It enables us to analyze exactly deformations in the zones where they are significant and neglect them in the zones where they are small. In the second part of our paper, (Rojek and Kleiber 1993) we shall present more details concerning the computer implementation and illustrate the problem discussed with results of the numerical analysis of a tractor with a protective cab subjected to an impact loading.

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