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Transient wave propagation in piezoelectric hollow spheres subjected to thermal shock and electric excitation

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Abstract. An analytical method is presented to solve the problem of transient wave propagation in a transversely isotropic piezoelectric hollow sphere subjected to thermal shock and electric excitation. Exact expressions for the transient responses of displacements, stresses, electric displacement and electric potentials in the piezoelectric hollow sphere are obtained by means of Hankel transform, Laplace transform, and inverse transforms. Using Hermite non-linear interpolation method solves Volterra integral equation of the second kind involved in the exact expression, which is caused by interaction between thermo-elastic field and thermo-electric field. Thus, an analytical solution for the problem of transient wave propagation in a transversely isotropic piezoelectric hollow sphere is obtained. Finally, some numerical results are carried out, and may be used as a reference to solve other transient coupled problems of thermo-electro-elasticity.

Key words: thermo-electro-elastic; wave propagation; piezoelectric hollow sphere; electric excitation; thermal shock.

1. Introduction

In recent years, the applications for a transversely isotropic hollow sphere have continuously increased in some engineering areas, including aerospace, offshore and submarine structures, chemical vessel and civil engineering structures. An exact solution of spherically isotropic shells subjected to both internal and external uniform pressures was introduced by Lave (1927) and Lekhniskii (1981). Hu (1954) first initiated use a separation method and presented a general theory of elasticity for an spherically isotropic medium. Many subsequently important analyses were inspired based on Hu's elegant investigations on some static problems such as a concentrated force in an infinite medium, stress concentration due to a spherical cavity, and a steadily rotating shell. Sternberg and Chakravorty (1959) obtained an exact closed-form solution for the dynamic problem of a sudden temperature change at the surface of a spherical cavity in an infinite solid. Hata (1991, 1993, 1997) obtained the dynamic thermal stress responses in a uniformly heated isotropic spherical shell and solid sphere, as well as transversely isotropic solid sphere by using the ray theory. Wang

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(2000) discussed the thermal stress concentration in an isotropic solid sphere. Recently, Wang *et al*. (2001) obtained dynamic thermal stress in a transversely isotropic hollow sphere. Ding *et al.* (2003) investigated on stress-focusing in a uniformly heated solid sphere. However, investigations on thermo-electro-elastic interaction in piezoelectric structures subjected to thermal shock and transient electric excitation have been a few.

This paper presents an analytical method for thermo-electro-elastic interaction in a transversely isotropic piezoelectric hollow sphere subjected to thermal shock and transient electric excitation. The thermo-electro-elastic dynamic equation of the transversely isotropic piezoelectric hollow sphere may be decomposed into a quasi-static homogeneous equation with inhomogeneous boundary conditions and an inhomogeneous dynamic equation with homogeneous boundary conditions. Firstly, using the method described by Lekhniskii (1981), we can solve the quasi-static question. Secondly, the solution to the inhomogeneous dynamic question which satisfies homogeneous boundary conditions is obtained by utilizing the finite Hankel transforms (Cinelli 1965), and the Laplace transforms. Then, using Hermite non-linear interpolation method solves Volterra integral equation of the second kind caused by interaction between thermo-elastic field and electric field. Thus, the exact expressions for the transient responses of displacements, stresses, electric displacement and electric potentials in the transversely isotropic piezoelectric hollow sphere are obtained.

2. Formation of the problem

A spherical coordinate system (r, θ, φ) with the origin identical to the center of the sphere is used for a spherically symmetric problem where a transversely isotropic piezoelectric hollow sphere with internal radius a and external radius b as shown in Fig. 1, is subjected to a rapid change in temperature $T(r, t)$, so that the strain-displacement relations are expressed as

$$\varepsilon_{rr} = \frac{\partial u_r(r,t)}{\partial r}, \quad \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{u_r(r,t)}{r}, \quad \varepsilon_{r\theta} = \varepsilon_{\theta\varphi} = \varepsilon_{r\varphi} = 0 \tag{1}$$

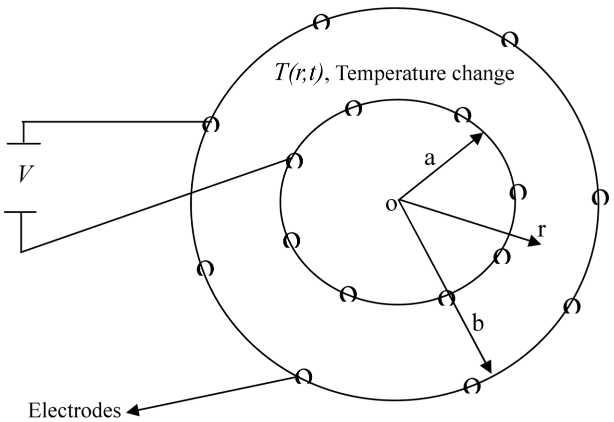

Fig. 1 A geometric graph of piezoelectric hollow sphere subjected to thermal shock $T(r, t)$ and electric excitation $\phi(r, t)$

where $\varepsilon_{ij}(i, j = r, \theta, \varphi)$ are strain components, and $u_r(r, t)$ expresses a radial displacement. The constitutive relations of a spherically transversely isotropic pyroelectric medium are expressed as (Sinha 1962, Chen and Shioya 2001)

$$\sigma_{rr} = c_{11}\varepsilon_{rr} + 2c_{12}\varepsilon_{\theta\theta} + e_{11}\frac{\partial\phi}{\partial r} - \lambda_1 T(r, t) \tag{2a}$$

$$\sigma_{\theta\theta} = c_{12}\varepsilon_{rr} + (c_{22} + c_{23})\varepsilon_{\theta\theta} + e_{12}\frac{\partial\phi}{\partial r} - \lambda_2 T(r, t) \tag{2b}$$

$$D_{rr} = e_{11}\varepsilon_{rr} + 2e_{12}\varepsilon_{\theta\theta} - \beta_{11}\frac{\partial\phi}{\partial r} + p_{11}T(r, t) \tag{2c}$$

$$\lambda_1 = c_{11}\alpha_r + 2c_{12}\alpha_\theta, \quad \lambda_2 = c_{12}\alpha_r + (c_{22} + c_{23})\alpha_\theta \tag{2d}$$

where c_{ij}, e_{ij}, α_i, β_{ij}, and p_{11} are elastic constants, piezoelectric constants, thermal expansion coefficients, dielectric constants, and pyroelectric coefficients, respectively. σ_{ii} and D_{rr} are the component of stress and radial electric displacement, respectively.

In absence of free charge density, the charge equation of electrostatics is expressed as Dunn and Taya (1994)

$$\frac{\partial D_{rr}(r, \tau)}{\partial r} + \frac{2D_{rr}}{r} = 0 \tag{3}$$

Solving Eq. (3), yields

$$D_{rr}(r, t) = \frac{1}{r^2}d(t) \tag{4}$$

where $d(t)$ is an undetermined function.

