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Axisymmetric analysis of multi-layered transversely isotropic elastic media with general interlayer and support conditions

J. S. Lee† and L. Z. Jiang‡

Department of Civil and Environmental Engineering, Clarkson University, Potsdam, NY 13676, USA

Abstract. Based on the transfer matrix approach and integral transforms, a solution method is developed for the stress analysis of axisymmetrically loaded transversely isotropic elastic media with generalized interlayer and support conditions. Transfer functions (Green's functions in the transformed domain) are obtained in explicit integral form. For several problems of practical interest with different loading and support conditions, solutions are worked out in detail. For the inversion operation, an efficient technique is introduced to remedy the slow convergence of numerical integrals involving oscillating functions. Several illustrative examples are considered and numerical results are presented.

Key words: stress analysis; anisotropy; elastic layers; transfer matrix.

1. Introduction

The increasing use of composite materials in many engineering applications has resulted in considerable interest in the stress analysis of anisotropic layered media. For multi-layered media, the problem of stress analysis becomes very complicated since solutions to the elasticity problem for all layers are required. These solutions must also satisfy both the boundary and interlayer continuity conditions. The complexity of such procedure is attested by solutions such as the ones given by Burminster (1945) and Chen (1971) who considered a semi-infinite medium composed of isotropic layers by using potential functions and integral transforms. A serious limitation of the conventional procedure is the fact that the number of simultaneous equations to be solved for displacements and stresses at a prescribed field point increases as the number of layers increases.

Other solution procedures for a layered isotropic or transversely isotropic (cross-anisotropic) half-space have also been reported. Gerrard (1967), for example, applied a Fourier series to the solution of stresses and displacements in a layered transversely isotropic soil deposit subjected to an axially symmetric load. Fares and Li (1988) introduced the generalized image method to consider isotropic, plane layered media. Lee and Zhang (1992) utilized the image method in conjunction with the boundary integral formulation to analyze a problem of layered half-space which contains a cylindrical cavity. As for multi-layered media, however, most of these approaches suffer the same limitation as the conventional one in that analytical opera-

† Assistant Professor
‡ Research Assistant

tions become excessively complicated as the number of layers grows.

To overcome the limitation shared by above-mentioned formulations, various alternative approaches have also been developed. In the stiffness matrix method (e.g., Kausel and Roesett 1981, Seale and Kausel 1989, Choi and Thangjitham 1991), a local matrix equation is constructed in terms of field variables at the upper and lower surfaces of each layer, and the global matrix equation is assembled by imposing interlayer continuity conditions. The unknown interfacial variables are then obtained by solving the global matrix equation. This process results in a significant reduction in the number of equations that must be solved when compared to the conventional formulation.

Another efficient alternative for the analysis of multi-layered media is the transfer matrix approach (Bahar 1972). Based on the mixed formulation of elasticity proposed by Vlasov and Leontev (1966), this approach converts the boundary value problem to an equivalent initial value problem in terms of state variables. Once the transfer matrix for each layer is found, the global matrix can be assembled when interlayer contact conditions and boundary conditions are introduced. The order of the global transfer matrix does not depend on the number of layers since the transfer matrix is multiplicative in nature for certain interlayer contact conditions. To the best of the authors' knowledge, however, this method has not been fully exploited in the stress analysis despite of its intrinsic merit for solving problems involving multi-layered media.

In this paper, based on the transfer matrix method and integral transforms, a solution method for problems of axisymmetrically loaded, multi-layered transversely isotropic elastic media with general interface and boundary support conditions is developed. The transfer matrix is obtained in the way different from the usual state space method. This alternative technique is useful for problems in which it is difficult to find transfer functions explicitly. In contrast to most previous studies where perfectly bonded (or welded) interlayer contact conditions are typically assumed, this generalized transfer matrix method can readily accommodate general interlayer contact and support conditions. Moreover, the number of simultaneous equations to be solved remains constant regardless of the number of layers in the media for the case of perfect (welded) or smooth (non-welded) interlayer contact conditions. For the assembly of the global matrix for other contact conditions, a systematic procedure is developed and described in some detail. For the inversion operation of integral transform, an efficient technique is introduced to remedy the prohibitively slow convergence of infinite integrals involving Bessel functions. Several numerical examples are presented to demonstrate the effectiveness of the proposed technique.

2. Basic equations

The equilibrium equations for an axisymmetric elastic solid without body forces are given by

$$
\begin{aligned}
&\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0\\
&\frac{\partial \sigma_z}{\partial z}+\frac{\partial \tau_{rz}}{\partial r}+\frac{\tau_{rz}}{r}=0
\end{aligned}
\tag{1}
$$

where σ and τ are the normal and shear stress respectively, and r, θ, and z are the radial, circumferential, and axial coordinate respectively (Fig.1). For transversely isotropic materials,

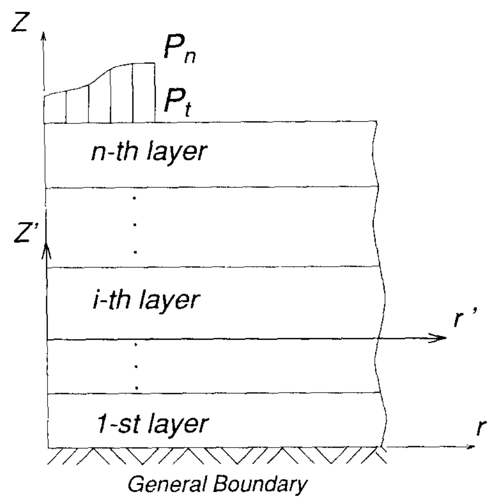

Fig. 1 Configuration of multi-layered medium.

