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Finite element vibration and damping analysis of a partially covered cantilever beam

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Abstract. There are several ways of decreasing the vibration energy of structures. One of which is special damping layers made of various viscoelastic materials are widely applied in structures subjected to dynamic loading. In this study, a cantilever beam, partially covered by damping a constraining layers, is investigated by using Finite Element method (FEM). The frequency and system loss factor are evaluated. The effects of different physical and geometrical parameters on the natural frequency and system loss factors are discussed.

Key words: damping; sandwich beam; finite element.

1. Introduction

To increase energy loss in beams and plates under flexural vibrations, viscoelastic layers have been commonly used. Viscoelastic materials used in such application are used to dissipate energy when subjected to alternating deformation. Several papers have been published recently on the vibration of three-layered sandwich beams and plates. Hajela and Lin (1991) have studied the enhancement of damping characteristics of beam structures through application of viscoelastic damping material as a multi criterion optimization problem. Marcelin *et al.* (1992) dealt with optimal damping of beams constrained by a viscoelastic layer when only one or several portions of the beam are covered. The design variables are the dimensions and prescribed locations of the viscoelastic layers, and the objective is to maximize the damping factor. Their paper mainly deals with finding out the locations of maximum strain, but they have not given any results pertaining to optimal configuration for increasing the damping and the paper deals with only the clamped free boundary condition. Rikards *et al.* (1993) studied the Finite Element analysis of damping the vibration of laminated composites. By excluding all degree of freedom in the nodes of the middle layer, they formed a superelement stiffness matrix of the sandwich beam or plate. To estimate the damping of structure, they used two methods, which are the complex frequency and the energy method, to calculate modal loss factors. They stated that both methods give sufficiently good results. Levy and Chen (1994) and Chen and Levy (1994) analyzed the natural vibration of a cantilever beam, and also studied the same beam with a tip mass that is partially covered by a double sandwich-type viscoelastic material. They employed Hamilton's principle to derive the differential

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equations of motion, and evaluated the frequency and the loss factor of the system. They also discussed the effects of different physical and geometrical parameters on the complex natural frequency and system loss factor. It was indicated that the double sandwich system provides better damping in most cases. Baz and Ro (2001) studied the vibration control of rotating beams with an active constrained damping layer, in which the dynamics of a rotating beam were described with a finite-element model that provides means for predicting the damping characteristics of the active treatment at different setting angles and controller gains. They claim that the results obtained clearly demonstrate the attenuation capabilities of actively controlled constrained layer damping and suggest its potential in suppressing the vibration of practical systems such as helicopter rotor blades. Teng and Hu (2001) analyzed the design parameters for constrained layer damping structures by employing the Ross-Kerwin-Ungar (RKU) model. They also discussed the effects of temperature, frequency and the dimensions of damped structures on vibration damping characteristics. They pointed out that a laminate with viscoelastic damping treatment has a better damping capability than that without the addition of viscoelastic layer. Sisemore and Darvennes (2002) studied transverse vibration of elastic - viscoelastic - elastic sandwich beams for which a new analytical model for compressional damping was proposed and compared with experimental results, with the Mead and Markus shear damping model, and with the Douglas and Yang compressional damping model. They claimed that the proposed compressional model was a better predictor of resonance frequencies for the cantilever beams tested. Sun and Tong (2002) presented a detailed model for the beams with partially debonded active constraining damping (ACLD) treatment, in which the transverse displacement of the constraining layer was considered to be non-identical to that of the host structure. They pointed out that the numerical results showed that edge debonding can lead to a reduction of both passive and active damping, and the hybrid damping may be more sensitive to the debonding of the damping layer than the passive damping. Lin and Chen (2003) studied the dynamic stability of a rotating sandwich beam with a constrained damping layer by the finite element method. The effects of rotating speed, setting angle, hub radius ratio and core thickness ratio were examined, and the regions of dynamic instability for various parameters were presented. Rikards *et al.* (1993) studied the Finite Element analysis of damping the vibration of laminated composites. By excluding all degree of freedom in the nodes of the middle layer, they formed a superelement stiffness matrix of the sandwich beam or plate. To estimate the damping of structure, they used two methods, which are the complex frequency and the energy method, to calculate modal loss factors. They stated that both methods give sufficiently good results. Shi *et al.* (2004) proposed a practical procedure to overcome technical problems of structure control due to a large model size. They claimed that the results indicated the efficiency of the proposed procedure.