Substituting Eq. (4) into Eq. (2c), gives

$$\frac{\partial\phi}{\partial r} = \frac{e_{11}}{\beta_{11}}\frac{\partial u_r}{\partial r} + \frac{2e_{12}}{\beta_{11}}\frac{u_r}{r} - \frac{1}{\beta_{11}}\frac{d(t)}{r^2} + \frac{p_{11}}{\beta_{11}}T(r, t) \tag{5}$$

Substituting Eq. (5) into Eqs.(2a) and (2b), yields

$$\sigma_{rr} = \left(c_{11} + \frac{e_{11}^2}{\beta_{11}}\right)\frac{\partial u_r}{\partial r} + 2\left(c_{12} + \frac{e_{11}e_{12}}{\beta_{11}}\right)\frac{u_r}{r} - \frac{e_{11}}{\beta_{11}}\frac{d(t)}{r^2} - T_{1p}(r, t) \tag{6a}$$

$$\sigma_{\theta\theta} = \left(c_{12} + \frac{e_{11}e_{12}}{\beta_{11}}\right)\frac{\partial u_r}{\partial r} + \left(c_{22} + c_{23} + \frac{2e_{12}^2}{\beta_{11}}\right)\frac{u_r}{r} - \frac{e_{12}}{\beta_{11}}\frac{d(t)}{r^2} - T_{2p}(r, t) \tag{6b}$$

Where

$$T_{ip}(r, t) = \left(\lambda_i - \frac{e_{1i}p_{11}}{\beta_{11}}\right)T(r, t) \quad (i = 1, 2) \tag{7a,b}$$

Substituting Eq. (6) into the motion equation of a spherically symmetric problem (Lekhniskii 1981), the basic displacement equation of thermo-electro-elastic motion of a transversely isotropic piezoelectric hollow sphere is expressed as

$$\frac{\partial^2 u_r(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial u_r(r,t)}{\partial r} - \frac{H^2 u_r(r,t)}{r^2} = \frac{1}{C_L^2}\frac{\partial^2 u_r(r,t)}{\partial t^2} + I\frac{d(t)}{r^2} + g(r,t) \tag{8a}$$

where

$$H^2 = \frac{2(c_{22}+c_{23})\beta_{11} + 4e_{12}^2 - 2c_{12}\beta_{11} - 2e_{11}e_{12}}{m\beta_{11}}, \quad I = -\frac{2e_{12}}{m\beta_{11}}, \quad m = c_{11} + \frac{e_{11}^2}{\beta_{11}}$$

$$C_L^2 = \frac{m}{\rho}, \quad g(r,t) = \frac{1}{m}\left[\lambda_1 - \frac{e_{11}p_{11}}{\beta_{11}}\right]\frac{\partial T}{\partial r} + 2\left(\frac{e_{12}-e_{11}}{\beta_{11}}p_{11} + \lambda_1 - \lambda_2\right)\frac{T}{r} \tag{8b}$$

Boundary conditions of stress and electric are expressed as

$$\sigma(r,t)_{r=j} = \left.\frac{\partial u_r(r,t)}{\partial r} + h\frac{u_r(r,t)}{r}\right]_{r=j} = \theta_j(t) \quad (j = a, b) \tag{9a,b}$$

$$\phi(a,t) = \phi_a(t) \qquad \phi(b,t) = \phi_b(t) \tag{9c,d}$$

where

$$h = \frac{2(c_{12}\beta_{11} + e_{11}e_{12})}{m\beta_{11}}, \quad \theta_j(\tau) = \frac{1}{m}\left[\frac{e_{11}}{\beta_{11}j^2}d(t) + T_{1p}(j,t)\right] \quad (j = a, b) \tag{9e,f}$$

Initial conditions are

$$[u_r(r,t)]_{t=0} = u_0(r) \qquad \left[\frac{\partial u_r(r,t)}{\partial t}\right]_{t=0} = v_0(r) \tag{10a,b}$$

3. Solution of the problem

The general solution of the basic displacement Eq. (8a) may be decomposed into the form as follows

$$u_r(r,t) = u_q(r,t) + u_d(r,t) \tag{11}$$

where $u_q(r,t)$ and $u_d(r,t)$ are, respectively, defined as the quasi-static term which satisfies inhomogeneous boundary conditions and dynamic term which satisfies homogeneous boundary conditions.

The quasi-static term $u_q(r,t)$ satisfies Eq. (12) as follows

$$\frac{\partial^2 u_q(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial u_q(r,t)}{\partial r} - \frac{H^2}{r^2}u_q(r,t) = I\frac{d(t)}{r^2} + g(r,t) \tag{12a}$$

$$\left[\frac{\partial u_q(r,t)}{\partial r} + h\frac{u_q(r,t)}{r}\right]_{r=j} = \theta_j(t) \qquad (j = a, b) \tag{12b,c}$$

Solving Eq. (12), we have

$$u_q(r,t) = A_1(r,t) + A_2(r)d(t) \tag{13}$$

where

$$A_1(r,t) = g_1(r,t) + L_1L_3r^{n-0.5} + L_2L_4r^{-(n+0.5)} \tag{14a}$$

$$A_2(r) = L_1L_5\left[\frac{b^{-(n+1.5)}}{a^2} - \frac{a^{-(n+1.5)}}{b^2}\right]r^{n-0.5} + L_2L_5\left[\frac{b^{(n-0.5)}}{a^2} - \frac{a^{(n-0.5)}}{b^2}\right]r^{-(n+0.5)} - \frac{I}{H^2r} \tag{14b}$$

$$n = \sqrt{0.25+H^2}, \quad g_1(r,t) = r^{-n-0.5}\int_a^r r^{2n-1}\left[\int_a^r r^{-n+1.5}g(r,t)dr\right]dr$$

$$g_2(r,t) = g_1'(r,t) + \frac{h}{r}g(r,t), \quad L_1 = \frac{1}{(n+h-0.5)[a^{n-0.5}b^{-(n+1.5)} - b^{n-0.5}a^{-(n+1.5)}]}$$

$$L_2 = \frac{1}{(n-h+0.5)[a^{n-0.5}b^{-(n+1.5)} - b^{n-0.5}a^{-(n+1.5)}]}, \quad L_5 = \frac{e_{11}}{mg_{11}} + \frac{(h-1)I}{H^2}$$

$$L_3 = \frac{1}{m}[T_{1p}(a,t)b^{-(n+1.5)} - T_{1p}(b,t)a^{-(n+1.5)}] - [g_2(a,t)b^{-(n+1.5)} - g_2(b,t)a^{-(n+1.5)}]$$

$$L_4 = \frac{1}{m}[T_{1p}(a,t)b^{n-0.5} - T_{1p}(b,t)a^{n-0.5}] - [g_2(a,t)b^{n-0.5} - g_2(b,t)a^{n-0.5}] \tag{14c-j}$$