stresses are related to strains by

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{12} & C_{11} & C_{13} & 0 \\ C_{13} & C_{13} & C_{33} & 0 \\ 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix} \tag{2}$$

where C_{ij} are elastic parameters, and strains are given by

$$\varepsilon_r = \frac{\partial u}{\partial r}, \ \varepsilon_\theta = \frac{u}{r}, \ \varepsilon_z = \frac{\partial w}{\partial z}, \ \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{3}$$

where u and w are displacements in the radial and thickness direction respectively. Eliminating stresses from (1) using (2) and (3) leads to

$$\begin{gathered} C_{11}\frac{\partial^2 u}{\partial r^2} + C_{11}\frac{\partial}{\partial r}\left(\frac{u}{r}\right) + C_{44}\frac{\partial^2 u}{\partial r^2} + (C_{13} + C_{44})\frac{\partial^2 w}{\partial_r \partial_z} = 0 \\ (C_{13} + C_{44})\frac{1}{r}\frac{\partial^2 (ru)}{\partial_r \partial_z} + C_{44}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + C_{33}\frac{\partial^2 w}{\partial z^2} = 0 \end{gathered} \tag{4}$$

3. Transfer matrix in integral transform space

The Hankel transform pair is defined by

$$\overline{F}(\xi, z) = \int_0^\infty F(r, z)\, rJ_k(\xi r)dr \tag{5.a}$$

$$F(r, z) = \int_0^\infty \overline{F}(\xi, z)\, \xi J_k(\xi r)d\xi \tag{5.b}$$

where F represents u, w, τ_{rz} or σ_z, and J_k is the Bessel's function of the first kind of order k. Note that $k=1$ for u and τ_{rz}, and $k=0$ for w and σ_z. The Laplace transform pair is defined by

$$\widetilde{F}(\xi, s) = \int_0^\infty \overline{F}(\xi, z) e^{-sz} dz \tag{6.a}$$

$$\overline{F}(\xi, z) = \frac{1}{2\pi j} \int_{\gamma - j\infty}^{\gamma + j\infty} \widetilde{F}(\xi, s) e^{sz} ds \tag{6.b}$$

where $j = \sqrt{-1}$

After applying the Hankel transform and then the Laplace transform, (4) can be written in the following matrix form

$$\begin{bmatrix} C_{44} s^2 - C_{11} \xi^2 & -(C_{13} + C_{44})\xi s \\ (C_{13} + C_{44})\xi s & C_{33} s^2 - C_{44} \xi^2 \end{bmatrix} \begin{Bmatrix} \tilde{u} \\ \tilde{w} \end{Bmatrix}$$
$$= \begin{bmatrix} C_{44} & 0 \\ 0 & C_{33} \end{bmatrix} \begin{Bmatrix} \overline{u}' \\ \overline{w}' \end{Bmatrix}_{z=0} + \begin{bmatrix} C_{44}\, s & -(C_{13} + C_{44})\xi \\ (C_{13} + C_{44})\xi & C_{33}\, s \end{bmatrix} \begin{Bmatrix} \overline{u} \\ \overline{w} \end{Bmatrix}_{z=0} \tag{7}$$

in which $\overline{u}' = \partial\overline{u}/\partial z$ and $\overline{w}' = \partial\overline{w}/\partial z$. In order to eliminate the derivatives $\overline{u}'$ and $\overline{w}'$ from (7), we first introduce the following relations:

$$\begin{aligned} \tau_{rz} &= C_{44}\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right) \\ \sigma_z &= C_{13}\frac{\partial u}{\partial r} + C_{13}\frac{u}{r} + C_{33}\frac{\partial w}{\partial z} \end{aligned} \tag{8}$$

After applying the Hankel transform, the above equations can be written in matrix form as

$$\begin{Bmatrix} \overline{\tau}_{rz} \\ \overline{\sigma}_z \end{Bmatrix} = \begin{bmatrix} C_{44} & 0 \\ 0 & C_{33} \end{bmatrix} \begin{Bmatrix} \overline{u}' \\ \overline{w}' \end{Bmatrix} + \begin{bmatrix} 0 & -C_{44}\,\xi \\ C_{13}\,\xi & 0 \end{bmatrix} \begin{Bmatrix} \overline{u} \\ \overline{w} \end{Bmatrix} \tag{9.a}$$

or

$$\begin{Bmatrix} \overline{u}' \\ \overline{w}' \end{Bmatrix} = \frac{1}{C_{33} C_{44}} \begin{bmatrix} C_{33} & 0 \\ 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \overline{\tau}_{rz} \\ \overline{\sigma}_z \end{Bmatrix} + \frac{1}{C_{33}} \begin{bmatrix} 0 & C_{33}\,\xi \\ -C_{13}\,\xi & 0 \end{bmatrix} \begin{Bmatrix} \overline{u} \\ \overline{w} \end{Bmatrix} \tag{9.b}$$

