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Unified theory of reinforced concrete-A summary

Thomas T. C. Hsu †

Dept. of Civil and Enrironmental Engineering, University of Houston, Houston, Texas, U.S.A.

Abstract. A unified theory has recently been developed for reinforced concrete structures (Hsu 1993), subjected to the four basic actions - bending, axial load, shear and torsion. The theory has five components, namely, the struts-and-ties model, the equilibrium (or plasticity) truss model, the Bernoulli compatibility truss model, the Mohr compatibility truss model and the softened truss model. Because the last three models can satisfy the stress equilibrium, the strain compatibility and the constitutive laws of materials, they can predict not only the strength, but also the load-deformation history of a member. In this paper the five models are summarized to illustrate their intrinsic consistency.

Key words: axial load; bending moment; compatibility; constitutive laws; equilibrium; reinforced concrete; shear; structural design; torsion; truss models.

1. Introduction

The structural engineering of a reinforced concrete structure includes four steps as shown in Table 1. First, a structural analysis is made to find the bending moment diagram M, the

Table 1 Unified theory of reinforced concrete structures

<table>
<tr><td colspan="3" rowspan="2">(1) STRUCTURAL ANALYSIS</td><td colspan="5">WHOLE STRUCTURE (ELASTIC OR INELASTIC)</td></tr>
<tr><td colspan="5">BENDING (M), AXIAL LOAD (N), SHEAR (V), TORSION (T)</td></tr>
<tr><td colspan="3">(2) DIVISION OF REGIONS</td><td>LOCAL REGIONS</td><td colspan="4">MAIN REGIONS</td></tr>
<tr><td colspan="3" rowspan="2">(3) DESIGN ACTIONS</td><td rowspan="2">BOUNDARY STRESSES</td><td colspan="4">SECTIONAL ACTIONS</td></tr>
<tr><td>M.N.V.T</td><td>M.N</td><td colspan="2">V.T</td></tr>
<tr><td rowspan="5">(4) PRINCIPLES OF ANALYSIS AND DESIGN</td><td colspan="2">EQUILIBRIUM</td><td>STRUTS & TIES MODEL</td><td>EQUILIBRIUM (PLASTICITY) TRUSS MODEL</td><td rowspan="3">BERNOULLI COMPATIBILITY TRUSS MODEL (UNI-AXIAL)</td><td rowspan="4">MOHR COMPATIBILITY TRUSS MODEL (BI-AXIAL)</td><td rowspan="5">SOFTENED TRUSS MODEL</td></tr>
<tr><td rowspan="2">UNI-AXIAL</td><td>BERNOULLI COMPATIBILITY</td><td colspan="2"></td></tr>
<tr><td>NON-SOFTENED LAWS OF MATERIAL</td><td colspan="2"></td></tr>
<tr><td rowspan="2">BI-AXIAL</td><td>MOHR COMPATIBILITY</td><td colspan="3"></td></tr>
<tr><td>SOFTENED LAWS OF MATERIAL</td><td colspan="4"></td></tr>
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† Professor

axial load diagram N, the shear diagram V and the torsional moment diagram T. Second, the structure is divided into two types of regions, namely, the main regions and the local regions. Third, the design actions are determined for these two regions according to the four action diagrams. The main regions are subjected to the sectional actions M, N, V and T, while the local regions are treated as a free body subjected to boundary stresses. Fourth, the size and the reinforcement of the main regions are designed according to the sectional actions, while the reinforcement in the local regions are designed according to the boundary stresses. For this final and most important step a unified theory is developed (Hsu 1993).

A rational design and analysis of a structure should be based on the three fundamental principles of the mechanics of materials, namely, the stress equilibrium, the strain compatibility and the constitutive laws of materials. The main regions are those regions where the stresses and strains vary so regularly that they are governed by simple equilibrium and compatibility conditions. In the case of beams and columns subjected to bending and/or axial load, the main region should satisfy parallel stress equilibrium and Bernoulli's linear compatibility condition. In the case of shear and torsion, the main region should satisfy two–dimensional stress equilibrium and Mohr's circular strain compatibility.

In contrast, the local regions include the connections among beams and columns, the corbels, the regions adjacent to concentrated loads, the ledgers and the dapped ends of beams, etc. In such regions the stresses and strains are so disturbed and irregular that the compatibility condition is difficult to apply. The reinforcing bars in these local regions are designed according to the equilibrium condition alone. Although compatibility conditiion could be maintained using numerical analyses (such as finite element method), such tedious analyses are employed only occasionally for very important structures.

The two types of regions (main and local) and the four sectional actions M, N, V and T can all be treated by a unified theory. The unified theory includes five component models as shown in Table 1. Each model is named to reflect the most significant principle(s) embodied in each. Some models are particularly suitable for the service load stage or the ultimate load stage. The basic principles and the scope of applications of each model are explained as follows:

(1) ***Struts–and–Ties Model:*** satisfies equilibrium condition only; applicable to the design of local regions.

(2) ***Equilibrium (Plasticity) Truss Model:*** satisfies *equilibrium* condition and the theory of *plasticity*; applicable to the design of M, N, V and T in the main regions at the ultimate load stage.

(3) ***Bernoulli Compatibility Truss Model:*** satisfies equilibrium condition, *Bernoulli's compatibility* condition and the uniaxial constitutive laws of concrete and reinforcement. The constitutive laws may be linear or nonlinear. It is applicable to the design of M and N in the main regions at both the service and ultimate load stages.