Substituting Eq. (11) into Eq. (8) and utilizing Eq. (12) provides an inhomogeneous dynamic equation with homogeneous boundary conditions and the corresponding initial conditions for $u_d(r, t)$

$$\frac{\partial^2 u_d(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial u_d(r,t)}{\partial r} - \frac{H^2}{r^2}u_d(r,t) = \frac{1}{C_L^2}\left[\frac{\partial^2 u_d(r,t)}{\partial t^2} + \frac{\partial^2 u_q(r,t)}{\partial t^2}\right] \tag{15a}$$

$$\left[\frac{\partial u_d(r,t)}{\partial r} + h\frac{u_d(r,t)}{r}\right]_{r=j} = 0 \qquad (j = a, b) \tag{15b,c}$$

$$u_d(r,0) + u_q(r,0) = u_0 \qquad \frac{\partial u_d(r,0)}{\partial t} + \frac{\partial u_q(r,0)}{\partial t} = v_0 \tag{15d,e}$$

In order to transform Eq. (15a) into a normal Bessel equation, a new function $f(r, t)$ is introduced by

$$u_d(r,t) = r^{-0.5}f(r,t) \tag{16}$$

Substituting Eq. (16) into Eq. (15), yields

$$\frac{\partial^2 f(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial f(r,t)}{\partial r} - \frac{R^2}{r^2}f(r,t) = \frac{1}{C_L^2}\left[\frac{\partial^2 f(r,t)}{\partial t^2} + \frac{\partial^2 u_{q1}(r,t)}{\partial t^2}\right] \tag{17a}$$

$$\frac{\partial f(a,t)}{\partial r} + h_a f(a,t) = 0, \quad \frac{\partial f(b,t)}{\partial r} + h_b f(b,t) = 0 \tag{17b,c}$$

$$f(r, 0) + u_{q1}(r, 0) = u_1 \qquad \frac{\partial f(r, 0)}{\partial t} + \frac{\partial u_{q1}(r, 0)}{\partial t} = v_1 \tag{17d,e}$$

where

$$u_{q1}(r, t) = B_1(r, t) + B_2(r)d(t), \quad B_1(r, t) = r^{0.5}A_1(r, t), \quad B_2(r) = r^{0.5}A_2(r) \tag{18a-c}$$

$$R^2 = 0.25 + H^2, \quad u_1 = r^{0.5}u_0, \quad v_1 = r^{0.5}v_0, \quad h_a = \frac{(h - 0.5)}{a}, \quad h_b = \frac{(h - 0.5)}{b} \tag{18d-h}$$

A finite Hankel transform to $f(r, t)$ is defined as

$$\bar{f}(k_i, t) = H[f(r, t)] = \int_a^b rf(r, t)G_R(k_i r)dr \tag{19}$$

The inverse transform of Eq. (19) is given by

$$f(r, t) = \sum_{k_i} \frac{\bar{f}(k_i, t)}{F(k_i)} G_R(k_i r) \tag{20}$$

where

$$F(k_i) = \int_a^b r[G_R(k_i r)]^2 dr$$
$$= \frac{J_a^2}{J_b^2} \frac{2}{k_i^2 \pi^2}\left\{h_b^2 + k_i^2\left[1 - \left(\frac{R}{k_i b}\right)^2\right]\right\} - \frac{2}{k_i^2 \pi^2}\left\{h_a^2 + k_i^2\left[1 - \left(\frac{R}{k_i a}\right)^2\right]\right\} \tag{21a}$$

$$G_R(k_i r) = J_R(k_i r)Y_a - J_a Y_R(k_i r) \tag{21b}$$

where

$$J_a = k_i J'_R(k_i a) + h_a J_R(k_i a) \qquad J_b = k_i J'_R(k_i b) + h_b J_R(k_i b)$$
$$Y_a = k_i Y'_R(k_i a) + h_a Y_R(k_i a) \qquad Y_b = k_i Y'_R(k_i b) + h_b Y_R(k_i b) \tag{22}$$

$J_R(k_i r)$ and $Y_R(k_i r)$ are, respectively, the first and the second kind of the R^{th}-order Bessel function, where $k_i(i = 1, 2, \dots, n)$ express a series of positive roots for natural eigen-equation as follows:

$$J_a Y_b - J_b Y_a = 0 \tag{23}$$

The natural frequencies are expressed as

$$\omega_i = C_L k_i \tag{24}$$

Applying the finite Hankel transform (19) to Eq. (17a) and utilizing the homogeneous boundary condition (17b,c), we have

$$-k_i^2\bar{f}(k_i,t) = \frac{1}{C_L^2}\left[\frac{\partial^2\bar{f}(k_i,t)}{\partial t^2} + \frac{\partial^2\bar{u}_{q1}(k_i,t)}{\partial t^2}\right] \tag{25}$$

where

$$\bar{u}_{q1}(k_i,t) = H[u_{q1}(r,t)] \tag{26}$$

Applying Laplace transform to the two sides of Eq. (25) and utilizing the initial condition (17d,e), yields

$$\bar{f}^*(k_i,p) = -\bar{u}_{q1}^*(k_i,p) + \frac{\omega_i^2}{(\omega_i^2+p^2)}\bar{u}_{q1}^*(k_i,p) + \frac{p\bar{u}_1(k_i)}{(\omega_i^2+p^2)} + \frac{\bar{v}_1(k_i)}{(\omega_i^2+p^2)} \tag{27}$$

where p is the parameter of Laplace transform, and

$$\bar{u}_1(k_i) = H[\bar{u}_1(r)], \quad \bar{v}_1(k_i) = H[\bar{v}_1(r)]$$

The inverse Laplace transform of Eq. (27) is expressed as

$$\bar{f}(k_i,t) = -\bar{u}_{q1}(k_i,t) + \omega_i[\bar{u}_{q1}(k_i,t)\sin(\omega_i t)] + \bar{u}_1(k_i)\cos(\omega_i t) + \bar{v}_1(k_i)\frac{1}{\omega_i}\sin(\omega_i t) \tag{28}$$

where

$$\bar{u}_{q1}(k_i,t)\sin(\omega_i t) = \int_0^t \bar{u}_{q1}(k_i,\tau)\sin[\omega_i(t-\tau)]d\tau \tag{29}$$

Substituting Eq. (18a) into Eq. (26), yields

$$\bar{u}_{q1}(k_i,t) = \bar{B}_1(k_i,t) + \bar{B}_2(k_i)d(t) \tag{30}$$

where $\bar{B}_1(k_i,t) = H[B_1(r,t)]$, $\bar{B}_2(k_i) = H[B_2(r)]$.