Substitution of (9.b) into (7) leads to

$$\begin{Bmatrix} \tilde{u} \\ \tilde{w} \end{Bmatrix} = \frac{\boldsymbol{G}}{\det \boldsymbol{A}} \begin{Bmatrix} \overline{u} \\ \overline{w} \end{Bmatrix}_{z=0} + \boldsymbol{A}^{-1} \begin{Bmatrix} \overline{\tau}_{rz} \\ \overline{\sigma}_z \end{Bmatrix}_{z=0} \tag{10}$$

where

$$\boldsymbol{A} = \begin{bmatrix} C_{44} s^2 - C_{11} \xi^2 & -(C_{13} + C_{44})\, \xi s \\ (C_{13} + C_{44})\, \xi s & C_{33} s^2 - C_{44} \xi^2 \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} C_{33} C_{44} s^3 + C_{44} C_{13} \xi^2 s & C_{33} C_{44} \xi s^2 + C_{13} C_{44} \xi^3 \\ -C_{44} C_{13} \xi s^2 - C_{11} C_{44} \xi^3 & C_{33} C_{44} s^3 + (C_{13}^2 + C_{13} C_{44} - C_{11} C_{33}) s\, \xi^2 \end{bmatrix}$$

The inverse Laplace transform is then applied to (10) to obtain

$$\begin{Bmatrix} \overline{u} \\ \overline{w} \end{Bmatrix}_z = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} \overline{u} \\ \overline{w} \end{Bmatrix}_{z=0} + \begin{bmatrix} T_{13} & T_{14} \\ T_{23} & T_{24} \end{bmatrix} \begin{Bmatrix} \overline{\tau}_{rz} \\ \overline{\sigma}_z \end{Bmatrix}_{z=0} \tag{11}$$

in which

$$
\begin{aligned}
T_{11} &= f_3 + \frac{C_{13}}{C_{33}} \xi^2 f_1, & T_{12} &= \xi f_2 + \frac{C_{13}}{C_{33}} \xi^3 f_0 \\
T_{21} &= -\frac{C_{13}}{C_{33}} \xi f_2 - \frac{C_{11}}{C_{33}} \xi^3 f_0, & T_{22} &= f_3 + \frac{C_{13}^2 + C_{13} C_{44} - C_{11} C_{33}}{C_{33} C_{44}} \xi^2 f_1 \\
T_{13} &= \frac{1}{C_{44}} f^2 - \frac{1}{C_{33}} \xi^2 f_0, & T_{14} &= \frac{C_{13} + C_{44}}{C_{33} C_{44}} \xi f_1 \\
T_{23} &= -\frac{C_{13} + C_{44}}{C_{33} C_{44}} \xi f_1, & T_{24} &= \frac{1}{C_{33}} f_2 - \frac{C_{11}}{C_{33} C_{44}} \xi^2 f_0
\end{aligned}
\tag{12}
$$

In the above equations, f_i ($i = 0, 1, 2, 3$) are determined from the Laplace inverse transform operation, i.e.,

$$
f_i = L^{-1}\left[\frac{s^i}{N(s)} \right] \tag{13.a}
$$

where

$$
N(s) = s^4 + \frac{C_{13}^2 + 2C_{13} C_{44} - C_{11} C_{33}}{C_{33} C_{44}} \xi^2 s^2 + \frac{C_{11}}{C_{33}} \xi^4 \tag{13.b}
$$

Explicit expressions of f_i are given in Appendix. Substitution of (11) into (9.a) results in

$$
\begin{Bmatrix} \bar{\tau}_{rz} \\ \bar{\sigma}_z \end{Bmatrix}_z = \begin{bmatrix} T_{31} & T_{32} \\ T_{41} & T_{42} \end{bmatrix} \begin{Bmatrix} \bar{u} \\ \bar{w} \end{Bmatrix}_{z=0} + \begin{bmatrix} T_{33} & T_{34} \\ T_{43} & T_{44} \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{rz} \\ \bar{\sigma}_z \end{Bmatrix}_{z=0} \tag{14}
$$

in which

$$
\begin{aligned}
T_{31} &= \frac{2C_{13}C_{44}}{C_{33}} \xi^2 f_2 + \frac{C_{11}C_{44}}{C_{33}} \xi^4 f_0 + C_{44} \frac{\partial f_3}{\partial z}, \\
T_{32} &= \frac{C_{11}C_{33} - C_{13}^2}{C_{33}} \xi^3 f_1, & T_{33} &= f_3 + \frac{C_{13}}{C_{33}} \xi^2 f_1 \\
T_{34} &= \frac{C_{11}}{C_{33}} \xi^3 f_0 + \frac{C_{13}}{C_{33}} \xi f_2, & T_{41} &= \frac{C_{13}^2 - C_{11}C_{33}}{C_{33}} \xi^3 f_1 \\
T_{42} &= \frac{C_{13}^2 + 2C_{13} C_{44} - C_{11} C_{33}}{C_{44}} \xi^2 f_2 + C_{33} \frac{\partial f_3}{\partial z} + \frac{C_{13}^2}{C_{33}} \xi^4 f_0 \\
T_{43} &= -\xi f_2 - \frac{C_{13}}{C_{33}} \xi^3 f_0, & T_{44} &= f_3 + \frac{C_{13}^2 + C_{13} C_{44} - C_{11} C_{33}}{C_{33} C_{44}} \xi^2 f_1
\end{aligned}
\tag{15}
$$

Finally, Eqs. (11) and (14) can be combined into a single matrix equation as the following

$$
\begin{Bmatrix} \bar{u} \\ \bar{w} \\ \bar{\tau}_{rz} \\ \bar{\sigma}_z \end{Bmatrix}_z = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{Bmatrix} \bar{u} \\ \bar{w} \\ \bar{\tau}_{rz} \\ \bar{\sigma}_z \end{Bmatrix}_{z=0} \tag{16}
$$

or

$$
\boldsymbol{d}(z) = \boldsymbol{T}\, \boldsymbol{d}(0)
$$

in which $\boldsymbol{T}$ is the so-called transfer matrix whose elements shown in (12) and (15) depend only on the elastic constants and thickness of the layer.

