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Stress intensity factors for 3-D axisymmetric bodies containing cracks by p-version of F.E.M.

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Abstract. A new axisymmetric crack model is proposed on the basis of p-version of the finite element method limited to theory of small scale yielding. To this end, axisymmetric stress element is formulated by integrals of Legendre polynomial which has hierarchical nature and orthogonality relationship. The virtual crack extension method has been adopted to calculate the stress intensity factors for 3-D axisymmetric cracked bodies where the potential energy change as a function of position along the crack front is calculated. The sensitivity with respect to the aspect ratio and Poisson locking has been tested to ascertain the robustness of p-version axisymmetric element. Also, the limit value that is an exact solution obtained by FEM when degree of freedom is infinite can be estimated using the extrapolation equation based on error prediction in energy norm. Numerical examples of thick-walled cylinder, axisymmetric crack in a round bar and internal part-through cracked pipes are tested with high precision.

Key words: axisymmetric p-version model; stress intensity factor; virtual crack extension method; robustness; error prediction; Poisson locking.

1. Introduction

With the memory and speed of present day computers, determination of stress intensity factors of three-dimensional linear elastic cracked bodies is becoming almost commonplace. To this end, there have been many publications presenting methods for their determination and examining their accuracy. As the stress gradient in the vicinity of a crack tip is very steep, the energy based methods have been found to be most accurate. Two of these which have been employed with great success are the virtual crack extension method and the volume J-integral. In this study, the virtual crack extension method has been adopted to calculate the strain energy release rate G which was first proposed by Hellen(1975) for both two- and three- dimensional geometries. The volume J-integral yields identical results of G obtained by the virtual crack extension method within the scope of linear elastic fracture mechanics. Most of finite element approaches for axisymmetric cracked bodies are based on the conventional three-dimensional solid elements including singular elements. However, axisymmetric hierarchical elements used in this study are among an important class of finite elements for three-dimensional analysis of axisymmetric bodies. These elements provide an important numerical computation tool because a three-dimensional stress analysis problem can be handled in a two-dimensional manner.

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It is well recognized that there are three versions of the finite element method such as h -version, p-version and hp-version. The h-version is the standard one, where the degree p of the element is fixed, usually on low levels, typically $p=1$, 2 based on the shape functions of Lagrange family, and the accuracy is achieved by properly refining the mesh. The p-version, in contrast, fixes the mesh and achieves the accuracy by increasing the degree/p/of the element uniformly or selectively that is based on the shape functions of Legendre family. The shape functions of Legendre family are hierarchical and orthogonal. The hp-version in combination of both is the most desirable scheme in which the optimal mesh refinement is coupled with optimal p-distribution.

The polynomial approximation concept (p-approximation) has been developed by many investigators. In the beginning of the 1970s, Zienkiewicz, et al. (1970) presented the concept of higher order element approximation using Lagrange interpolation functions. In 1976, Peano(1976) presented the idea of hierarchical approximation functions. In 1977, Rossow, et al. (1976) developed a C° finite element based on hierarchical concept and precomputed arrays. In 1978, Szabo and Mehta(1978) applied p-version triangular finite element formulations to 2-D fracture mechanics problems. Woo(1989) and Basu(1990) analyzed the cylindrical shells with axial and circumferential cracks and solved the partial through cracked plates under tension and bending. In 1991, Surana, et al.(1991) developed hierarchical axisymmetric shell element. In this paper, we present a formulation of hierarchical p-version axisymmetric stress element based on integrals of Legendre polynomials for linear elastic fracture mechanics which are very difficult to model with 3-D solid elements. Numerical examples are presented to demonstrate the effectiveness, modeling convenience, accuracy, and overall superiority of present formulation over existing axisymmetric elements.