Substituting Eq. (30) into Eq. (29), gives

$$\bar{u}_{q1}(k_i,t)\sin(\omega_i t) = \int_0^t [\bar{B}_1(k_i,\tau) + \bar{B}_2(k_i)d(\tau)]\sin[\omega_i(t-\tau)]d\tau \tag{31}$$

Substituting Eq. (31) into Eq. (28), yields

$$\bar{f}(k_i,t) = I_{1i}(k_i,t) + \bar{B}_2(k_i)I_{2i}(k_i,t) + I_{3i}(k_i,t) \tag{32}$$

where

$$\begin{aligned} I_{1i}(k_i,\tau) &= -\bar{B}_1(k_i,t) + \omega_i\int_0^t \bar{B}_1(k_i,\tau)\sin[\omega_i(t-\tau)]d\tau \\ I_{2i}(k_i,t) &= -d(t) + \omega_i\int_0^t d(\tau)\sin[\omega_i(t-\tau)]d\tau \\ I_{3i}(k_i,t) &= \bar{u}_1(k_i)\cos(\omega_i t) + \bar{v}_1(k_i)\frac{1}{\omega_i}\sin(\omega_i t) \end{aligned} \tag{33}$$

Substituting Eq. (32) into Eq. (20), the dynamic solution for inhomogeneous dynamic Eq. (17) with homogeneous boundary conditions is given by

$$f(r,t) = \sum_{k_i} \frac{G_R(k_i r)}{F(k_i)}[I_{1i}(k_i,t) + \bar{B}_2(k_i)I_{2i}(k_i,t) + I_{3i}(k_i,t)] \tag{34}$$

Thus, from Eqs. (11), (13), (16) and (34), the solution of the basic displacement equation of thermo-electro-elastic motion in the piezoelectric hollow sphere is expressed as

$$u(r,t) = A_1(r,t) + A_2(r)d(t) + \sum_{k_i} \frac{r^{-0.5}G_R(k_i r)}{F(k_i)}[I_{1i}(k_i,t) + \bar{B}_2(k_i)I_{2i}(k_i,t) + I_{3i}(k_i,t)] \tag{35}$$

Noting that in the above expression $d(t)$ still is an unknown function which is relation to the electric displacement. It is necessary to determine $d(t)$ in the following.

Integrating Eq. (5) and utilizing the corresponding electric boundary condition (9c), yields

$$\phi(r,t) = \Phi_1(r,t) + \Phi_2(r)d(t) + \sum_i \Phi_{3i}(r)F_i(t) + \phi_a(t) \tag{36}$$

where

$$\Phi_1(r,t) = \frac{e_{11}}{\beta_{11}}\left[A_{11}(r,t) - A_1(a,t) - \sum_{k_i}\frac{(r^{-0.5}G_R(k_i r) - a^{-0.5}G_R(k_i a))}{F(k_i)}\bar{B}_1(k_i,t)\right]$$
$$+ \frac{2e_{12}}{\beta_{11}}\int_a^r \frac{1}{r}\left[A_1(r,t) - \sum_{k_i}\frac{r^{-0.5}G_R(k_i r)}{F(k_i)}\bar{B}_1(k_i,t)\right]dr + \frac{p_{11}}{\beta_{11}}\int_a^r T(r,t)dr \tag{37a}$$

$$\Phi_2(r) = \frac{e_{11}}{\beta_{11}}\left[A_2(r) - A_2(a) - \sum_{k_i}\frac{(r^{-0.5}G_R(k_i r) - a^{-0.5}G_R(k_i a))}{F(k_i)}\bar{B}_2(k_i)\right]$$
$$+ \frac{2e_{12}}{\beta_{11}}\int_a^r \frac{1}{r}\left[A_2(r,t) - \sum_{k_i}\frac{r^{-0.5}G_R(k_i r)}{F(k_i)}\bar{B}_2(k_i)\right]dr + \frac{1}{\beta_{11}r} \tag{37b}$$

$$\Phi_3(r) = \frac{e_{11}}{\beta_{11}}\frac{(r^{-0.5}G_R(k_i r) - a^{-0.5}G_R(k_i a))}{F(k_i)} + \frac{2e_{12}}{\beta_{11}}\int_a^r \frac{1}{r^{1.5}}\frac{G_R(k_i r)}{F(k_i)}dr \tag{37c}$$

$$F_i(t) = F_{1i}(t) + \bar{B}_2(k_i)\omega_i\int_0^t d(\tau)\sin[\omega_i(t-\tau)]d\tau \tag{37d}$$

$$F_{1i}(t) = \omega_i\int_0^t \bar{B}_1(k_i,t)\sin[\omega_i(t-\tau)]d\tau + \bar{u}_1(k_i)\cos(\omega_i t) + \bar{v}_1(k_i)\frac{1}{\omega_i}\sin(\omega_i t) \tag{37e}$$

Substituting $r = b$ into Eq. (36), yields

$$\phi_b(t) = \Phi_1(b,t) + \Phi_2(b)d(t) + \sum_i \Phi_{3i}(b)F_i(t) + \phi_a(t) \tag{38}$$

Substituting $t = 0$ into Eq. (38), we have

$$d(0) = \frac{\phi_b(0) - \phi_a(0) - \Phi_1(b, t) - \sum_i \Phi_{3i}(b)F_i(0)}{\Phi_2(b)} \tag{39}$$

Substitution of Eq. (37d) into Eq. (38), gives

$$\vartheta(t) = M_1 d(t) + \sum_i M_{2i}\int_0^t d(\tau)\sin[\omega_i(t-\tau)]d\tau \tag{40}$$

where

$$\vartheta(t) = \phi_b(t) - \phi_a(t) - \Phi_1(b, t) - \sum_i \Phi_{3i}(b)F_{1i}(t)$$

$$M_1 = \Phi_2(b), \qquad M_{2i} = \Phi_{3i}(b)\bar{B}_1(k_i)\omega_i \tag{41}$$

Time derivative of Eq. (40) gives

$$\dot{\vartheta}(t) = M_1\dot{d}(t) + \sum_i M_{2i}\omega_i\int_0^t d(\tau)\cos[\omega_i(t-\tau)]d\tau \tag{42}$$

It is seen that Eq. (40) is Volterra integral equation of the second kind (Kress 1989). In the following, Eq. (40) will be solved by using the recursion formula based on non-linear Hermite interpolation function. In order to describe the method solving the integral Eq. (40), the time interval [0, t] is divided into n subintervals and the discrete time points are $t_0 = 0, t_1, t_2, \ldots, t_n$. The interpolation function at the time interval $[t_{j-1}, t_j]$ may be expressed as

$$d(t) = E_j^0(t)d(t_{j-1}) + E_j^1(t)d(t_j) + E_j^2(t)\dot{d}(t_{j-1})E_j^3(t)\dot{d}(t_{j-1}) \qquad (j = 1, 2, \ldots, n) \tag{43}$$

where

$$E_j^0(t) = \left(1+2\frac{t-t_{j-1}}{t_j-t_{j-1}}\right)\left(\frac{t-t_j}{t_j-t_{j-1}}\right)^2, \; E_j^1(t) = \left(1+2\frac{t_j-t}{t_j-t_{j-1}}\right)\left(\frac{t-t_{j-1}}{t_j-t_{j-1}}\right)^2, \; E_j^2(t) = (t-t_{j-1})\left(\frac{t-t_j}{t_j-t_{j-1}}\right)^2$$

$$E_j^3(t) = (t-t_j)\left(\frac{t-t_{j-1}}{t_j-t_{j-1}}\right)^2, \quad \dot{d}(t_j) = (dd(t)/dt)_{t=t_j} \quad (j = 1, 2, \ldots, n) \tag{44}$$