4. Assembly of global matrix

Once the transfer matrix for each layer is formed following the procedure described above, the global matrix can be assembled if the interlayer contact conditions and boundary conditions of the layered medium are introduced. For the case where all layers are perfectly bonded, the transfer matrix can be easily determined and its order does not depend on the number of layers since the transfer matrix is multiplicative in nature. The global transfer matrix for a n-layered medium (see Fig. 1) can be obtained as a product of n matrices, i.e.

$$\boldsymbol{d}(z) = \boldsymbol{T}\,\boldsymbol{d}(0) = \boldsymbol{T}_n \cdots \boldsymbol{T}_i \cdots \boldsymbol{T}_1\,\boldsymbol{d}(0) \tag{17}$$

where $\boldsymbol{T}_i$ denotes the transfer matrix for the i-th layer. For more general interlayer contact conditions, such as a partial contact at the interface, however, a different procedure must be used. The procedure is described in the following: let us consider two adjacent layers, the i-th and $i+1$-th layers. According to (16), unknowns at the upper and lower surfaces of an individual layer can be related in the following forms:

$$\boldsymbol{d}_i^+ = \boldsymbol{T}_i\,\boldsymbol{d}_i \tag{18}$$

for the i-th layer,

$$\boldsymbol{d}_{i+1}^+ = \boldsymbol{T}_{i+1}\,\boldsymbol{d}_{i+1}^- \tag{19}$$

for the $i+1$-th layer where superscripts + and − denote the upper and lower surface of each layer respectively. Assembling (18) and (19) and introducing the general interlayer contact condition between the i and $i+1$-th layers, we obtain

$$\begin{bmatrix} -\boldsymbol{I} & \boldsymbol{T}_{i+1} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J} & \boldsymbol{K} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & -\boldsymbol{I} & \boldsymbol{T}_i \end{bmatrix}_{12\times 16} \begin{Bmatrix} \boldsymbol{d}_{i+1}^+ \\ \boldsymbol{d}_{i+1}^- \\ \boldsymbol{d}_i^+ \\ \boldsymbol{d}_i^- \end{Bmatrix}_{16\times 1} = \begin{Bmatrix} \boldsymbol{O} \\ \boldsymbol{O} \\ \boldsymbol{O} \end{Bmatrix}_{12\times 1} \tag{20}$$

or

$$\boldsymbol{H}_i^{i+1}\,\boldsymbol{D}_i^{i+1} = \boldsymbol{O} \tag{21}$$

where $\boldsymbol{I}$ is the 4x4 unit matrix, and $\boldsymbol{J}$ and $\boldsymbol{K}$ are the contact matrices which depend on the interlayer contact condition between the i-th and $i+1$-th layers. For a smooth contact, for example, the interlayer contact condition becomes

$$\overline{w}_{i+1}^- = \overline{w}_i^+,\ (\overline{\sigma}_z)_{i+1}^- = (\overline{\sigma}_z)_i^+ \quad \text{and} \quad (\overline{\tau}_{rz})_{i+1}^- = (\overline{\tau}_{rz})_i^+ = 0 \tag{22}$$

Therefore, the contact matrices in this case become

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{K} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{23}$$

If the boundary conditions on the upper surface of the n-th layer and the lower surface of the first layer are introduced, the global matrix for the n-layer medium can be assembled into

$$\begin{bmatrix} H_n & \cdot\cdot & O & \cdot\cdot & O \\ H^n_{n-1} & \cdot\cdot & O & \cdot\cdot & O \\ \cdot & \cdot\cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot\cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot\cdot & \cdot & \cdot\cdot & \cdot \\ O & \cdot\cdot & H^{i+1}_i & \cdot\cdot & O \\ \cdot & \cdot\cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot\cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot\cdot & \cdot & \cdot\cdot & \cdot \\ O & \cdot\cdot & O & \cdot\cdot & H^2_1 \\ O & \cdot\cdot & O & \cdot\cdot & H_o \end{bmatrix} \begin{Bmatrix} D^n_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ D^{i+1}_i \\ \cdot \\ \cdot \\ \cdot \\ D^2_1 \end{Bmatrix} = \begin{Bmatrix} \overline{P} \\ O \\ \cdot \\ \cdot \\ \cdot \\ O \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ O \end{Bmatrix} \tag{24}$$

in which H_n and H_o are the boundary matrices (for the top and bottom boundary respectively) and $\overline{P}$ is the load vector. If normal and tangential loads are acted on the top surface of the n-th layer simultaneously,

$$\overline{P} = \frac{1}{2\pi}\begin{Bmatrix} p_t J_1(\xi a) \\ -p_n J_0(\xi a) \end{Bmatrix} \tag{25}$$

$$H_n = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

where p_t and p_n are the normal and tangential loads respectively. The boundary matrix H_o can take different forms depending on the boundary condition at the lower surface of the first layer as follows:

$$H_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad H_o = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad H_o = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -k & 0 & 1 \end{bmatrix} \tag{27}$$

for a fixed boundary, smooth rigid boundary, and Winkler base (k=spring constant) respectively. Note that elements of the main diagonal of the coefficient matrix in (24) may be zero, but the matrix is not singular and can be solved by any standard elimination method. When all unknowns on the interface are determined, displacements and stresses at any interior point lying in the i-th layer can be found from the following equation

$$\boldsymbol{d}(z) = \boldsymbol{T}_i(z)\, \boldsymbol{d}_i^- \tag{28}$$

in which z is the distance of the field point from the lower surface of the i-th layer. The Hankel inverse transform is then applied to obtain the stress and displacement in the virtual space.