2. Axisymmetric P-version model

2.1. Integrals of Legendre polynomial

The shape functions are defined on a standard square $[-1, 1]\times[-1, 1]$ in the local coordinates ξ, η. They can be grouped into three classes(Woo and Busu 1989). The first group is ***basic modes***, which are the lowest order shape functions($p=1$) like

$$Ni=\frac{(1+\xi\xi_i)(1+\eta\eta_i)}{4} \tag{1}$$

The subscript "i" refers to the four vertex nodes of standard element. The second class is ***edge modes.*** For each higher p-level four more edge shape functions are required to be added as

$$\begin{aligned}&(1\pm\eta)\ F_n(\xi);\ \text{for nodes along edges}\ \eta=\pm 1\\&(1\pm\xi)\ F_n(\eta);\ \text{for nodes along edges}\ \xi=\pm 1\end{aligned} \tag{2}$$

where

$$F_n(x)=\sqrt{\frac{2n-1}{2}}\int_{-1}^{x} P_{n-1}(t)\ dt \tag{3}$$

where $P_n(t)$ is the Legendre polynomial defined by Rodrigues formula.

$$P_n(t)=\frac{1}{2^n n!}\cdot\frac{d^n}{dt^n}\cdot(t^2-1)^n \tag{4}$$

The third class is ***bubble modes*** or ***internal modes*** defined by

$$N_i(\xi,\ \eta)=F_i(\xi)\cdot F_j(\eta)\quad i,\ j\geq 2 \tag{5}$$

which are identically zero on all edges of the elements. The completeness requirement is satisfied by introducing bubble or internal modes for $p\geq 4$ as $F_i(\xi)\cdot F_i(\eta)$ with the requirement that $i+j=p$ and $i, j\geq 2$.

2.2. Formulation of axisymmetric stress element

A solid of revolution is axially symmetric if its geometry and material properties are independent of the circumferential coordinate θ. The problem is physically three-dimensional but mathematically two-dimensional. Thus, the material points have only $\boldsymbol{u}$(radial) and $\boldsymbol{w}$ (axial) displacements. The analysis procedure is essentially that of plane stress, so essential changes consist of adding more terms to the $[\boldsymbol{B}]$ and $[\boldsymbol{D}]$ matrices. The stress-strain relationship is expressed by

$$\{\sigma_r\ \ \sigma_\theta\ \sigma_z\ \sigma_{zr}\}=[D][\{\varepsilon_r\ \varepsilon_\theta\ \varepsilon_z\ \varepsilon_{zr}\}-\{\varepsilon_o\}]+\{\sigma_o\} \tag{6}$$

where, if the material is isotropic, the elasticity matrix is shown by

$$[D]=\frac{E(1-v)}{(1+v)(1-2v)}\begin{bmatrix} 1 & \frac{v}{1-v} & \frac{v}{1-v} & 0 \\ & 1 & \frac{v}{1-v} & 0 \\ \text{Symm.} & & 1 & 0 \\ & & & \frac{1-2v}{2(1-v)} \end{bmatrix} \tag{7}$$

The strain-matrix is the form of Eq. (8).

$$[B]=\begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \tag{8}$$

The stiffness matrix is expressed by Eq. (9).

$$[K]^e_{ij}=2\pi\int B_i^T\cdot D\cdot B_j\ \ r\ dr\ dz \tag{9}$$

where

$$r = \sum_{i=1}^{NODE} N_i \cdot r_i \tag{10}$$

During stress computation, the indeterminate form $u/r = 0/0$ arises for points on the z-axis. We can avoid the trouble by finding u/r slightly away from the axis or by extrapolating Gauss point strains to the axis.

3. Error prediction in energy norm

In the presence of singularities, the asymptotic convergence behavior of the p-version permits a close estimate of the exact strain energy by extrapolation, and therefore can predict the error in energy norm under current mesh design and p-level. The procedure is based on the theorem which establishes the rate of convergence for the p-version and the corresponding inverse theorem as follows:

$$U(u_{ex}) - U(u_{fe}) \leq \frac{k}{N_P^{2\alpha}} \tag{11}$$

There are three unknowns U_{ex}, k, and α in Eq. (11). By performing three successive extension processes p-2, p-1, and p, we have three equations for computing the unknowns. Canceling α, k in Eq. (11), we have

$$\frac{LOG \dfrac{U_{ex} - U_p}{U_{ex} - U_{p-1}}}{LOG \dfrac{U_{ex} - U_{p-1}}{U_{ex} - U_{p-2}}} = \frac{LOG \dfrac{N_{p-1}}{N_p}}{LOG \dfrac{N_{p-2}}{N_{p-1}}} \tag{12}$$

where U_p, U_{p-1}, and U_{p-2} are the strain energies calculated from the FE solutions. N_p, N_{p-1}, and N_{p-2} are the number of degrees of freedom of each analysis.