Substituting Eq. (43) into Eqs. (40) and (42), gives

$$\vartheta(t_j) = M_1 d(t_j) + \sum_i M_{2i}\sum_{k=1}^{j}[R_{0ijk}d(t_{k-1}) + R_{1ijk}d(t_k) + R_{2ijk}\dot{d}(t_{k-1}) + R_{3ijk}\dot{d}(t_k)] \tag{45a}$$

$$\dot{\vartheta}(t_j) = M_1\dot{d}(t_j) + \sum_i M_{2i}\omega_i\sum_{k=1}^{j}[S_{0ijk}d(t_{k-1}) + S_{1ijk}d(t_k) + S_{2ijk}\dot{d}(t_{k-1}) + S_{3ijk}\dot{d}(t_k)] \tag{45b}$$

where

$$\begin{aligned} R_{lijk} &= \int_{t_{k-1}}^{t_k} E_k^l(\tau)\sin[\omega_i(t-\tau)]d\tau \\ S_{lijk} &= \int_{t_{k-1}}^{t_k} E_k^l(\tau)\sin[\omega_i(t-\tau)]d\tau \end{aligned} \quad (l = 0, 1, 2, 3;\ k = 1, 2, \ldots, j;\ j = 1, 2, \ldots, n) \tag{46}$$

From Eq. (45), the recursion formula for $d(t_j)$ and $\dot{d}(t_j)$ can be expressed as

$$d(t_j) = \frac{X_{1j}a_{22j} - X_{2j}a_{12j}}{a_{11j}a_{22j} - a_{12j}a_{21j}}, \quad \dot{d}(t_j) = \frac{X_{2j}a_{11j} - X_{1j}a_{21j}}{a_{11j}a_{22j} - a_{12j}a_{21j}} \quad (j = 1, 2, \ldots, n) \tag{47}$$

where

$$\begin{aligned} X_{1j} &= \vartheta(t_j) - \sum_{i=1}^{m} M_{2i} \sum_{k=1}^{j-1} [R_{0ijk}d(t_{k-1}) + R_{1ijk}d(t_k) + R_{2ijk}\dot{d}(t_{k-1}) + R_{3ijk}\dot{d}(t_k)] \\ &\quad - \sum_{i=1}^{m} M_{2i}[R_{0ijk}d(t_{j-1}) + R_{2ijk}\dot{d}(t_{j-1})] \\ X_{2j} &= \dot{\vartheta}(t_j) - \sum_{i=1}^{m} M_{2i}\omega_i \sum_{k=1}^{j-1} [S_{0ijk}d(t_{k-1}) + S_{1ijk}d(t_k) + S_{2ijk}\dot{d}(t_{k-1}) + S_{3ijk}\dot{d}(t_k)] \\ &\quad - \sum_{i=1}^{m} M_{2i}\omega_i[S_{0ijk}d(t_{j-1}) + S_{2ijk}\dot{d}(t_{j-1})] \\ a_{11j} &= M_1 + \sum_{i=1}^{m} M_{2i}R_{1ijj}, \quad a_{12j} = \sum_{i=1}^{m} M_{2i}R_{3ijj} \\ a_{21j} &= \sum_{i=1}^{m} M_{2i}\omega_i S_{1ijj}, \quad a_{22j} = M_1 + \sum_{i=1}^{m} M_{2i}\omega_i S_{3ijj} \end{aligned} \tag{48}$$

Substituting $d(0)$ in Eq. (40) and $\dot{d}(0)$ in Eq. (42) into Eq. (47), we can obtain $d(t_j)$ and $\dot{d}(t_j)$, $(j = 1, 2, \ldots, n)$ step by step, and determine $d(t)$. Substituting $d(t)$ in Eq. (47) into Eq. (35), gives the exact expression of the solution, $u(r, t)$, for the basic equation of thermo-electro-elastic motion in the transversely isotropic piezoelectric hollow sphere. Thus, the corresponding transient stresses $\sigma_{rr}(r, t)$, $\sigma_{\theta\theta}(r, t)$, the transient electric displacement $D_r(r, t)$ and the transient electric potential $\phi(r, t)$ are easily obtained from Eqs. (4) to (6).

4. Numerical results and discussions

Thermo-electro-elastic interaction in a transversely isotropic piezoelectric hollow sphere subjected to thermal shock and electric potential is considered. A transitory temperature change produced by a sudden electric current pulse or by absorption of electromagnetic wave, is typically of a duration much less than 1 μs and may be expressed as

$$T(r,t) = T_0 H(t) \tag{49}$$

Where $H(t)$ expresses the Heaviside function.

In numerical calculations, material constants for the transversely isotropic piezoelectric hollow sphere are taken as $c_{11} = c_{33} = 111.0$ GPa, $c_{22} = 125.6$ GPa, $c_{12} = 77.8$ GPa, $c_{13} = c_{23} = 74.3$ GPa, $e_{11} = 15.1\ (C/\mathrm{m}^2)$, $e_{12} = e_{13} = -5.2\ (C/\mathrm{m}^2)$, $\alpha_1 = \alpha_3 = 2.0\times10^{-5}\ (1/k)$, $\alpha_2 = 2.0\times10^{-6}\ (1/k)$, $\beta_{11} = 5.62\times10^{-9}\ (C^2/\mathrm{Nm}^2)$, and $p_{11} = -2.5\times10^{-5}\ (C^2/\mathrm{m}^2K)$. In all results, the dimensionless time is taken as $\tau = C_L t/b-a$, the dimensionless radial coordinate is taken as, $S = r-a/b-a$, the response time is taken as $\tau = 10$, and the ratio of the external radii to the internal radii is taken as $b/a = 2$.

Example 1. Thermo-electro-elastic interaction in the transversely isotropic piezoelectric hollow sphere, with homogeneous electric boundary conditions, permeated by change thermal field, $T(r,t)$, in Eq. (49), is considered. The homogeneous electric boundary conditions are

$$\phi_a(a,t) = 0, \quad \phi_b(b,t) = 0 \tag{50}$$

$\sigma_i = \dfrac{\sigma_{ii}}{\alpha_r T_0 c_{11}} (i = r, \theta)$, $D_r = \dfrac{D_{rr}}{\alpha_r T_0 \sqrt{c_{11}\beta_{11}}}$ and $\phi^* = \dfrac{\phi}{\alpha_r T_0 \sqrt{c_{11}\beta_{11}}}$ are introduced in Figs. 2-6.