5. Selected solutions

Solutions to some problems of practical interest are worked out in the following:

(1) Normal loads

We consider a layered medium of finite depth H resting on either a fixed rigid base or a Winkler mattress. We assume that a normal load of p_n is applied on an annular area of radius a on the surface. The boundary conditions at the top (z=H) then become

$$\bar{\sigma}_z = \int_0^\infty \sigma_z(r, z) r J_0(\xi r) dr = -\frac{p_n}{2\pi} J_0(\xi a) \quad \text{and} \quad \bar{\tau}_{rz} = 0 \tag{29}$$

for both cases. The boundary conditions at the bottom ($z=0$) become

$$\bar{u} = 0 \quad \text{and} \quad \bar{w} = 0 \tag{30}$$

for a fixed base, or

$$\bar{\tau}_{rz} = 0 \quad \text{and} \quad \bar{\sigma}_z = k\bar{w} \tag{31}$$

for a Winkler base, in which k is the spring coefficient of the Winkler mattress.

The surface displacements in the Hankel transform space are then given by

$$\begin{Bmatrix} \bar{u} \\ \bar{w} \end{Bmatrix} = -\frac{p_n J_0(\xi a)}{2\pi(T_{33}T_{44} - T_{34}T_{43})} \begin{Bmatrix} T_{33}T_{14} - T_{13}T_{34} \\ T_{33}T_{24} - T_{23}T_{34} \end{Bmatrix} \tag{32}$$

for the case of fixed base and

$$\begin{Bmatrix} \bar{u} \\ \bar{w} \end{Bmatrix} = \frac{-p_n J_0(\xi a)}{2\pi[T_{31}T_{42} - T_{32}T_{41} + k(T_{31}T_{44} - T_{41}T_{34})]} \begin{Bmatrix} T_{12}T_{33} - T_{11}T_{32} + k(T_{14}T_{33} - T_{11}T_{34}) \\ T_{22}T_{33} - T_{21}T_{32} + k(T_{24}T_{33} - T_{21}T_{34}) \end{Bmatrix} \tag{33}$$

for the case of Winkler base.

(2) Tangential loads

We consider now a case where a tangential load of p_t is applied on an annular area on the top surface of the same layered medium. The boundary conditions at the top surface($z=H$) become

$$\bar{\tau}_{rz} = \int_0^\infty \tau_{rz} r J_1(\xi r) dr = \frac{p_t}{2\pi} J_1(\xi a) \quad \text{and} \quad \bar{\sigma}_z = 0 \tag{34}$$

The boundary conditions at the bottom ($z=0$) are given by (30) or (31). The surface displacements in the transformed space are then given by

$$\begin{Bmatrix} \bar{u} \\ \bar{w} \end{Bmatrix} = \frac{p_t J_1(\xi a)}{2\pi(T_{33}T_{44} - T_{34}T_{43})} \begin{Bmatrix} T_{13}T_{44} - T_{14}T_{43} \\ T_{23}T_{44} - T_{24}T_{43} \end{Bmatrix} \tag{35}$$

for the case of fixed base, and

$$\begin{Bmatrix} \bar{u} \\ \bar{w} \end{Bmatrix} = \frac{p_t J_1(\xi a)}{2\pi[T_{31}T_{42} - T_{32}T_{41} + k(T_{31}T_{44} - T_{41}T_{34})]} \begin{Bmatrix} T_{11}T_{42} - T_{12}T_{41} + k(T_{11}T_{44} - T_{14}T_{41}) \\ T_{21}T_{42} - T_{22}T_{41} + k(T_{21}T_{44} - T_{24}T_{41}) \end{Bmatrix} \tag{36}$$

for the case of Winkler base. The solutions for a normal concentrated load can be reduced from Eq.(32) or (33) by setting $a=0$. The solutions for other boundary conditions can also be obtained in similar fashion.

6. Convergence of numerical integration

If the Hankel inverse transform is applied to Eqs.(32), (33), (35) or (36), the solutions in the virtual space can be written in the following form:

$$F = \int_0^\infty f(\xi) J_j(\xi a) J_j(\xi r) d\xi \tag{37}$$

where i and j take on 0 or 1. It should be noted that convergence of the above integral is very slow if no special treatment is taken.

It can be shown that, for a lager ξ, the integrand $f(\xi)$ in (37) approaches asymptotically to a non-zero constant C for given a and r. Thus we can rewrite F as

$$F = C\int_0^\infty J_i(\xi a)J_j(\xi r)d\xi + \Delta F \tag{38}$$

Since the integral in the right hand side of (39) can be found in closed form (Abramowitz and Stegun 1964), ΔF can be written as the following:

$$\Delta F = \int_0^{\bar{\xi}} [f(\xi) - C]J_i(\xi a)J_j(\xi r)d\xi + \int_{\bar{\xi}}^\infty [f(\xi) - C]J_i(\xi a)J_j(\xi r)d\xi \tag{39}$$

Noting that the second integral in (39) vanishes, (39) becomes

$$\Delta F \approx \int_0^{\bar{\xi}} [f(\xi) - C]J_i(\xi a)J_j(\xi r)d\xi \tag{40}$$

where $\bar{\xi}$ can be determined by considering the permissible relative error of the numerical integration. ΔF converges rapidly enough to be evaluated by using the usual numerical integration scheme.