4. Computation of stress intensity factors

The finite element method has been used by a number of investigators to determine elastic stress intensity factors for cracked bodies. The characteristic elastic square root singularity has been represented by the use of virtual crack extension method in this work. For the virtual crack extension method(Hellen 1975), it may be shown in the absence of body forces.

$$\int_0^L G(s) \cdot \delta A(s)\, ds = -\frac{1}{2} \{u\}^T \cdot \triangle[K] \cdot \{u\} \tag{13}$$

where $G(s)$ is Griffith's energy, equal to J, and a function of position s along the crack front, a is the length of the crack front, $\delta A(s)$ is the incremental crack surface shown in Fig. 1, and $\{u\}$ is the vector of nodal point displacements found from the finite element computation. The change in the stiffness matrix $\Delta[K]$, for a given virtual crack extension may be written as a forward difference, namely

$$\triangle[K] \simeq [K]_{a+\delta a} - [K]_a \tag{14}$$

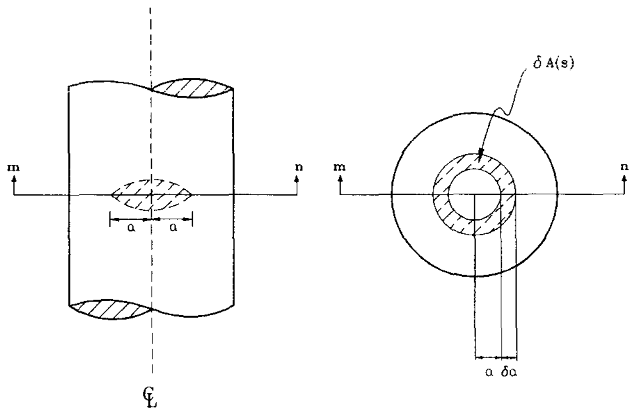


Fig. 1 A virtual crack extension employed in axisymmetric stress analysis

Consider a crack length a which advances by an incremental amount δa, thereby causing a release of strain energy of amount δU. Thus the incremental crack surface $\delta A(s)$ for axisymmetric bodies is defined by;

$$\delta A(s) = \pi\,(a+\delta a)^2 - \pi a^2 \tag{15}$$

Therefore, the strain energy release rate $G(s)$ for axisymmetric cracked bodies can be expressed as;

$$G(s) = \frac{\delta U}{\delta A(s)} \tag{16}$$

then the stress intensity factors are directly related to the value of $G(s)$ caused by a crack extension in the appropriate mode. In particular

$$K_I = \left[\frac{8\mu G_I}{1+\kappa}\right]^{1/2} \qquad K_{II} = \left[\frac{8\mu G_{II}}{1+\kappa}\right]^{1/2} \tag{17}$$

in which G_I and G_{II} are the strain energy release rates under mode I and mode II actions, respectively and μ is the shear modulus of the material and

$$\kappa = \begin{cases} 3-4\nu & \text{for plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{for plane stress} \end{cases} \tag{18}$$

However, since the quantity $G(s)$ is very sensitive to the crack length increment δa, the sensitivity test was investigated to obtain the consistent range of strain energy release rate between $G(s)$ and δa. From this, δa was adopted by $10^{-7}a$.

5. Robustness of axisymmetric stress element

5.1. Aspect ratio

Robinson(1976) suggests a single-element test in which response is examined as element aspect ratio changed. It is advised that element aspect ratios should not exceed roughly 7 for good displacement results and roughly 3 for good stress results. Since the large aspect ratio is one of sources to cause the numerical errors(Meyer 1987), it is desirable to make aspect ratio one, especially, in the higher stress regions. The relative errors of displacements with p-version model($p=8$, 1-element) are shown in Fig. 2 as aspect ratio, a/t, varies from 10 to 6000. The relative errors to exact solution by Timoshenko are below 2.0% until $a/t=3000$, 3.7% when $a/t=4000$, 6.8% when $a/t=5000$, and 22.0% when $a/t=6000$. From these, it is noted that the p-version model tolerates the large aspect ratio up to 4000 if we use 5% accuracy.

5.2. Nearly incompressible materials

In the case of linear isotropic elasticity with homogeneous material properties, the Lame' parameter λ is defined from Eq. (7) as follows;

$$\lambda = \frac{Ev}{(1+v)(1-2v)} \tag{19}$$

As $v \rightarrow 1/2$, $\lambda \rightarrow \infty$. When p is low and mesh refinement is used to control the error of approximation, then Poisson locking may occur. However, Poisson locking does not occur when p-versions or hp-versions are used.