Figs. 2, 3 and Fig. 4 show, respectively, the response histories of radial, and hoop stresses at different radial points. Because of the small wall thickness, the reflected effects of wave between the inner-wall and outer-wall occur. From Fig. 2 and Fig. 3, it is shown that except the radial stresses at the internal and external surfaces of the transversely isotropic piezoelectric hollow sphere satisfy the given zero boundary condition, the stresses at other points oscillate dramatically because of the reflected effect of wave. From Fig. 4 and Fig. 5, it is shown that the peak values of hoop stresses and electric displacement decrease gradually from inner-wall to outer-wall at the identical time τ. The distribution of electric potential ϕ^* in the transversely isotropic piezoelectric hollow sphere subjected to only thermal shock is seen in Fig. 6. The electric potential ϕ^* at the internal and

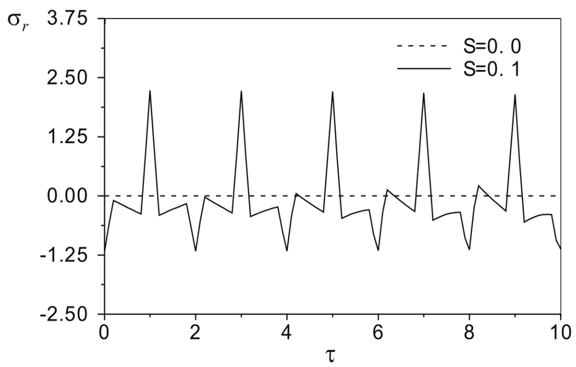

Fig. 2 Response histories of the transient radial stresses σ_r, where $S = \dfrac{r-a}{b-a}$, $\tau = \dfrac{C_L t}{a}$, $\sigma_r = \dfrac{\sigma_{rr}}{\alpha_r T_0 c_{11}}$, and $\phi_b = 0$

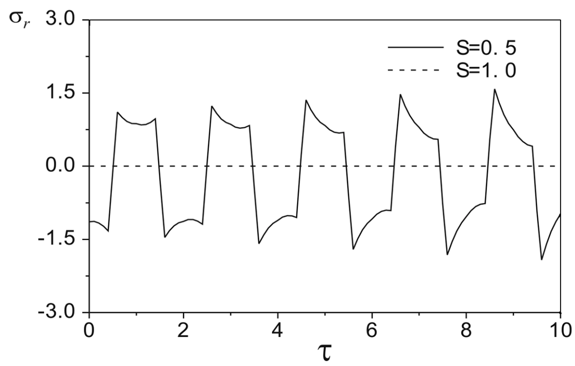

Fig. 3 Response histories of the transient radial stresses σ_r, where $S = \dfrac{r-a}{b-a}$, $\tau = \dfrac{C_L t}{a}$, $\sigma_r = \dfrac{\sigma_{rr}}{\alpha_r T_0 c_{11}}$, and $\phi_b = 0$

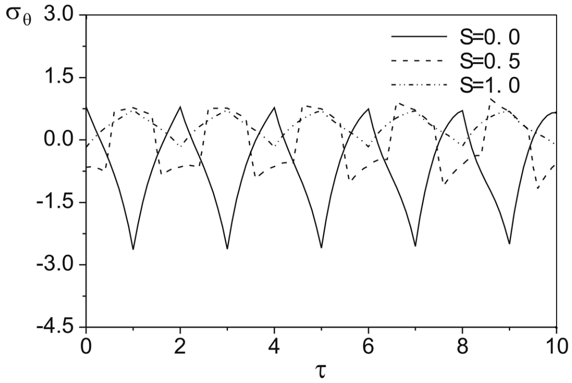

Fig. 4 Response histories of the transient hoop stresses σ_r^* at $R = 0.5$ and $R = 1$, where $R = \frac{r-a}{b-a}$, $\tau 1 = \frac{C_L t}{a}$, $\sigma_\theta = \frac{\sigma_\theta}{\alpha_r T_0 c_{11}}$ and $\phi_b = 0$

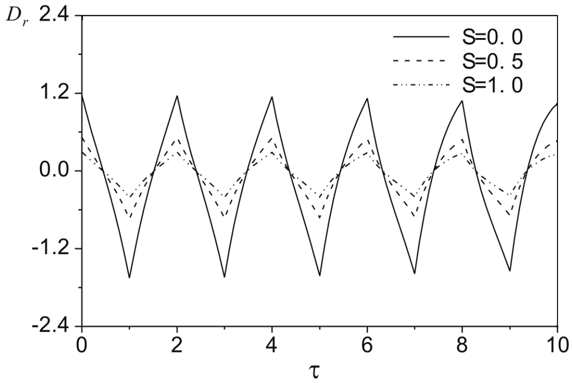

Fig. 5 Response histories of the transient electric displacements D_r, where $S = \frac{r-a}{b-a}$, $\tau = \frac{C_L t}{a}$, $D_r = \frac{D_{rr}}{\alpha_r T_0 \sqrt{c_{11}\beta_{11}}}$ and $\phi_b = 0$

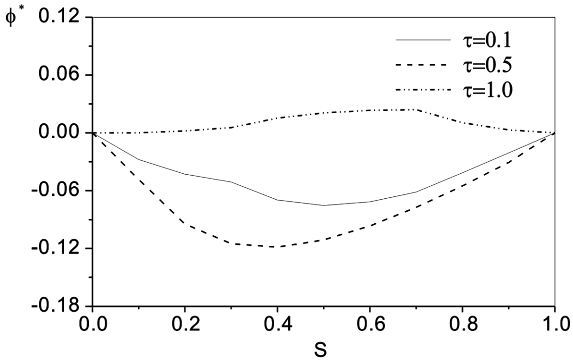

Fig. 6 Distributions of the transient electric potentials ϕ^*, where $S = \frac{r-a}{b-a}$, $\tau = \frac{C_L t}{b-a}$, $\phi^* = \frac{\phi}{\alpha_r T_0 \sqrt{c_{11}\beta_{11}}}$ and $\phi_b = 0$

external boundary equal zero, which satisfy the prescribed electric boundary conditions in Eq. (50). The distribution of the electric potential ϕ^* along radius is non-linear as time τ.

Example 2. Consider that the thermo-electro-elastic interaction in the transversely isotropic piezoelectric hollow sphere is induced by both thermal shock, $T(r, t)$ in Eq. (49) and electric excitation in inhomogeneous electric boundary conditions which is expressed as

$$\phi_a(a, t) = 0, \quad \phi_b(b, t) = H(\tau) \tag{51}$$

In the numerical calculation, $\sigma_i = \frac{\sigma_{ii}}{\alpha_r T_0 c_{11}} (i = r, \theta)$, $D_r = \frac{D_{rr}}{\alpha_r T_0 \sqrt{c_{11}\beta_{11}}}$ and $\phi^* = \frac{\phi}{\alpha_r T_0 \sqrt{c_{11}\beta_{11}}}$ are introduced.

From Figs. 7, 8 and Fig. 13, it is seen that the radial stresses and the electric potential at the boundaries R = 0, 1 satisfy the given boundary conditions. Except the points at the given boundary condition, transient responses at other points oscillate dramatically as shown in Figs. 7-13. It is seen from Figs. 7, 8 that the maximum amplitude of radial compression stress is smaller than that of radial tension stress. Fig. 9 shows that the amplitude of hoop compression stress of the transversely isotropic piezoelectric hollow sphere is smaller than that of hoop tension stress. The response histories of electric displacement always are negative as shown in Figs. 10-12. It is shown from Figs. 10-12 that the peak values of electric displacement decrease gradually from inner-wall to outer wall at the identical time τ. It is seen in Fig. 13 that the electric potential ϕ^* at the external boundary equals to 1, which satisfy the prescribed electric boundary conditions (51), and the distribution of electric potential ϕ^* along radius is weak non-linear at different non-dimensional time τ.