(4) ***Mohr Compatibility Truss Model:*** satisfies equilibrium condition, *Mohr's compatibility* condition and Hooke's uniaxial constitutive law for both concrete and steel. It is applicable to the design of V and T in the main regions at the service load stage.

(5) ***Softened Truss Model:*** satisfies equilibrium condition, Mohr's compatibility condition and the *softened* biaxial constitutive laws of concrete. The constitutive law of reinforcement may be linear or nonlinear. It is applicable to the design of V and T in the main region at both the service and ultimate load stages.

2. Development of unified theory

Concrete is a material that is very strong in terms of compressive strength but weak in tensile strength. When concrete is used in a structure to carry loads, the tensile region will crack and must be reinforced by materials having high tensile strength, such as steel. The amount of steel bars is designed to pick up the tensile forces that could not be resisted by concrete. This concept of utilizing the concrete to resist compression and the steel to carry the tension gave rise to the **struts–and–ties model**. In this model, concrete compression struts and the steel tension ties form a stable truss which is capable of resisting the applied loadings. Ever since Joseph Monier, the French gardener, used steel meshes to reinforce his flower pots in 1857, the struts–and–ties model has been used intuitively by engineers to design reinforced concrete structures.

Although struts–and–ties model is applicable to both the main and the local regions, it is useful primarily in the local regions governed by equilibrium alone. A free–form truss is formed in the local region consisting of concrete compression struts and steel tension ties oriented in arbitrary angles of inclination. The truss is stable under the boundary stresses. The analysis of the local region is focused on the flow of the compressive and tensile stresses, the "node" where the struts and the ties intersect, the dimensioning of the concrete struts, and the specifications of the permissible stresses under various stress conditions.

When the struts–and–ties model is applied to the main regions, it is known as a truss model. In a truss model the angle of inclination of the concrete struts becomes a constant. This angle may be obtained from the stress equilibrium before or after cracking, or from the strain compatibility. The analysis of a main region is focused on the change of this angle under loading, as well as the stresses and strains associated with this change of angle.

The truss model concept was most conveniently applied to the main region of a beam. In a beam subjected to bending, vertical cracks are expected to form. The compressive stresses in the upper part of a beam will be resisted by concrete in the form of a compression stringer, while the tensile stress in the lower part is taken by the bottom steel in the form of a tension stringer. The forces in the concrete and in the steel must be equal, and they form a couple to resist the applied bending moment. The determination of the stress distributions in the concrete and in the steel, however, could not be determined by equilibrium alone. The solution was obtained by borrowing Bernoulli's hypothesis from the analysis of homogeneous beams. This hypothesis stated that a plane section before bending would remain a plane after bending. The application of Bernoulli's compatibility condition to the analysis of reinforced concrete beams gave rise to the **Bernoulli compatibility truss model**. This is the first rational model that satisfies the three fundamental principles of the mechanics of materials, namely, equilibrium, compatibility and the constitutive laws of materials. Although not clearly documented, this model had been used by engineers since the late nineteenth century, serving as the fundamental theory of reinforced concrete for more than a hundred years. This rational model could, of course, be easily extended to columns subjected to bending and axial load.

The first application of the concept of truss model to shear was proposed by Ritter (1989) and Morsch (1902) in connection with a reinforced concrete beam subjected to bending and shear. In their concept, a reinforced concrete beam after cracking acts like a parallel–stringer truss to resist bending. At the same time the shear stress is resisted by the web region which has developed diagonal cracks at an angle α inclined to the longitudinal steel. These cracks would separate the concrete into a series of diagonal concrete struts. To resist the applied shear forces after cracking, the transverse steel bars in the web will be subjected to tensile forces and the diagonal concrete struts will be taking compressive forces. The transverse steel, therefore, serves as the tensile web members in the truss while the diagonal concrete struts becomes the diagonal compression web members.

The plane truss model concept for a beam was extended to treat members subjected to torsion by Rausch (1929). In Rausch's concept, a torsional member is idealized as a space truss formed by connecting a series of component plane trusses capable of resisting shear action. The circulatory shear stresses, developed in the space truss, form an internal torsional moment capable of resisting the applied torsional moment.

The rudimentary truss model of Ritter, Morsch and Rausch, unfortunately, could not explain some behavior of reinforced concrete. Further research, therefore, did not follow this line until the late 1960's when Nielson (1967) and Lampert and Thurlimann (1968) derived the three fundamental equilibrium equations for shear based on the theory of plasticity. The interaction relationship of bending, shear and torsion was further obtained by Elfgren (1972). All these theories were known as the **plasticity truss model** because they were based on the yielding of steel. In the unified theory, the name **equilibrium truss model** would be more appropriate, because it indicated that only the equilibrium condition was considered. The compatibility condition and the constitutive laws of material were not taken into account.

The next important advancement was the determination of the angle of inclination of the concrete struts by Collins (1973) using the compatibility condition in a reinforced concrete element subjected to shear. Because the average strain condition in a membrane element must satisfy Mohr's circle, a **Mohr compatibility truss model** was established. This model satisfies the two-dimensional equilibrium of membrane stresses, Mohr's circular compatibility and Hooke's law. From Mohr's circular geometric relationship, three compatibility equations can be established. Because Hooke's law is used, Mohr compatibility truss model is consistent with the theory of elasticity.