For example, a thick-walled cylinder is subjected to a unit internal pressure and $E=1.0E+6$ psi, inner radius $r_i=3.0$ in, and outer radius $r_o=9.0$ in. The radial displacements with different Poisson's ratio from $v=0.3$ to 0.4999999 are plotted in Fig. 3 using 1-element p-version axi symmetric model of ninth order. In Table 1, the normalized radial displacements and stresses by the ninth one-element p-version model are compared with those by SAP90 ASOLID elements as Poisson's ratio varies from 0.3 to 0.49999999.

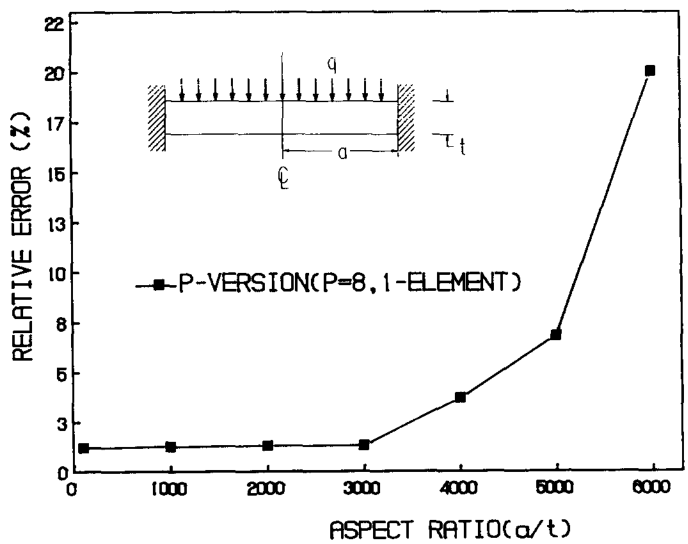


Fig. 2 Relative error in displacements with respect to aspect ratio a/t.

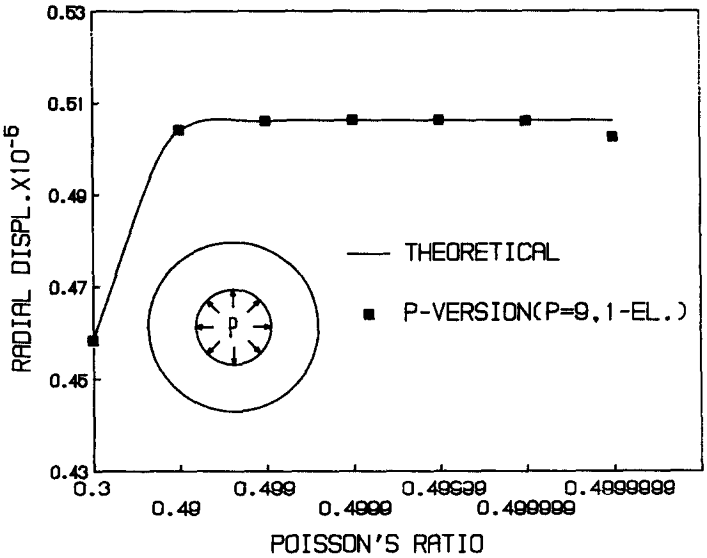


Fig. 3 Radial displacements for incompressible materials.

Table 1 Normalized radial displacements and stresses

Poisson's ratio	Radial Displacement		Radial Stress	
	SAP 90	P-version	SAP 90	P-version
0.3	1.000	1.000	0.98	1.00
0.49	1.000	1.000	0.78	0.99
0.499	0.986	1.000	1.12	0.94
0.4999	0.879	1.000	17.82	0.38
0.49999		1.000		5.21
0.499999		0.999		61.13
0.4999999		0.993		
0.49999999		0.932		

6. Numerical examples

6.1. Thick-walled cylinder

A thick cylinder of 2.5 inch inner radius and 17.5 inch wall thickness, subjected to internal pressure of 1000 psi are considered. The theory of elasticity solutions for these thick cylinder problem under plane strain condition are available by Timoshenko and Goodier (1984). The formula for radial displacement are expressed by Eq. (20).

$$u=\frac{(1+\nu)\cdot q\cdot R_i^2}{E\cdot(R_o^2-R_i^2)}\left[\frac{R_o^2}{r}+(1-2\nu)\cdot r\right] \tag{20}$$

where q represents inner pressure and R_i and R_o are inner radius and outer radius respectively. The test problem is modeled with one-element and p-level which varies from 1 to 10. The comparisons of radial displacements and stresses by the present solution and those by the elasticity solution are evident from Fig. 4 and Fig. 5. P-levels, degrees of freedom N and radial displacement u along the thickness for p-model and displacements obtained using analytical solution are presented in Table 2.