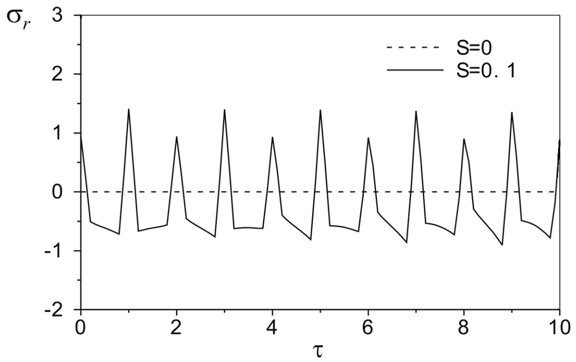

Fig. 7 Response histories of the transient radial stresses σ_r, where $S=\dfrac{r-a}{b-a}$, $\tau=\dfrac{C_L t}{a}$, $\sigma_r=\dfrac{\sigma_{rr}}{\alpha_r T_0 c_{11}}$, $\phi_b = 1$

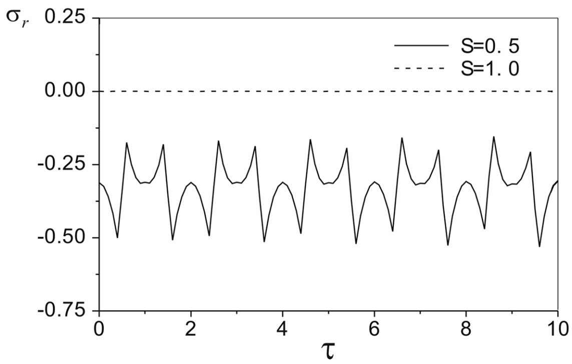

Fig. 8 Response histories of the transient radial stresses σ_r, where $S=\dfrac{r-a}{b-a}$, $\tau=\dfrac{C_L t}{a}$, $\sigma_r=\dfrac{\sigma_{rr}}{\alpha_r T_0 c_{11}}$, $\phi_b = 1$

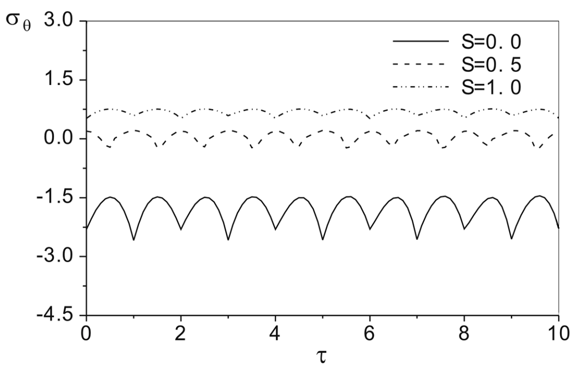

Fig. 9 Response histories of the transient hoop stresses σ_θ^*, where $S=\dfrac{r-a}{b-a}$, $\tau=\dfrac{C_L t}{a}$, $\sigma_\theta=\dfrac{\sigma_\theta}{\alpha_r T_0 c_{11}}$, $\phi_b = 1$

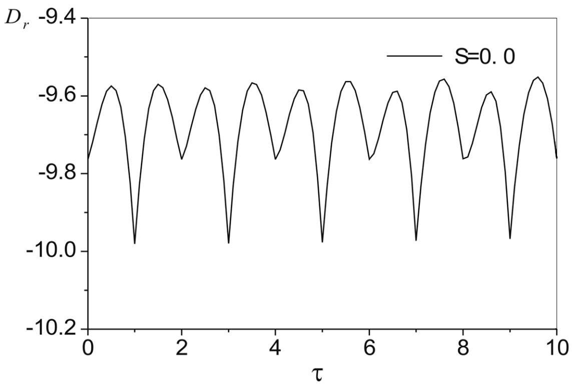

Fig. 10 Response histories of the transient electric displacements D_r, where $S=\dfrac{r-a}{b-a}$, $\tau=\dfrac{C_L t}{b-a}$, $D_r=\dfrac{D_{rr}}{\alpha_r T_0\sqrt{c_{11}\beta_{11}}}$, $\phi_b = 1$

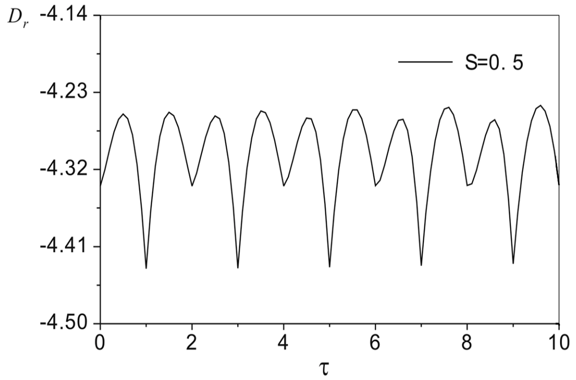

Fig. 11 Response histories of the transient electric displacements D_r, where $S=\dfrac{r-a}{b-a}$, $\tau=\dfrac{C_L t}{b-a}$, $D_r=\dfrac{D_{rr}}{\alpha_r T_0\sqrt{c_{11}\beta_{11}}}$, $\phi_b=1$

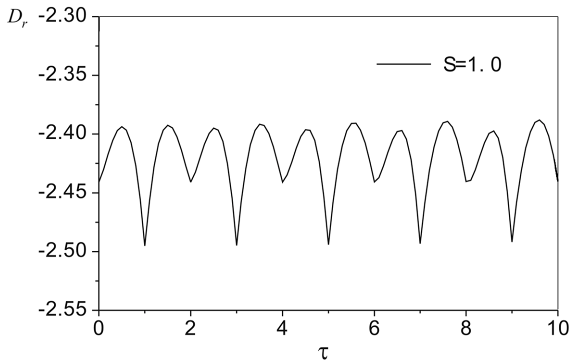

Fig. 12 Response histories of the transient electric displacements D_r, where $S=\dfrac{r-a}{b-a}$, $\tau=\dfrac{C_L t}{b-a}$, $D_r=\dfrac{D_{rr}}{\alpha_r T_0\sqrt{c_{11}\beta_{11}}}$, $\phi_b=1$