A fundamental breakthrough in the understanding of shear and torsion was the discovery of the softening of the concrete struts by Robinson and Demorieux (1972) and the first quantification of this phenomenon by Vecchio and Collins (1981). Prior to 1972, the stress–strain curve of the concrete struts was assumed to be the same as that obtained from the uniaxial compression tests of standard concrete cylinders. This assumption led to a severe overestimation of the shear and torsional strengths. Robinson and Demorieux observed that a reinforced concrete panel subjected to compression in one direction was softened by tension in the perpendicular direction. This softening phenomenon was quantified by Vecchio and Collins who proposed a stress–strain curve incorporating a softening coefficient.

By combining the equilibrium, compatibility, and the softened stress–strain relationship of concrete, the author and his colleagues developed a theory that can predict with good accuracy the behavior of various types of structures subjected to shear. The problems solved include low-rise shear walls (Hsu and Mo 1985d, Mau and Hsu 1986), framed shear walls (Mau and Hsu 1987a), shear transfer (Hsu, Mau and Chen 1987) and deep beams (Mau and Hsu 1987b). By including an additional equilibrium equation and four additional compatibility equations, the theory became applicable to torsion (Hsu and Mo 1985a, 1985b, 1985c). This theory, which unified shear and torsion (Hsu 1988), had been called the **softened truss model**. The theory can predict not only the shear and torsional strengths but also the load–deformation behavior of a structure throughout its loading history.

3. Struts-and-ties model

Because the struts–and–ties model was found to be a powerful method for the design of local regions, it received renewed interest in the 1980s. In the modern design concept, the local region is isolated as a free body and is subjected to boundary forces obtained from the action diagrams. The local region itself is imagined to be a free–form truss made up of compression struts and tension ties. The struts and ties are arranged so that the internal forces are in equilibrium with the boundary forces. In this design method the compatibility condition is not satisfied, and the serviceability criteria may not be ensured. Understanding of the stress flows and the steel anchorage requirement in a local region can help to improve the serviceability and to prevent premature failures. In other words, the assumed crack angle for the compression struts should be close to the actual crack angle. A good design depends on the experience of the engineer.

Proficiency in the application of this design method requires practice. An excellent treatment of the struts–and–ties method was given by Schlais, Schafer and Jennewein (1987). This 77–page paper provides many examples to illustrate the application of the method.

For important structures, design of local regions by struts–and–ties method may be supplemented by a numerical analysis (such as finite element method), because such an analysis can be made to satisfy the compatibility condition. Although numerical analyses can illustrate the stress flow and guide the placement of reinforcement, it is quite tedious even for first–order linear analysis.

4. Equilibrium(plasticity) truss model

Equilibrium(plasticity) truss model is based on the equilibrium condition and the yielding of the steel reinforcement. Consequently, no compatibility equations are required. The basic equilibrium equations for bending, shear and torsion have been derived (Thurlimann 1979) as follows:

4.1. Basic equilibrium equations

Bending $$M_0 = A_s f_y (jd) = N_{bly} d_v \qquad (1)$$

Shear $$V_0 = 2d_v \sqrt{\left(\frac{2N_{tly}}{d_v}\right) n_{ty}} \qquad (2)$$

Torsion $$T_0 = 2A_0 \sqrt{\left(\frac{2N_{tly}}{p_0}\right) n_{ty}} \qquad (3)$$

where

M_0 = yield moment in pure bending,

V_0 = yield force in pure shear,

T_0 = yield moment in pure torsion,

d_v = lever arm in bending, measured from center of bottom stringer to center of top stringer,

A_0 = cross-sectional area within the centerline of the shear flow,
p_0 = perimeter of the centerline of the shear flow,
N_{bly} = tensile yield force in bottom longitudinal stringer,
N_{tly} = tensile yield force in top longitudinal stringer,
n_{ty} = tensile yield force per unit length in transverse steel.

4.2. Interaction of bending, shear and torsion

When a reinforced concrete member is subjected simultaneously to a bending moment M, a shear force V and a torsional moment T, it could fail in one of three modes, resulting in three interaction equations. The three interation equations were first derived from the three failure modes of rectangular sections by Elfgren (1972) and are summarized as follows:

First Failure Mode —— failure is caused by yielding in the bottom stringer and in the transverse reinforcement on the side where shear flows due to shear and torsion are additive:

$$\frac{M}{M_0} + \left(\frac{V}{V_0}\right)^2 R + \left(\frac{T}{T_0}\right)^2 R = 1 \tag{4}$$

Second Failure Mode —— failure is caused by yielding in the top stringer and in the transverse reinforcement on the side where shear flows due to shear and torsion are additive:

$$-\left(\frac{M}{M_0}\right)\frac{1}{R} + \left(\frac{V}{V_0}\right)^2 + \left(\frac{T}{T_0}\right)^2 = 1 \tag{5}$$

Third Failure Mode —— failure is caused by yielding in the top bar, in the bottom bar and in the transverse reinforcement, all on the side where shear flows due to shear and torsion are additive:

$$\left(\frac{V}{V_0}\right)^2 + \left(\frac{T}{T_0}\right)^2 + \left(\frac{VT}{V_0 T_0}\right) 2\sqrt{\frac{2d_v}{p_0}} = \frac{(1+R)}{2R} \tag{6}$$