In this research, the finite element method of a partially covered beam was investigated, for which at first, a superelement stiffness matrix of the sandwich beam was formed, excluding all degrees of freedom in the nodes of the middle layer. Then a finite element analysis of the structure was performed with the help of these sandwich superelements of reduced degrees of freedom. Modal loss factors were calculated with the exact method of complex eigenvalues.

The damping theory of vibration of viscoelastic structures contains two main models: a model of viscous resistance and a model of complex modulus of elasticity. The latter model was used for this analysis. An extensive review can be seen in this topic (Nashif *et al*. 1985)

Energy dissipation in harmonic vibrations is taken into account for the complex elasticity $E^{\bullet}$ and shear $G^{\bullet}$ moduli for isotropic linear viscoelastic materials (Rikards *et al*. 1993),

$$E^{\bullet} = E + iE^{\circ} = E(1 + i\eta_E)$$

$$G^{\bullet} = G + iG^{\circ} = G(1 + i\eta_G)$$

where η_E and η_G are the material loss factors in tension-compression and shear, respectively

$$\eta_E = \frac{E^{\circ}}{E}, \quad \eta_G = \frac{G^{\circ}}{G}$$

E, G and, E°, G° represent the real and complex parts of elasticity and shear moduli, respectively. The equation of motion has the following form for the complex elasticity modulus model

$$[M]\{\ddot{u}^{\bullet}\} + [K^{\bullet}]\{u^{\bullet}\} = \Re\{t\} \tag{1}$$

where $[K^{\bullet}] = [K + iK^{\circ}]$ is a complex stiffness matrix. The nodal displacements $u^{\bullet}$ are also complex. The viscoelastic properties of composite materials are defined by a complex stiffness tensor.

2. Finite element model

Let us take a laminated composite beam (Fig. 1). The layers of the beam consist of an isotropic or anisotropic, linear elastic or viscoelastic material. For the linear elastic dynamic analysis, it is possible to take into account the laminated beam as a simple beam with one kinematic hypothesis for the multilayered structure. A shear correction coefficient is used to consider the real transverse shear stiffness. If one or some of the layers are viscoelastic, each layer is considered as a simple beam finite element, and then a laminated superelement can be formed from these simple finite elements. If a simple Timoshenko beam finite element with four nodes and length ℓ in arbitrary spatial deformation (Fig. 2) is considered, coordinates x, y, z may be in an arbitrary point of the cross-section of the beam. There are three degrees of freedom in every node: two displacements u and w and one rotation γ_y. Timoshenko's kinematic relations are given in bending and tension for the displacements u_x and u_z at an arbitrary point of the cross-section of the beam as (Rikards *et al.* 1993, Bathe 1996):

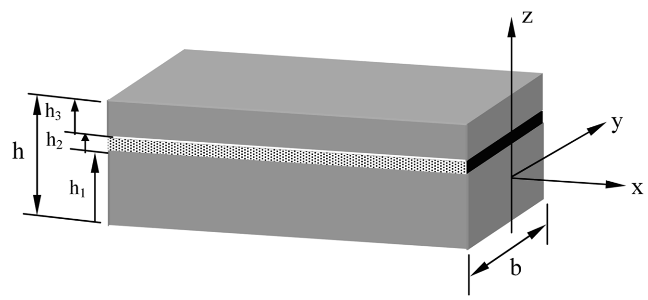

Fig. 1 Laminated composite beam

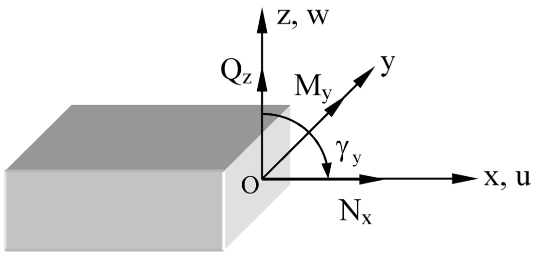

Fig. 2 Sign convention in the Timoshenko's beam

$$u_x(x,z) = u(x) + z \cdot \gamma_y(x)$$
$$u_z(x,z) = w(x) \tag{2}$$

The strains in bending and tension-compression are expressed for the arbitrary point of the cross-section of the beam as follows:

$$e_x = \frac{\partial u_x}{\partial x} = \varepsilon_x + z\chi_y \tag{3}$$

$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \gamma_y + \frac{\partial w}{\partial x} \tag{4}$$

where

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \chi_y = \frac{\partial \gamma_y}{\partial x} \tag{5}$$