Moreover, since the term $f(\xi)-C$ in (40) decays exponentially, it is possible to avoid the numerical integration at all. We note that

$$f(\xi) - C \approx \sum_{k=1}^{K} a_k\, e^{-(kh\xi)^2} \tag{41}$$

where K depends on the permissible relative errors of ΔF, h is the average thickness of layer of the multilayered medium, and a_k can be obtained in a recurrent analysis or the least square approximation. If we replace $f(\xi) - C$ in (40) with (41), ΔF can be determined in closed form by using the Wallis expansion (e.g., Arfken 1970). For the case when $a=0$, for example,

$$\Delta F \approx \int_0^\infty e^{-(kh\xi)^2} J_0(r\xi)d\xi = \frac{\sqrt{\pi}}{kh} M\left(\frac{1}{2}, 1, -\frac{r^2}{4(kh)^2}\right) \tag{42}$$

in which M stands for the confluent hypergeometric function (Abramowitz and Stegun 1964).

7. Numerical examples

As an illustration, three problems are considered and their results are presented in the following:

Example 1: In order to verify the method developed herein, we first consider a simple example of an isotropic layer of depth H on a fixed rigid base under the action of a concentrated vertical load P (see Fig. 2). Fig. 3 shows the dimensionless transverse displacement $w\pi ER/P(1-v^2)$ as a function of the normalized thickness H/R for given Young's modulus (E) and Poisson's ratio (v), where R denotes the distance between the source point and the prescribed field point. The results are in excellent agreement with the analytical solution (Timoshenko and Goodier 1970). Fig. 4 shows u/w at the prescribed field point as a function of v for selected values of the normalized thickness H/R. For an isotropic half space ($H/R=\infty$), the exact solu-

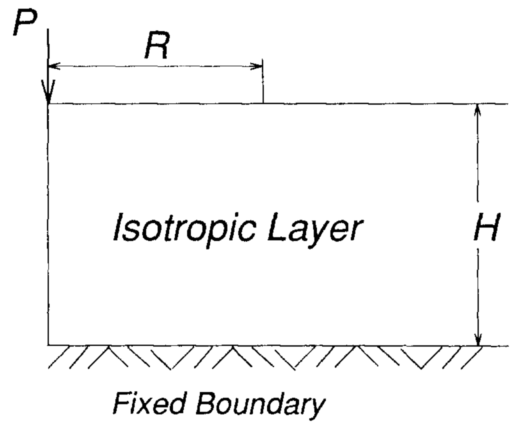

Fig. 2 Example 1: isotropic elastic layer on fixed base.

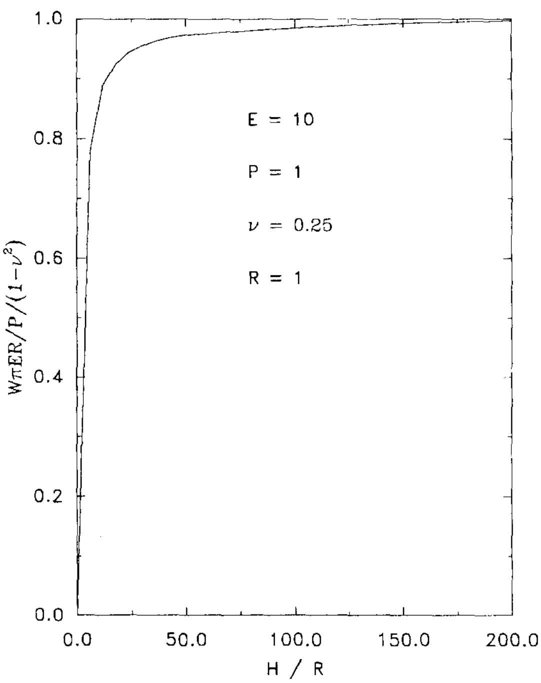

Fig. 3 W/R vs. H/R with $E = 10$ and $v = 0.25$

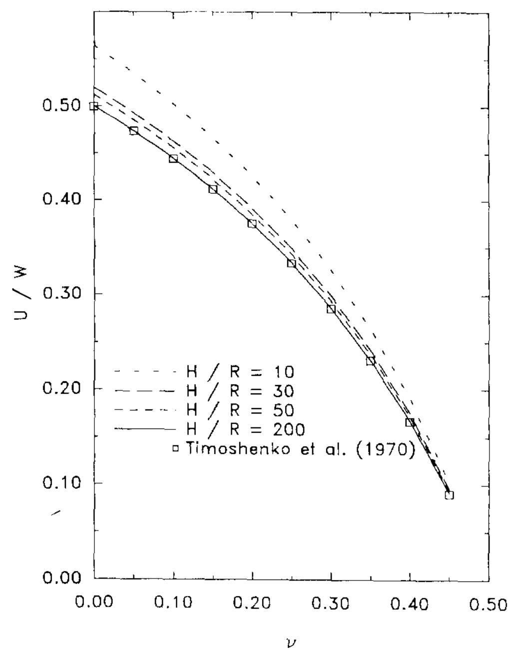

Fig. 4 U/W vs. v for selected values of H/R.

tion becomes $u/w = (1-2v)/(2-2v)$ (Timoshenko and Goodier 1970), which is also plotted in the figure. When $H/R = 200$, the results from the present method is shown to be identical to the exact solution.