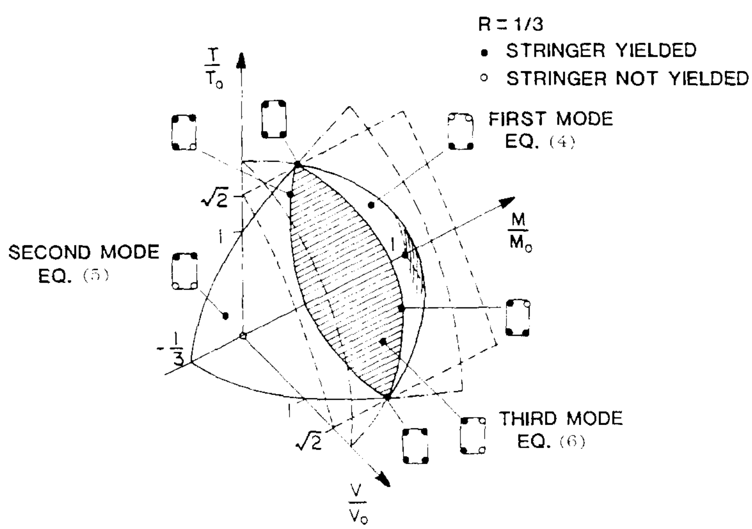


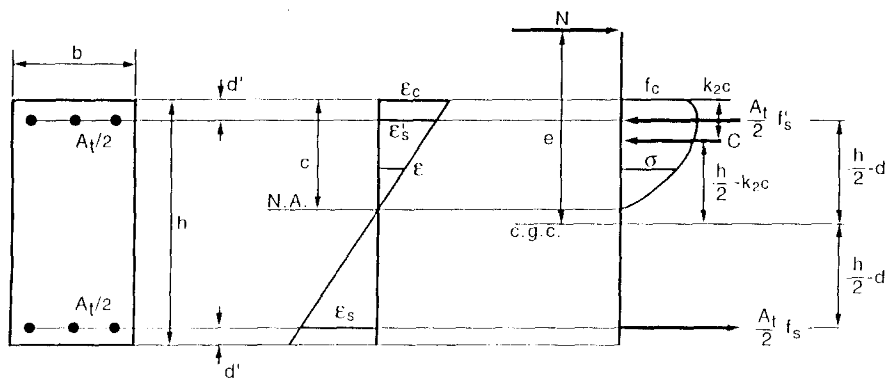
Fig. 1 Interaction Surface for Bending, Shear and Torsion

where $R = N_{tly}/N_{bly}$. The above three equations provides the three portions of an interaction surfaces as shown in Fig. 1.

Because the equilibrium (plasticity) truss model is conceptionally very clear and elegant, it is particularly convenient to express the interaction of bending, shear and torsion. However, in view of the fact that strain compatibility condition is not taken into account, this model can not predict the bending, shear or torsional deformation of a member.

5. Bernoulli compatibility truss model

Bernoulli compatibility truss model is applicable to bending of a member with or without axial load. Fig. 2 shows a symmetrically reinforced column section subjected to a bending moment and an axial compression load. To find the stresses and strains in the concrete and steel, we can utilize the two equilibrium equations for the parallel force system on the column, as well as the two compatibility equations from the Bernoulli's hypothesis. Bernoulli compatibility condition is based on a constant 90° crack angle, which is shown by tests to be correct. The stress–strain relationships of both the concrete and the steel are assumed to be nonlinear.



(a) Cross Section (b) Strain Diagram (c) Stress Diagram

Fig. 2 Column Section Subjected to Bending and Axial Load

Equilibrium equations

Force equil. $$N = C + \frac{A_t}{2}(f_s' - f_s) \tag{7}$$

Moment equil. $$M = Ne = C\left(\frac{h}{2} - k_2c\right) + \frac{A_t}{2}(f_s' + f_s)\left(\frac{h}{2} - d'\right) \tag{8}$$

Compatibility equations

Tension steel $$\frac{\varepsilon_c}{\varepsilon_c + \varepsilon_s} = \frac{c}{h - d'} \tag{9}$$

Compression steel $$\frac{\varepsilon_s'}{\varepsilon_c} = \frac{c - d'}{c} \tag{10}$$

Constitutive laws

Concrete	$f_c = f_1(\varepsilon_c)$	(11)
Tension steel	$f_s = f_2(\varepsilon_s)$	(12)
Compression steel	$f'_s = f_3(\varepsilon'_s)$	(13)

where

A_t = total cross–sectional area of steel in column,
b = width of cross section,
c = depth from neutral axis to extreme compression fiber,
d' = distance from center of top or bottom steel to adjacent edge of cross section,
e = eccentricity of compression force N, measured from centroidal axis of cross section,
f_c = stress in concrete at extreme compression fiber,
f_s = stress in tension steel,
f'_s = stress in compression steel,
h = total height of cross section,
M = bending moment $= Ne$,
N = axial compression force,
ε_c = strain in concrete at extreme compression fiber,
ε_s = strain in tensile steel,
ε'_s = strain in compression steel.