The strain energy for the finite element of the length ℓ is

$$\Pi = \frac{1}{2}\int_0^{\ell} \{N_x\varepsilon_x + M_y\chi_y + Q_z\gamma_{xz}\}dx \tag{6}$$

where N_x and Q_z are the axial and shear forces, respectively while M_y is the bending moment (Rikards *et al.* 1993, Bathe 1996). The constitutive equations of the Timoshenko beam have the following form, considering Hooke's law

$$\begin{Bmatrix} N_x \\ M_y \end{Bmatrix} = \begin{bmatrix} A_{xx} & B_{xy} \\ B_{xy} & D_{yy} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \chi_y \end{Bmatrix} \tag{7}$$
$$\{Q_z\} = [kG_{xz}]\{\gamma_{xz}\}$$

where A_{xx}, D_{yy}, B_{xy}, and G_{xz} are the tension compression, bending, tension bending and shear stiffness respectively, while "k" is the shear correction coefficient. The stiffnesses are evaluated in a usual manner, integrating over the cross-section A of the beam

$$A_{xx} = \int_A C_{11} dA \qquad B_{xy} = \int_A z C_{11} dA$$
$$D_{yy} = \int_A z^2 C_{11} dA \qquad G_{xz} = \int_A C_{55} dA \tag{8}$$

where C_{ij} are the components of the stiffness matrix of the material (Rikards *et al.* 1993, Bathe 1996).

For isotropic beams with a rectangular cross-section, the value of the shear correction coefficient is $k = 5/6$.

Substituting Eq. (7) into Eq. (6) the strain energy is obtained as a functional of three unknowns: two displacements (u, w) and one rotation (γ_y). Third-order polynominal shape functions are used to approximate for all the unknown functions. A finite element stiffness matrix with 12 degrees of freedom was obtained by usual numerical integration. The first two terms in the strain energy Eq. (6) are fourth-order polynominals which had been integrated by the Gaussian quadrature with three integration points. This is a full or exact integration. On the other hand, the last term in strain energy Eq. (6) is a sixth-order polynominal. In order to avoid locking of the Timoshenko's beam finite element, it is necessary to use a reduced integration technique with only three Gaussian points instead of exact integration with four integration points for these terms (Rikards *et al.* 1993).

A mass matrix was obtained in similar way. The kinetic energy functional of Timoshenko's beam finite element is expressed as:

$$T = \frac{1}{2}\int_0^{\ell}\int_A \rho(\dot{u}_x^2 + \dot{u}_z^2) dA dx \tag{9}$$

where ρ is the density of the material. Substituting Eq. (2) into Eq. (9), after integrating, the mass matrix of the Timoshenko beam finite element was obtained. This simple beam finite element can be used in dynamic analysis of elastic beams, as well as sandwich or laminated structures. However, the superelement approach must be used for damping analysis of beams with viscoelastic layers.

A beam with three layers is given in Fig. 3. A simple beam finite element is used to represent each of the three layers. The coordinates chosen are connected with the lower (first) layer. When bending in one plane is considered, there are three degrees of freedom in each node of simple beam finite element: longitudinal displacement u. transverse displacement w and rotation $\gamma = \gamma_y$. The simple beam finite element has 12 degrees of freedom, and the vector of nodal displacements can be defined as (Rikards *et al.* 1993)

$$\delta^T = \{u_1, w_1, \gamma_1, u_2, w_2, \gamma_2, u_3, w_3, \gamma_3, u_4, w_4, \gamma_4\} \tag{10}$$

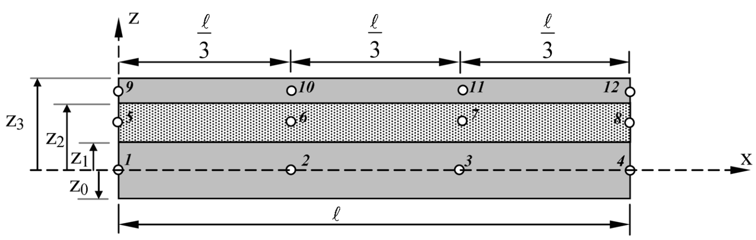

Fig. 3 Beam superelement with simple beam finite element nodes

Stiffness matrices $\mathbf{K}_i$ (i =1, 2, 3) presents for each layer. There are also displacement continuity conditions between the layer boundaries. The vector of independent nodal displacements for the superelement is denoted Θ. By considering the displacement continuity conditions between the layers, the relations between vector Θ and vectors δ_i (i =1, 2, 3) are given as follow (Rikards *et al.* 1993)

$$\left. \begin{aligned} u_x^{(1)} &= u_x^{(2)} \\ w^{(1)} &= w^{(2)} \end{aligned} \right|_{z = z_1} \qquad \left. \begin{aligned} u_x^{(2)} &= u_x^{(3)} \\ w^{(2)} &= w^{(3)} \end{aligned} \right|_{z = z_2} \tag{11}$$