6.2. Axisymmetric crack in a round bar

Two standard problems subjected to axial tension shown in Fig. 6 are solved using the hierarchical axisymmetric four element mesh with different p-level. The first problem is axisymmetric notch in a round bar and the second problem is axisymmetric penny-shaped crack in a round bar.

The limit values of strain energy release rate G with respect to $1000/NDF$ are plotted in Fig. 7 which are the estimated exact solutions based on the asymptotic convergence behavior of the p-version of F.E.M. by Eq. (12). From this plot, it is noted that there is strong linearity between energy release rate G and reciprocal of number of degree of freedom from $p=6$. In both cases, the correction factors with different a/b ratios by p-version model give an excellent

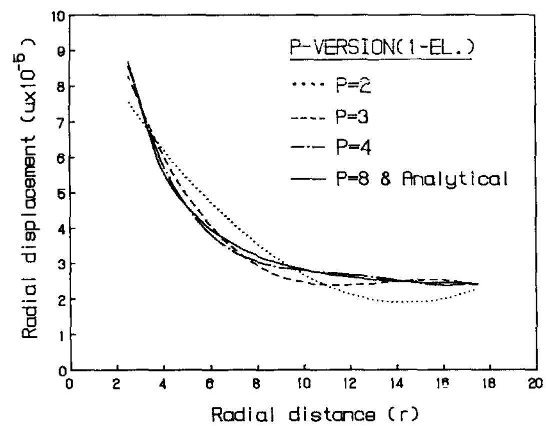


Fig. 4 Radial displacements along the thickness.

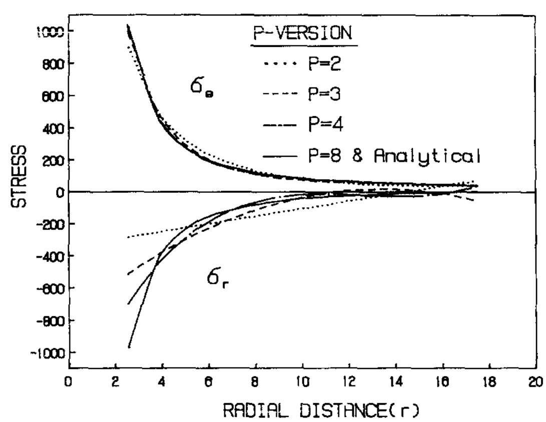


Fig. 5 Radial and circumferential stresses along the thickness with different p-level.

agreement with those by Benthem which are plotted in Fig. 8. The stress intensity factors of axisymmetric notch and penny-shaped crack are defined below, respectively, where $F(a/b)$ is called correction factor or shape factor.

$$K_I = \sigma_{net}\sqrt{\pi(b-a)}\, F(a/b) \tag{21}$$

Axisymmetric Notch

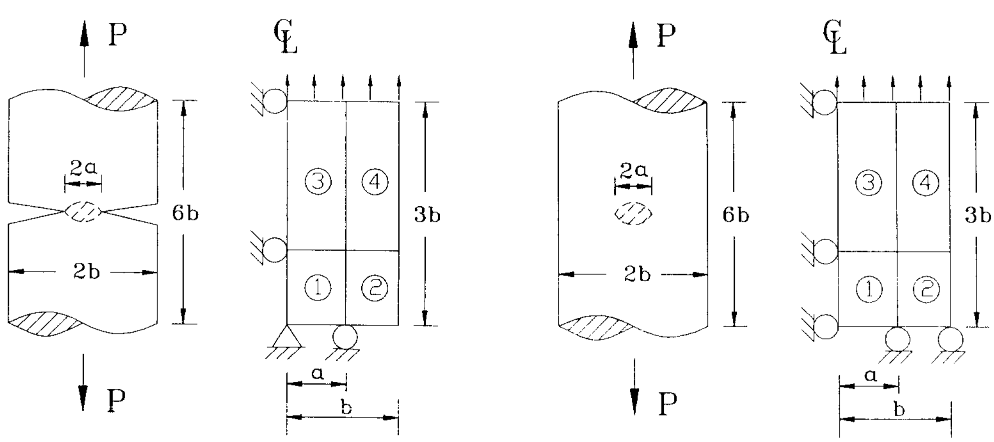
$$\sigma_{net} = \frac{P}{\pi \cdot a^2} \tag{22}$$