The nonlinear functions f_1, f_2 and f_3 are determined from the uniaxial tests. The resultant of concrete compressive stress C and its location, represented by the coefficient k_2, can be calculated by integrating $\sigma = f_1(\varepsilon)$ according to Eq. (11):

$$C = \frac{bc}{\varepsilon_c}\int_0^{\varepsilon_c} \sigma \, d\varepsilon = f_4(\varepsilon_c, c) \tag{14}$$

$$k_2 = 1 - \frac{1}{\varepsilon_c}\frac{\int_0^{\varepsilon_c} \sigma \, \varepsilon \, d\varepsilon}{\int_0^{\varepsilon_c} \sigma \, d\varepsilon} = f_5(\varepsilon_c) \tag{15}$$

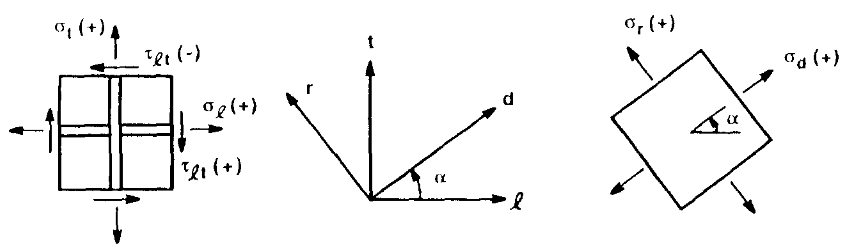
It can be seen that C is a function of ε_c and c, and k_2 is a function of ε_c only. The calculations of C and k_2 have been simplified in various code provisions.

As indicated in Fig. 2 this problem involves a total of 15 variables (N, b, h, d', A_t, f_c, f_s, f'_s, ε_c, ε_s, ε'_s, e, C, k_2 and c) and nine available equations (Eqs. 7 to 15). In the case of analysis, the first five variables representing the action and the cross sectional dimensions (N, b, h, d', A_t) are given. If one strain variable, usually ε_c, is selected, then the last nine unknown variable (f_c, f_s, f'_s, ε_s, ε'_s, e, C, k_2, c) can be solved by the nine equations. A series of solutions for an increasing sequence of selected ε_c values allow us to trace the whole bending load–deformation history.

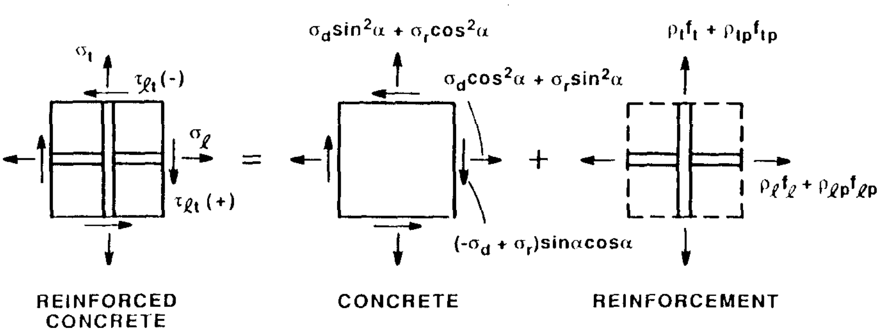
The generic trial–and–error procedures can be used to solve these nine equations. First, select a value of ε_c and assume a value of c, then the two steel strains, ε_s and ε'_s can be calculated from the two compatibility equations (Eqs. 9 and 10). Second, inserting the strains ε_c, ε_s, ε_s' and the depth c into the three stress–strain equations (Eqs. 11, 12 and 13) gives the three stresses f_c. f_s and f_s', as well as the resultant C and the coefficient k_2 (Eqs. 14 and 15). Third, substituting the stresses f_s, f_s' and the resultant C into the force equilibrium equations (Eqs. 7), the depth of the neutral axis c can be solved. If c is the same as assumed, we have a solution. If not, assume another value of c and repeat the cycle. The cycles are repeated until the calculated c is sufficiently close to the assumed c and a solution is obtained. Fourth, substituting the final values of f_s, f_s', C and k_2 into the moment equilibrium equation (Eq. 8), the final eccentricity e is calculated.

6. Mohr compatibility truss model

Mohr compatibility truss model is applicable to reinforced concrete membrane elements subjected to shear and normal stresses as shown in Fig. 3(a). This theory is based on the equilibrium and compatibility conditions of a membrane element, assumed to behave elastically at the service load stage. Using the concept of trasformation of stresses and strains in the membrane element and assuming the superposition of concrete stresses and steel stresses as shown in Fig. 3(b), the three equilibrium equations and the three compatibility equations can be derived (Hsu 1988, 1993) as follows:



(a) Definitions of stresses and coordinate system



(b) Superposition of concrete stresses and steel stresses

Fig. 3 Reinforced Concrete Membrane Elements Subjected to Shear and Normal Stresses