Substituting Timoshenko's kinematic relation, the followings are obtained

$$u_x^{(i)} = u^{(i)} + z \cdot \gamma^{(i)} \qquad i = 1, 2, 3 \tag{12}$$

Therefore, for the superelement there are 16 relations (11) for 36 nodal displacements $u^{(i)}$, $w^{(i)}$ and $\gamma^{(i)}$ (i = 1, 2, 3). It results in 20 independent nodal displacements for vector Θ. The relation between vectors δ_i and Θ is as follow (Rikards *et al.* 1993):

$$\delta_i = \Re_i \Theta \qquad i = 1, 2, 3 \tag{13}$$

where the dimension of matrices $\Re_i$, is 12×20. Displacements and rotations of the upper and lower layers are chosen as independent nodal displacements for the superelement. For the lower layer there are three displacements in each node: u, w, γ while for the upper layer there are only two independent displacements in each node: u, γ (Fig. 4). Thus, the stiffness matrix for the superelement is obtained as (Rikards *et al.* 1993):

$$\Pi = \sum_{i=1}^{3} \Pi_i = \frac{1}{2} \sum_{i=1}^{3} \delta_i^T K_i \delta_i$$

$$\Pi = \frac{1}{2} \Theta^T K \Theta \tag{14}$$

where

$$K = \sum_{i=1}^{3} \Re_i^T K_i \Re_i \tag{15}$$

The mass matrix [**M**] for the superelement is obtained in the same way.

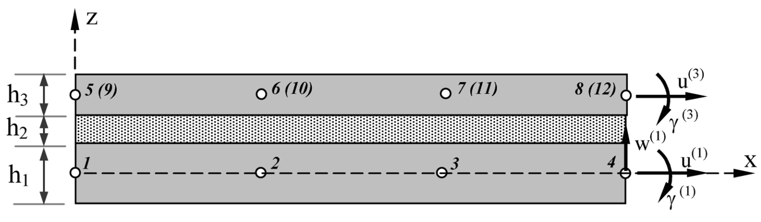

Fig. 4 Beam superelement with eight nodes and 20 degrees of freedom

3. Method of numerical analysis

The damping of the structure was examined by a model of complex modulus of elasticity representing the viscoelastic properties of material of the layers. The equation of motion (1) was solved using the method of complex eigenvalues for dynamic analysis of the structure. The eigenvalues of the damped structure can be determined from Eq. (1) of free vibrations

$$[M]\{\ddot{u}^\bullet\} + [K^\bullet]\{u^\bullet\} = 0 \tag{16}$$

Using $u^\bullet = u_0^\bullet e^{i\omega^\bullet t}$, for free vibrations the following is obtained:

$$[K^\bullet]\{u_0^\bullet\} = \lambda^\bullet[M]\{u_0^\bullet\} \tag{17}$$

where $\lambda^\bullet = \omega^{2\bullet}$ is the complex eigenvalue, $\omega^\bullet$ is the complex frequency, $u_0^\bullet$ is the complex eigenvector. Eq. (17) was solved by using MATLAB (2002). As a result, the complex eigenvalues and the complex frequencies for the damped structure are obtained:

$$\lambda^\bullet = \lambda + i\lambda^\circ, \qquad \omega^\bullet = \omega + i\omega^\circ \tag{18}$$

The modal loss factors η_n for each frequency can be determined from the relation

$$\eta_n = \frac{\lambda_n^\circ}{\lambda_n} \tag{19}$$

where λ_n and λ_n° are the real and imaginary parts of the complex eigenvalue $\lambda_n^\bullet$ (Rikards *et al.* 1993), respectively.