Example 2: We consider next a cross-anisotropic elastic layer on a Winkler mattress with spring constant k which is subjected to a normal concentrated load P as shown in Fig. 5. It is noted that $k = 0$ and $k = \infty$ represent the free and fixed boundary support respectively. The elastic constants used are as follows: $C_{11} = 8.68$, $C_{13} = 6.66$, $C_{33} = 9.13$, and $C_{44} = 1.124$. Flg. 6 shows the non-dimensional transverse displacements at the prescribed field point on the surface ($R = 0.4H$) for selected values of the nondimensionalized spring constant $K = kH^3/P$. As

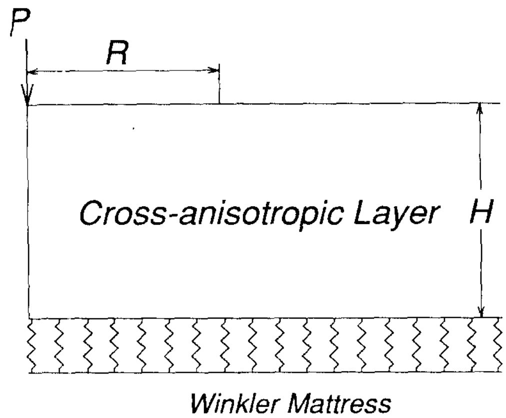

Fig. 5 Example 2: a anisotropic elastic layer on Winkler mattress.

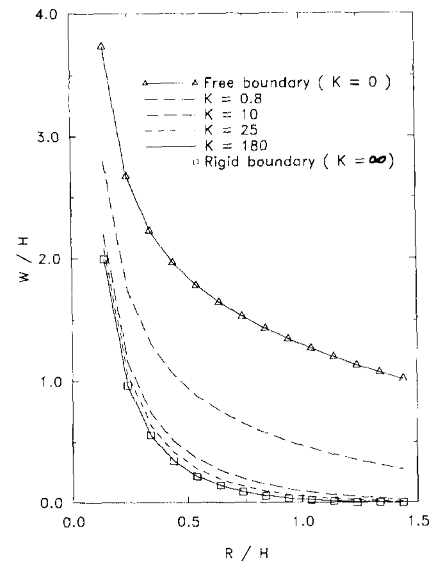

Fig. 6 W/H vs. R/H for selected values of K.

expected, the two extreme cases corresponding to the the free and fixed support (i.e., $k=0$ and $k=\infty$) provide the upper and lower limits for the elastic response as shown in Fig. 6. It is also shown in the figure that, when $K=180$, the result is almost identical to that of the case with rigid base. For a large K (>25), the difference in displacement between the Winkler base and the rigid base is shown to be negligible.

Example 3: As the last example, we consider a cross-anisotropic 3-layer system under the action of a normal concentrated load P as shown in Fg. 7. Elastic constants for three cross-anisotropic materials are shown in Table 1. Three cases are considered and the sequence (from top to bottom) of materials for the three cases are as follows: CASE 1 (A-B-C), CASE 2 (B-A-C), and CASE 3 (C-B-A). Fig. 8 shows the variation of the non-dimensional vertical stress ($\sigma_z H^2/P$) with depth (z/H) for three cases with perfectly bonded interlayer conditions and fixed support condition. As shown in the figure, the vertical stress reaches its maximum in the middle layer for CASE 1 and 3, whereas CASE 2 has its maximum at the bottom of the top layer. Fig. 9 shows the same variation of the non-dimensional vertical stress with depth for the three cases with smooth interlayer contacts and smooth rigid support. As expected, curves in Fig. 9 are not smooth due to the smooth interlayer contact condition which allows a jump in the radial displacement. This feature is most pronounced for CASE 1 at $z/H=1/3$. The figure also shows that the dimensionless normal stress essentially becomes constant in the bottom layer.

Table 1 Elastic constants of three materials for Example 3.

Material	C_{11}	C_{13}	C_{33}	C_{44}
A	8.69	6.66	9.13	1.12
B	18.50	9.94	19.47	4.45
C	45.25	17.22	51.93	15.63

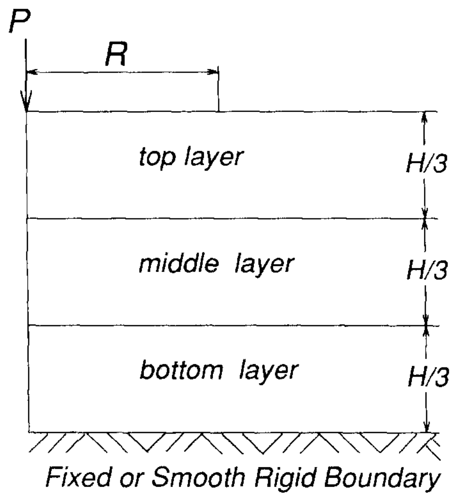

Fig. 7 Example 3: a 3-layer cross-anisotropic elastic medium on fixed or smooth rigid base.

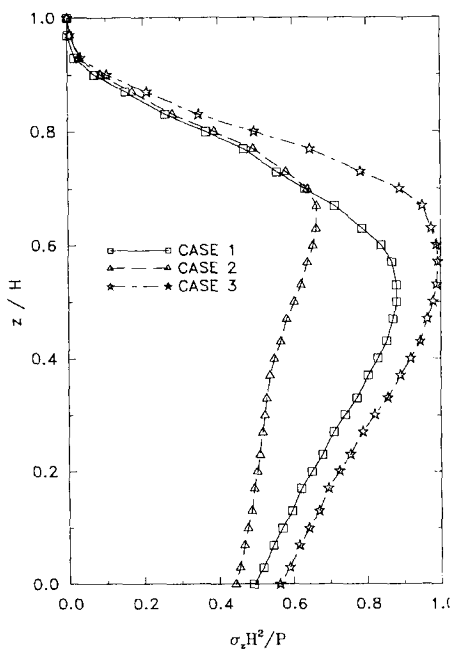

Fig. 8 Variation of $\sigma_z H^2/P$ for welded interlayer contact and fixed base.