Equilibrium equations

$$\sigma_l = \sigma_d \cos^2\alpha + \sigma_r \sin^2\alpha + \rho_l f_l + \rho_{lp} f_{lp} \tag{16}$$

$$\sigma_t = \sigma_d \sin^2\alpha + \sigma_r \cos^2\alpha + \rho_t f_t + \rho_{tp} f_{tp} \tag{17}$$

$$\tau_{lt} = (-\sigma_d + \sigma_r) \sin\alpha \cos\alpha \tag{18}$$

Compatibility equations

$$\varepsilon_l = \varepsilon_d \cos^2\alpha + \varepsilon_r \sin^2\alpha \tag{19}$$

$$\varepsilon_t = \varepsilon_d \sin^2\alpha + \varepsilon_r \cos^2\alpha \tag{20}$$

$$\gamma_{lt} = 2(-\varepsilon_d + \varepsilon_r)\sin\alpha \cos\alpha \tag{21}$$

where

σ_l, σ_t = normal stresses in the l and t directions, respectively (positive for tension),
τ_{lt} = shear stress in the l–t coordinate (positive as shown in Fig. 3),
σ_d, σ_r = principal stresses in the d and r directions, respectively (positive for tension),
α = angle of inclination of the d–axis with respect to l=axis,
ρ_l, ρ_t = reinforcement ratios in the l and t directions, respectively,
f_l, f_t = steel stresses in the l and t directions, respectively,
ρ_{lp}, ρ_{tp} = prestressed steel ratio in the l and t directions, respectively,
f_{lp}, f_{tp} = prestressed steel stresses in the l and t directions, respectively,
$\varepsilon_l, \varepsilon_t$ = average normal strains in the l and t directions, respectively (positive for tension),
γ_{lt} = average shear strains in l–t coordinate (positive as shown in Fig. 3 for τ_{lt}),
$\varepsilon_d, \varepsilon_r$ = average principal strains in the d and r directions, respectively, (positive for tension).

The solution of the above six equations, (16) to (21), requires six stress–strain relationships for materials: one relates σ_d and ε_d for concrete in the d–direction (principal compression direction). One relates σ_r and ε_r in the r–direction (principal tension directon). Two for mild steel (f_l vs ε_l and f_t vs ε_t) and two for prestressed steel (f_{lp} vs ε_l and f_{tp} vs ε_t).

For the service load stage, the tensile strength of concrete may be neglected ($\sigma_r = 0$), while Hooke's laws are assumed for the concrete in compression, for the mild steel and for the prestressing steel. These simple linear stress–strain relationships are:

Constitutive laws

$$\sigma_d = E_c \varepsilon_d \tag{22}$$

$$f_l = E_s \varepsilon_l \tag{23}$$

$$f_t = E_s \varepsilon_t \tag{24}$$

$$f_{lp} = E_s (\varepsilon_{dec} + \varepsilon_l) \tag{25}$$

$$f_{tp} = E_s (\varepsilon_{dec} + \varepsilon_t) \tag{26}$$

where

E_c, E_S=Modulus of elasticity of concrete and steel, respectively.

ε_{dec} = strain in prestressing steel at decompression of concrete, a given constant representing the intensity of prestressing.

The solution algorithm of Eqs. 16 to 26 is quite straightforward (Hsu 1993), but the solution is applicable only to the service load stage. Although this model could be extended to torsion, it is seldom necessary, because torsion is best treated by the softened truss model.

7. Softened truss model

7.1. Membrane elements

The softened truss model is derived for membrane elements and torsion. The theory is based on the same six equilibrium and compatibility equations, Eqs. 16 to 21, for membrane elements. In the softened truss model theory, however, the constitutive laws of materials are improved so that the theory is applicable not only to the service load stage, but also to the ultimate load stage. In fact, it can describe the entire load–deformation history of a member. The improved constitutive laws include: (1) A softened stress–strain relationship for concrete in compression. The softening effect is represented by a softening coefficient, ζ, which is found from biaxial tests to be a function of the strains in the r–directions, ε_r . (2) A stress–strain relationship for concrete in tension. (3) An average stress–strain relationship for steel bars stiffened by concrete. The suggested bilinear stress–strain curve is quite different from the elastic–perfectly plastic stress–strain curve obtained from a bare reinforcing bar. First, the yield stress is lowered; and second, the yield plateau is replaced by a sloped post–yield stress–strain curve; and (4) A stress–strain relationship for prestressed strands, utilizing the Richard–Abbott expression. These improved constitutive laws are suggested as follows (Belarbi and Hsu 1991, Pang and Hsu 1992):

Constitutive laws

Concrete in compression

$$\sigma_d = \zeta f_c'\left[2\left(\frac{\varepsilon_d}{\zeta\varepsilon_0}\right) - \left(\frac{\varepsilon_d}{\zeta\varepsilon_0}\right)^2 \right] \qquad \varepsilon_d/\zeta\varepsilon_0 \leq 1 \tag{27a}$$

$$\sigma_d = \zeta f_c'\left[1 - \left(\frac{\varepsilon_d/\zeta\varepsilon_0 - 1}{2/\zeta - 1}\right)^2 \right] \qquad \varepsilon_d/\zeta\varepsilon_0 > 1 \tag{27b}$$

$$\zeta = \frac{0.9}{\sqrt{1+600\varepsilon_r}} \tag{28}$$

where

f_c' = maximum compressive strength of standard 6 in. by 12 in. concrete cylinder,

ε_o = concrete strain at maximum compressive strength, taken as 0.002,

ζ = Softening coefficient.