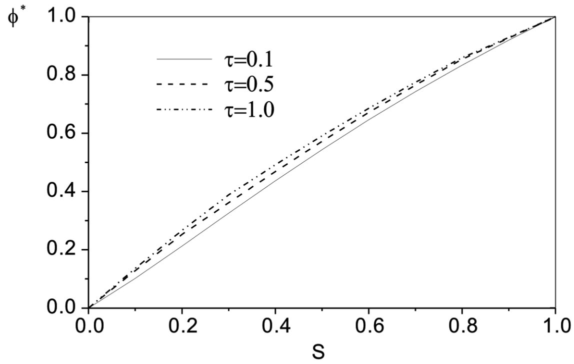

Fig. 13 Distributions of the transient electric potentials ϕ^*, where $S=\dfrac{r-a}{b-a}$, $\tau=\dfrac{C_L t}{b-a}$, $\phi^*=\dfrac{\phi}{\alpha_r T_0\sqrt{c_{11}\beta_{11}}}$, $\phi_b=1$

Example 3. Although the transient responses of coupled fields have been studied by a number of authors, no published experiment results can be used for a comparison with the present model. In fact, most of the previous experiment works have focused on the electro-elastic transient responses of beam and plate structures. To our knowledge, no experiment results on the problem of transient wave propagation in a transversely isotropic piezoelectric hollow sphere subjected to thermal shock and electric excitation are available in the literature. This is apparently due to the fact that the experiment research on the transient wave propagation in a piezoelectric hollow sphere under thermal shock and electric excitation remains a formidable task.

In order to prove the validity of these numerical results further, the present method can be applied to solve the transient problem of isotropic hollow sphere subjected to only thermal shock (Hata 1991). The equations of motion of this problem (Hata 1991) can be rewritten as

$$\frac{\partial^2 U(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial U(r,t)}{\partial r} - \frac{2}{r^2}U(r,t) = \frac{1}{C^2}\frac{\partial^2 U(r,t)}{\partial t^2} + \alpha\frac{1+v}{1-v}\frac{\partial T(r,t)}{\partial r}$$

$$\sigma_r(r,t) = \left[(\lambda+2\mu)\frac{\partial U}{\partial r} + \frac{2\lambda}{r}U - \frac{ET\alpha}{1-2v}\right]_{r=a,b} = 0$$

$$U(r,0) = 0, \quad \left[\frac{\partial U(r,t)}{\partial t}\right]_{t=0} = 0 \tag{52}$$

By using the present method, one can express the solution of Eq. (52) as

$$\sigma_r^* = \sum_i \left\{\frac{\bar{U}_{TB}}{F}\left[\frac{1-v}{1+v}\frac{\partial C_R}{\partial r} + \frac{2v}{(1+v)r}C_R\right]\cos(\xi_i Ct)\right\}$$

$$\sigma_\theta^* = \sum_i \left\{\frac{\bar{U}_{TB}}{F}\left[\frac{v}{1+v}\frac{\partial C_R}{\partial r} + \frac{1}{(1+v)r}C_R\right]\cos(\xi_i Ct)\right\} \tag{53}$$

where $\bar{U}_{TB} = -Hankle[r^{3/2}],\ C_R = r^{-1/2}[J_{3/2}(\xi_i r)Y_a - Y_{3/2}(\xi_i r)J_a]$

$$F = \int [J_{3/2}Y_a - J_{3/2}J_a]^2 dr \tag{54}$$

For ease of comparison with reference (Hata 1991), the same parameters are taken: $t^* = Ct/a$, $\sigma_i^* = \sigma_i/(\varepsilon_r T_0 \rho C^2)$, $\varepsilon_r T_0 \rho C^2 = \alpha E T_0/(1-2v)$ and $v = 0.3$. From Fig. 14, one can see that the results from the two different methods are nearly the same. Note that solving the hollow sphere problem, the number of eigenvalue terms was taken to be only 40, and the error of the results obtained is less than 1%.

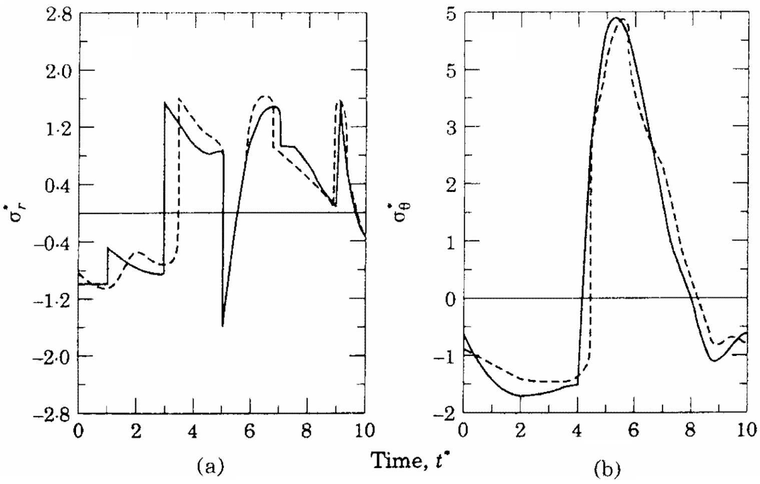

Fig. 14 Response histories of isotropic elastic hollow sphere subjected to only thermal shock, for $b/a = 5$, $t^* = Ct/a$, where —— expresses the result in Hata (1991) and ––– expresses the result in this paper. (a) $r/a = 2$, (b) $r/a = 1$

5. Conclusions

1. Comparing Example 1 with Example 2, it is seen that the response histories and distributions of stresses, electric displacement and electric potential in a transversely isotropic piezoelectric hollow sphere are obviously different for two kinds of electric excitation which is, respectively, shown in Eq. (50) and Eq. (51). Thus, it is possible to control the response histories and distribution of thermal stresses in the transversely isotropic piezoelectric hollow sphere by applying a suitable electric excitation load to the structure, or to assessment the response histories and distribution of thermal stresses in the transversely isotropic piezoelectric hollow sphere by measuring the response histories of electric potential in the structure.
2. It is concluded from the above analyses and discussions that the presented method is simple and effective. So it may be used as a reference to solve other coupling problems in a piezoelectric hollow sphere. From the knowledge of the response histories of transient stresses, electric displacement and electric potential in a piezoelectric hollow cylinder, one can design various thermo-electro-elastic elements subjected to thermal shock and electric excitation to meet specific engineering requirements.

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Notation

ε_{ij}	: components of strains
$u_r(r, t)$	: radial displacement [m]
c_{ij}, e_{ij}, α_i, p_{ij}	: elastic constants [N/m^2], piezoelectric constants [C/m^2], thermal expansion coefficients [1/k] and dielectric constants[C^2/Nm^2]
λ_i, β_{ii}	: thermal modulus [N/m^{2}k], and pyroelectric coefficient [C/m^{2}k]
σ_{ij}, D_{rr}	: components of stresses [N/m^2] and radial electric displacement [C/m^2]
$\varphi(r, t)$	: electric potential [W/A]
$T(r, t)$	: temperature change function [k]
ρ, t	: mass density [kg/m^3] and time variable [s]
r	: radial variable [m]
a, b	: inner and outer radii of piezoelectric hollow cylinder [m]
C_L	: electroelastic wave speed [m/s]
ω	: inherent frequency of piezoelectric hollow cylinder [1/s]