4. Results and discussion

The finite element analysis of vibration and damping of partially covered cantilever beam as shown in Fig. 5 was solved and the numerical results are given in Figs. 6-12. The geometrical properties, the physical properties and core loss factor are chosen as the parameters for the solution. The values chosen for the solution, unless stated otherwise, are $E_1 = E_3 = 20.6 \times 10^{10}$ N/m^2, $G_1 = G_3 = 79.2 \times 10^9$ N/m^2, $\rho_1 = \rho_3 = 7850$ kg/m^3, $\eta_1 = \eta_3 = 0$, $\eta_2 = 1$, $E_2 = 0.525 \times 10^{10}$ N/m^2, $\rho_2 = 1150$ kg/m^3, $\nu_2 = 0.5$, $L = 0.4$ m, $b = 0.25$ m, $h_1 = 0.002$ m, $h_3 = h_1/2$.

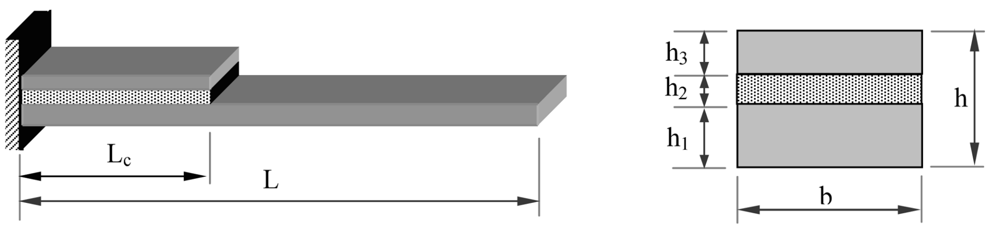

Fig. 5 The partially covered sandwich-type cantilever beam

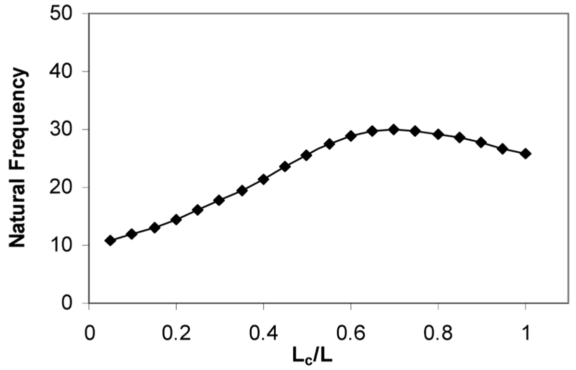

Fig. 6 The effects of the damping layer length on natural frequency for $h_2 = 0.0015$

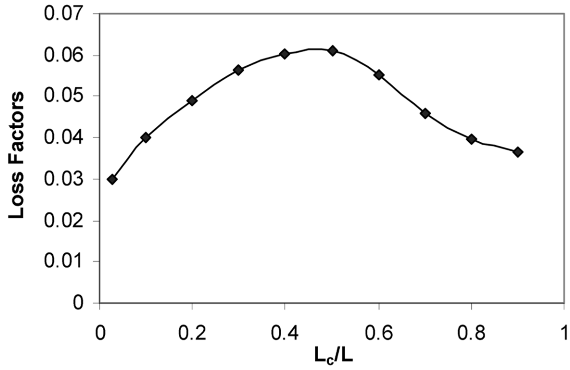

Fig. 7 The effects of the damping layer length on the system loss factor for $h_2 = 0.0015$

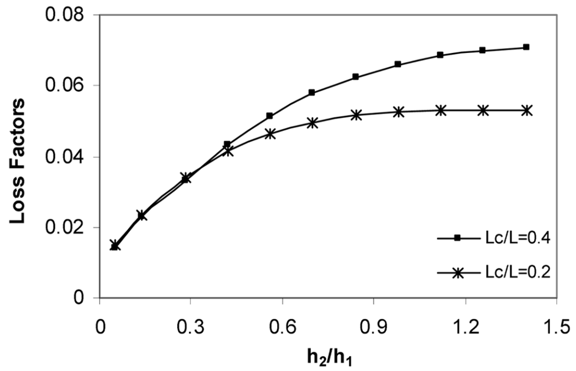

Fig. 8 The effects of the damping layer thickness on the system loss factor

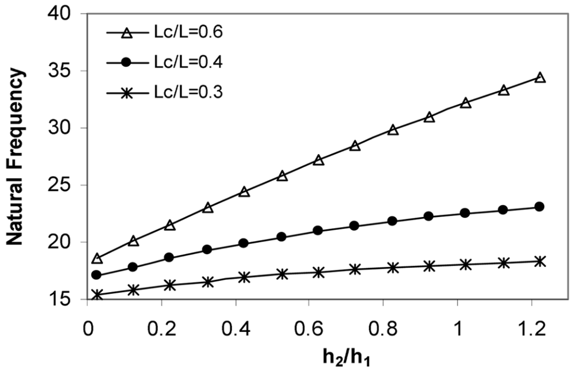

Fig. 9 The effects of the damping layer length on natural frequency

The effect of the damping layer length on the real frequency is shown in Fig. 6 for $h_3/h_1 = 0.5$. It appears that initially the real frequency at the first mode increases as the damping length increases, which is contrary to expectations: as coverage increases, the real frequency is expected to decrease, which occurs after $L_c/L = 0.7$.