Concrete in tension

$$\sigma_r = E_c \varepsilon_r \qquad \varepsilon_r \leq 0.00008 \tag{29a}$$

$$\sigma_r = f_{cr}\left(\frac{0.00008}{\varepsilon_r}\right)^{0.4} \qquad \varepsilon_r > 0.00008 \tag{29b}$$

where
E_c = modulus of elasticity of concrete, taken as 47,000 $\sqrt{f'_c}$ (f'_c and $\sqrt{f'_c}$ are in *psi*).
f_{cr} = cracking stress of concrete, taken as 3.75 $\sqrt{f'_c}$ (f'_c and $\sqrt{f'_c}$ are in *psi*).

Mild steel

$$f_s = E_s \varepsilon_s \qquad f_s \leq f'_y \qquad \text{(30a) and (31a)}$$

$$f_s = \left(1 - \frac{2-\alpha_2/45^\circ}{\rho}\right)[(0.91-2B)f_y + (0.02+0.25B)\,E_s \varepsilon_s] \qquad f_s > f'_y \qquad \text{(30b) and (31b)}$$

where
B = a parameter defined as $\frac{1}{\rho}\left(\frac{f_{cr}}{f_y}\right)^{1.5}$,
E_s = modulus of elasticity of steel bars, taken as 29,000 *ksi*,
f_s = stress in mild steel, f_s becomes f_l or f_t when applied to longitudinal steel or transverse steel, respectively,
f_y = yield strength of steel reinforcement,
f'_y = $(0.93 - 2B)f_y$,
α_2 = angle between the applied principal compression stress (2–direction) and the longitudinal rebars (l–direction),
ε_s = strain in the mild steel. ε_s becomes ε_l or ε_t, when applied to the longitudinal and transverse steel, respectively,
ρ = percentage of steel reinforcement.

Prestressing steel

$$f_p = E_{ps}(\varepsilon_{dec} + \varepsilon_s) \qquad f_p \leq 0.7\, f_{pu} \qquad \text{(32a) and (33a)}$$

$$f_p = \frac{E'_{ps}(\varepsilon_{dec} + \varepsilon_s)}{\left[1 + \left\{\frac{E'_{ps}(\varepsilon_{dec} + \varepsilon_s)}{f_{pu}}\right\}^m\right]^{\frac{1}{m}}} \qquad f_p > 0.7\, f_{pu} \qquad \text{(32b) and (33b)}$$

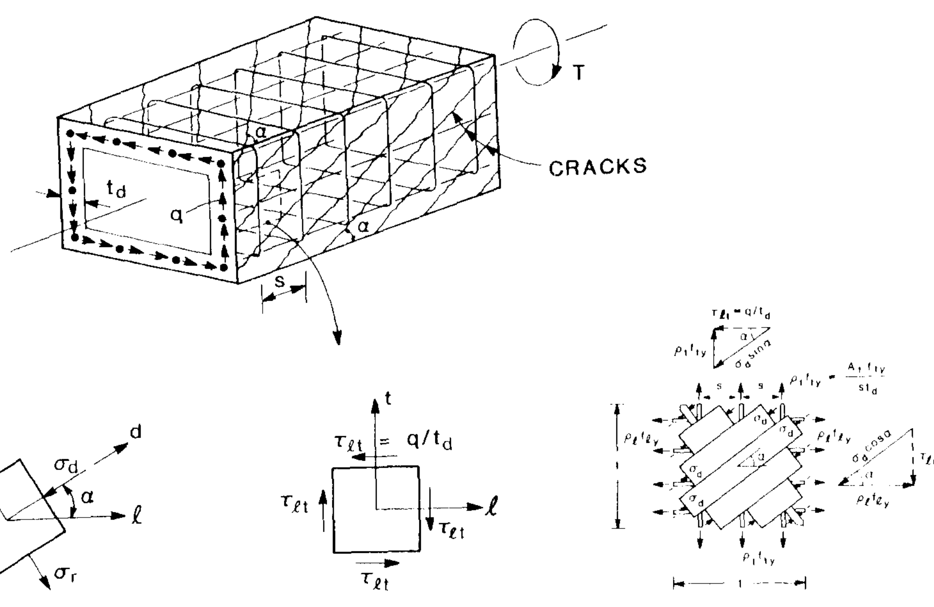
where
E_{ps} = modulus of elasticity of prestressed steel, take as 29,000 *ksi*,
E'_{ps} = tangential modulus of Ramberg–Osgood curve at zero load (taken as 31,060 *ksi*),
f_p = stress in prestressing steel. f_p becomes f_{lp} or f_{tp} when applied to the longitudinal and transverse steel, respectively,
f_{pu} = ultimate strength of prestressing steel,
m = shape parameter (taken as 4 for 250 *ksi* and 270 *ksi* prestressing strands),
ε_{dec} = strain in prestressing steel at decompression of concrete, usually taken as 0.005 for fully prestressed strands.

The 13 governing equations, 16 to 21 and 27 to 33, contain 16 unknown variables. These unknown variables include 9 stresses(σ_l, σ_t, τ_{lt}, σ_d, σ_r, f_l, f_t, f_{lp}, f_{tp}) and 5 strains (ε_l, ε_t, γ_{lt}, ε_d, ε_r), as well as the angle α and the material coefficients ζ. If three unknown variables are given (usually the two applied stresses, σ_l, σ_t and a selected strain ε_d), then the remaining 13 unknown variables can be solved by the 13 equations. An efficient algorithm to solve this set of 13 equations has been developed (Hsu 1991a, 1993) and the shear load–deformation relationship can be traced.