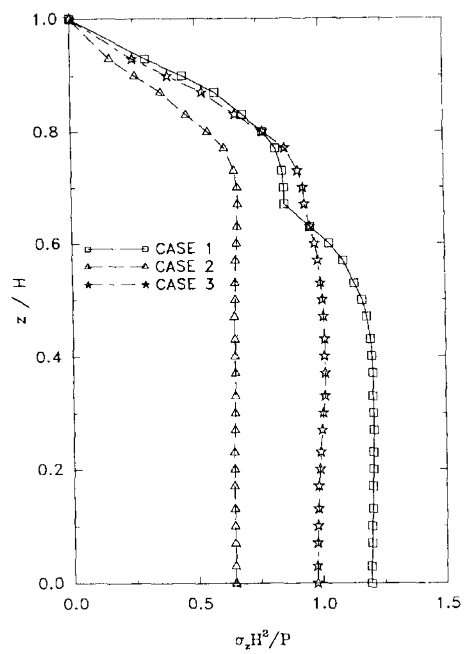

Fig. 9 Variation of $\sigma_z H^2/P$ for smooth interlayer contact and smooth rigid base.

8. Conclusions

Fundamental solutions to problems of multi-layered transversely isotropic elastic media to localized axisymmetric loads have been obtained by introducing a generalized transfer matrix method. In contrast to the conventional formulation, the number of simultaneous equations to be solved remains constant regardless of the number of layers in the media for the case of perfect or smooth interlayer contact conditions. For other contact conditions, a simple systematic procedure has been proposed for the assembly of the global transfer matrix. Since the transfer matrix is obtained in the way different from the usual state space method, this technique can be an effective tool for problems in which it is difficult to find the transfer matrix functions explicitly. Furthermore, in contract to most previous studies, this generalized transfer matrix method can accommodate combinations of general interlayer contact conditions, loads and support conditions. In addition, a special numerical technique has been introduced to efficiently evaluate the slowly converging infinite integrals. Several numerical examples have demonstrated the accuracy and effectiveness of the proposed technique. The solution method described in this paper can readily be extended to general 3-D anisotropic layered media.

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Appendix.

The Laplace inverse transform f_i of the function $s^i / N(s)$ is given here. By letting $s=\eta\xi$, the characteristic equation $N(s)=0$ (13.b) can be written as

$$\eta^4 + \frac{C_{13}^2 + 2C_{13}C_{44} - C_{11}C_{33}}{C_{33}C_{44}}\eta^2 + \frac{C_{11}}{C_{33}} = 0$$

Although the above equation can possess more than two classes of roots, only two cases need be given in the following since the elastic constants should satisfy certain conditions (Gerrard 1976). (Note sh≡sinh and ch ≡cosh).

CASE I: different real roots: $s=\pm\eta_1\xi,\ \pm\eta_2\xi$

$$f_0 = \frac{1}{(\eta_1^2-\eta_2^2)\xi^3}\left[\frac{1}{\eta_1}\mathrm{sh}(\eta_1\xi z) - \frac{1}{\eta_2}\mathrm{sh}(\eta_2\xi z)\right], \qquad \frac{\partial f_0}{\partial z} = \frac{\mathrm{ch}(\eta_1\xi z) - \mathrm{ch}(\eta_2\xi z)}{(\eta_1^2-\eta_2^2)\xi^2}$$

$$f_1 = \frac{\partial f_0}{\partial z}, \qquad \frac{\partial f_1}{\partial z} = \frac{\eta_1\,\mathrm{sh}(\eta_1\xi z) - \eta_2\,\mathrm{sh}(\eta_2\xi z)}{(\eta_1^2-\eta_2^2)\xi}$$

$$f_2 = \frac{\partial f_1}{\partial z}, \qquad \frac{\partial f_2}{\partial z} = \frac{\eta_1^2\,\mathrm{ch}(\eta_1\xi z) - \eta_2^2\,\mathrm{ch}(\eta_2\xi z)}{\eta_1^2-\eta_2^2}$$

$$f_3 = \frac{\partial f_2}{\partial z}, \qquad \frac{\partial f_3}{\partial z} = \frac{\xi[\eta_1^3\,\mathrm{sh}(\eta_1\xi z) - \eta_2^3\,\mathrm{sh}(\eta_2\xi z)]}{\eta_1^2-\eta_2^2}$$

CASE II: two pairs of equal real roots: $s=\pm\eta\xi,\ \pm\eta\xi$

$$f_0 = -\frac{\mathrm{sh}(\eta\xi z)}{2(\eta\xi)^3} + \frac{z\,\mathrm{ch}(\eta\xi z)}{2(\eta\xi)^2}, \qquad \frac{\partial f_0}{\partial z} = \frac{z\,\mathrm{sh}(\eta\xi z)}{2\eta\xi}$$

$$f_1 = \frac{\partial f_0}{\partial z}, \qquad \frac{\partial f_1}{\partial z} = \frac{\mathrm{sh}(\eta\xi z) + \eta\xi z\,\mathrm{ch}(\eta\xi z)}{2\eta\xi}$$

$$f_2 = \frac{\partial f_1}{\partial z}, \qquad \frac{\partial f_2}{\partial z} = \mathrm{ch}(\eta\xi z) + \frac{1}{2}\eta\xi z\,\mathrm{sh}(\eta\xi\zeta)$$

$$f_3 = \frac{\partial f_2}{\partial z}, \qquad \frac{\partial f_3}{\partial z} = \frac{3}{2}\eta\xi\,\mathrm{sh}(\eta\xi z) + \frac{1}{2}(\eta\xi)^2 z\,\mathrm{ch}(\eta\xi z)$$