Penny-Shaped Crack

$$\sigma_{net} = \frac{P}{\pi\,(b^2 - a^2)} \tag{23}$$

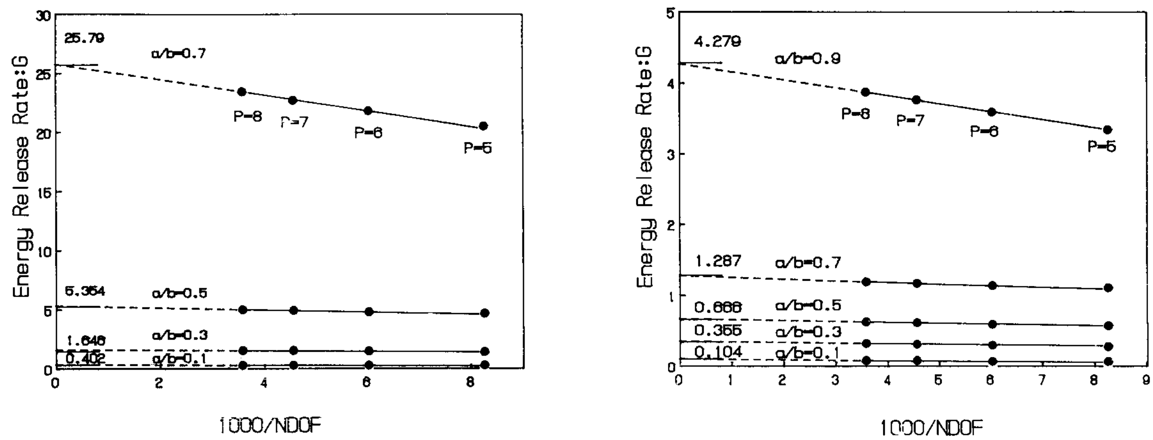
Table 2 p-Levels, degree of freedom(N) and displacements

P-Level	Degree of Freedom, N	Radial Displacement $u*10^3$ at r =2.5 in	r =10 in	r =17.5 in
1	4	0.0599415	0.0402047	0.0204678
2	10	0.0754816	0.0267963	0.0226878
3	16	0.0827921	0.0246125	0.0237322
4	24	0.0855428	0.0279312	0.0241252
5	34	0.0864383	0.0287415	0.0242531
6	46	0.0867044	0.0282061	0.0242911
7	60	0.0867788	0.0280880	0.0243017
8	76	0.0867987	0.0282089	0.0243046
9	84	0.0868038	0.0282357	0.0243053
10	114	0.0868051	0.0282120	0.0243055
Analytical		0.0868056	0.0282118	0.0243056



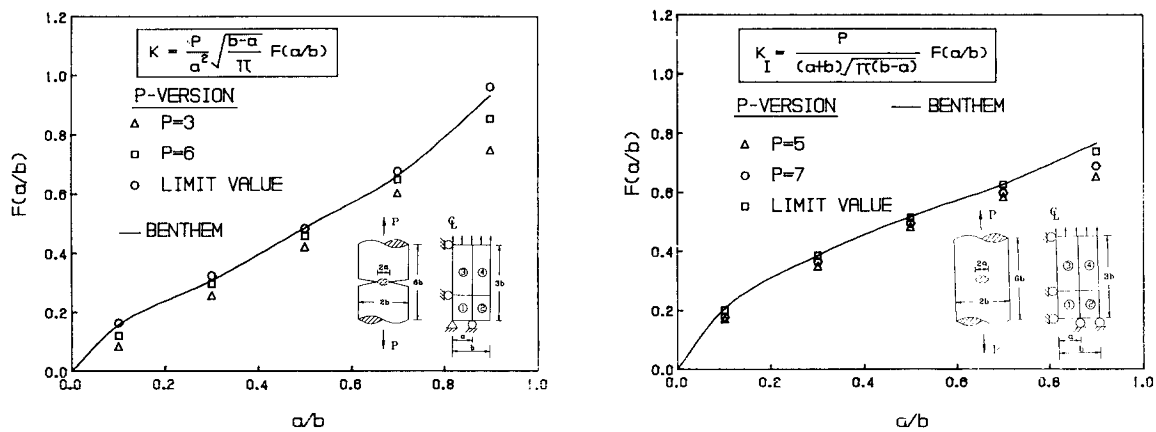
(a) axisymmetric notch (b) penny shaped crack