7.2. Torsion

The space truss model proposed by Rausch had captured the basic action of torsion as shown in Fig. 4. The torsional resistance of a reinforced concrete member is contributed by the circulatory shear flow in the outer ring zone of a cross section. An element in the shear flow zone should satisfy the three equilibrium equations (Eqs. 16 to 18) and the three compatibility equations (Eqs. 19 to 21). In addition to these six equations, one equation is required for the equilibrium of the whole cross section and four equations are needed for the compatibility of the whole member.

(a) General View



(b) Shear Flow Zone Element in $d-r$ Coordinate

(c) Shear Flow Zone Element in $l-t$ Coordinate

(d) Truss Model of Element in Shear Flow Zone

Fig. 4 Reinforced Concrete Member in Torsion

Additional equilibrium equation

The equilibrium of internal and external moment provides Bredt's relationship between the torsional moment and the internal shear stresses (Hsu 1984):

$$T = \tau_{lt}(2A_0 t_d) \tag{34}$$

where
T = torsional moment,
A_0 = lever arm area, which is the cross–sectional area within the centerline of the shear flow,
t_d = thickness of shear flow zone in a cross section, which is also the depth of the compression zone in the diagonal concrete struts.

Additional compatibility equations

Twisting of a member produces two types of compatibility conditions. The first is Bredt's compatibility equations relating the shear strain in the elements of the shear flow zone (γ_{lt}) to the angle of twist of the member (θ):

$$\theta = \gamma_{lt}\left(\frac{p_0}{2A_0}\right) \tag{35}$$

where p_0 is the periphery of the center line of shear flow.

The second compatibility condition is the warping of the element in the shear flow zone, resulting in the bending of the concrete struts. The angle of twist (θ) is then related to the curvature of the diagonal concrete strus (Ψ) (Hsu 1984, 1993) by:

$$\Psi = \theta \ \sin 2\alpha \tag{36}$$

Assuming that Bernoulli's linear strain distribution is applicable to the bending of the concrete struts, the curvature (Ψ) is related to the thickness of the shear flow zone (t_d) and to the strains in the concrete struts (ε_{ds} and ε_d) (Hsu 1984) by:

$$\varepsilon_{ds} = -\Psi t_d \tag{37}$$

$$\varepsilon_d = \frac{\varepsilon_{ds}}{2} \tag{38}$$

where
ε_{ds} = concrete compressive strain at the surface of diagonal concrete struts,
ε_d = average concrete compressive strain at the mid–depth of the shear flow zone.

Constitutive laws

As in shear, the softened constitutive laws of concrete in compression (Eqs. 27 to 28) should be applicable to torsion. Since the concrete struts are subjected to bending in addition to axial load, the average stress in the concrete struts (σ_d) is given by (Hsu 1984, 1988, 1993):

$$\sigma_d = k_1 \zeta f_c' \tag{39}$$

$$k_1 = \frac{1}{\zeta f_c' \varepsilon_{ds}} \int_0^{\varepsilon_{ds}} \sigma(\varepsilon, \zeta) d\varepsilon = f_6(\varepsilon_{ds}, \zeta) \tag{40}$$

where

f_c' = cylinder compressive strength of concrete,
k_1 = ratio of average stress to peak stress in the compression stress block,
ζ = softened coefficient for peak stress, given in Eq. 28

As shown in Eq. (40) the coefficient k_1 is obtained from the integration of Eq. 27. It is a function of ε_{ds} and ζ, and is given in Table 8.1. (Hsu 1993). For simplicity the center line of shear flow is assumed to be located at the mid–depth of the thickness of the shear flow zone (t_d). That is to say, the distance from the center line of shear flow to the extreme compression fiber is $0.5t_d$, or $k_2 = 0.5$.

The torsional load–deformation relationships of a reinforced concrete member can be obtained by solving 19 equations, Eqs. 16 to 21 and 28 to 40. Since 22 variables (σ_l, σ_t, τ_{lt}, σ_d, σ_r, f_l, f_t, f_{lp}, f_{tp}, T, ε_l, ε_t, γ_{lt}, ε_d, ε_r, α, θ, Ψ, t_d, ε_{ds}, ζ and k_1), are involved in the torsion problem, three variables must be given. In pure torsion, $\sigma_l = \sigma_t = 0$, and ε_d can be selected. Solution of the 19 equations for a series of ε_d values will give the entire torsional load–deformation history. An efficient algorithm for the solution of the 19 equations had also been developed (Hsu 1991(b), 1993).

8. Concluding remarks

The unified theory of reinforced concrete has been summarized in a systematic and concise manner. The struts–and–ties model is shown to be a powerful tool in guiding the design of local regions. The four truss models could be used for the design and analysis of the main regions. Because of the simplicity and clarity of concept, the equilibrium (plasticity) truss model could be utilized for the preliminary design of a member subjected to any combinations of the four actions–bending, axial load, shear and torsion. This truss model can predict the ultimate strength of a structure, but can not provide any information on the deformations.

The last three truss models are capable of predicting the various deformations because they satisfy not only the stress equilibrium, but also the strain compatibility and the constitutive laws of materials. The behavior of a member subjected to bending and/or axial load can be analyzed by Bernoulli compatibility truss model. The behavior of a structure subjected to shear and torsion can be predicted by Mohr compatibility truss model up to the service load stage, or by the softened truss model throughout the whole loading history.

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