Fig. 6 Configurations of axisymmetric crack in a round bar and p-version model



(a) axisymmetric notch (b) penny shaped crack

Fig. 7 Limit values of strain energy release rate G



(a) axisymmetric notch (b) penny shaped crack

Fig. 8 Correction factors F(a/b) with respect to a/b ratios

6.3. Internal part-through cracked pipes

The closed form expression for this crack geometry and remotely applied axial load shown in Fig. 9 are reported by Zahoor (1985). The K_I solutions are available for $a/t=0.125$, 0.25, 0.50, and 0.75, and $R_i/t=5$, 10, and 20. A closed form K_I expression covering this range of parameters may be expressed as

$$K_I=\sigma_t\sqrt{\pi a}\cdot F(R_i/t,\ a/t) \tag{24}$$

where

a : crack depth
t : pipe wall thickness
R_i : pipe inner radius
T : axial load
σ_t : $\dfrac{T}{2\pi Rt}$
F : $1.1+A\{1.948(a/t)^{1.5}+0.3342(a/t)^{4.2}\}$

The quantity A is defined below that depends only on the pipe geometry.

$$A : \{0.125(R_i/t) - 0.25\}^{0.25} \quad \text{for } 5 \leq R_i/t \leq 10$$
$$\{0.4(R_i/t) - 3.0\}^{0.25} \quad \text{for } 10 \leq R_i/t \leq 20$$

The four element p-version model is applied to compute the correction factors denoted by $F(R_i/t,\ a/t)$ with respect to crack advances. Fig.10 shows a good agreement of F-values for different crack lengths obtained by p-version model and the empirical solutions by Zahoor when R_i/t is 5 and 10. However, there are some differences in the case of R_i/t is 20. From this results, it may be concluded that the proposed crack model based on axisymmetric stress element yields good solutions for thick shell problems.

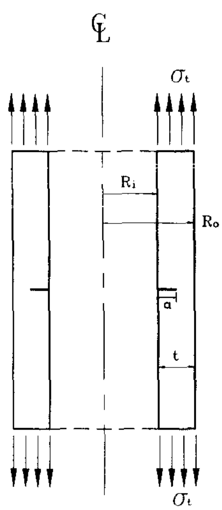


Fig. 9 Internal part-through cracked pipes

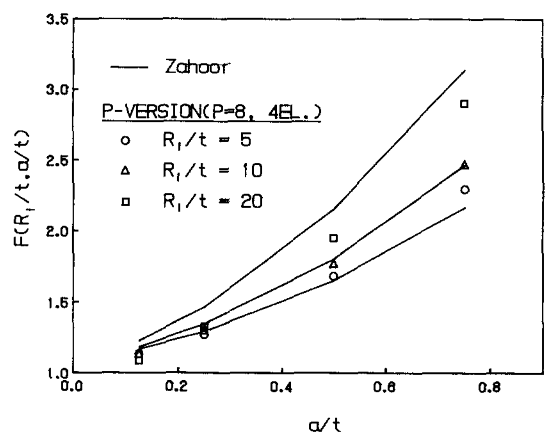


Fig. 10 Comparison of correction factors for internal part-through cracked pipes when R_i/t = 5, 10, and 20

7. Conclusions

Hierarchical axisymmetric elements using integrals of Legendre polynomials are proposed showing the superiority in terms of accuracy of the displacements, stresses, and stress intensity factors for the standard problem of axisymmetric crack in a round bar and internal part-through cracked pipes. The virtual crack extension method is used to compute the stress intensity factors for both cases. A rapid and highly regular energy convergence occurs as the polynomial order is increased on a fixed mesh. A stable value for the stress intensity factor is obtained with a crack length increment of $10^{-7}a$. Since the linearity of strain energy release rate versus the reciprocal of number of degree of freedom is very strong, the extrapolation method can be applied to determine the limit value. In view of the above, the p-version model developed herein is found to be eminently suited for calculating the stress intensity factor for axisymmetric 3-D cracked bodies.

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