It can be concluded that the effective beam length is shortened since the constraining layer whose length and parameters are the same as the damping layer and the beam, respectively, moves the fixed end of the beam closer to the free end. This, in turn, causes the real frequency of the effective fixed-free beam to increase, as indicated in the figure. After the point of $L_c/L > 0.7$, the effect of damping layer decreases the real frequency factor.

In Fig. 7, h_3/h_1 was taken as 0.5 and h_2 was 0.0015 m. It was observed that an optimum loss factor exists at $L_c/L = 0.5$, which suggests that a damping layer covering of $L_c/L = 0.5$ can optimize the system loss factor for the first mode.

The increase in the damping layer thickness also increases the system loss factor (Fig. 8). The optimum value of η occurs for the different values of L_c/L, when h_2/h_1 is in the range of 0.9-1. As expected, the increase in the damping layer thickness also raises the real frequency of the system to higher values. If the operating frequency was near the lowest value of the real frequency of the

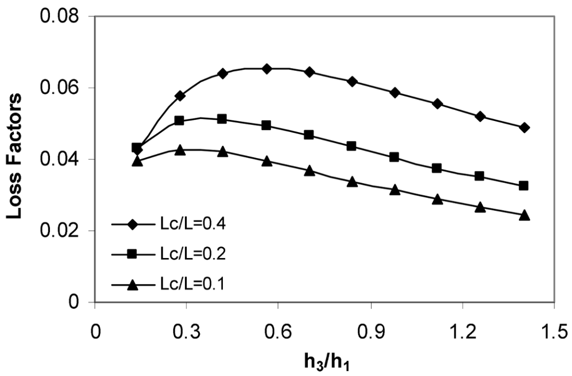

Fig. 10 The effects of the constraining layer thickness on the system loss factor for $h_2 = h_1$

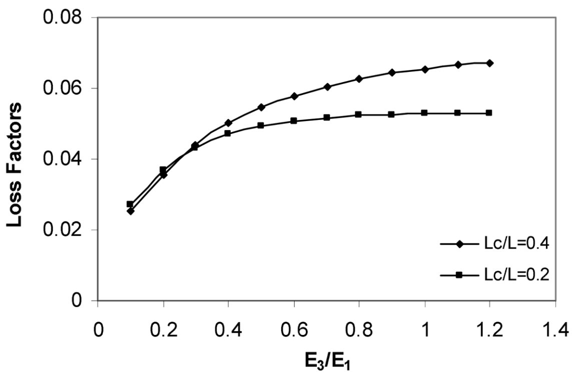

Fig. 11 The effects of Young's modulus on the system loss factor for $h_2 = h_1$

unconstrained beam, increasing the thickness of the damping layer would result in a safer design since the difference between the natural and operating frequency increases. The variation of the damping layer thickness versus the real frequency for the lowest mode is shown in Fig. 9. The same tendency can be observed for the constraining layer thickness (Fig. 10). However, to obtain a good damping, the height ratio should be different from the damping layer thickness ratio. If the ratio of h_3/h_1 is less than 0.5, increasing the damping layer coverage increases the system loss factor. In this case, the effect of the viscoelastic layer becomes dominant causing an increase in the loss factor. The beam acts as if the cantilever end has been moved closer to the free end, increasing the real frequency and the system loss factor. If $h_3/h_1 > 0.5$, as L_c/L is larger than 0.4, η decreases, as shown in Fig. 10. However, as h_3/h_1 increases, the decrease in η is offset by the fixed end condition effect described above. For the given parameters in the example, if the ratio of h_3/h_1 is selected around 0.5, larger loss factors are obtained for smaller damping layer coverage. The variation of the system loss factor with the Young's modulus is shown in Fig. 11. It can be seen from the figure that if the Young's modulus of the constraining layer (E_3) is too small, the system loss factors become very small, which indicates that there is an unconstrained beam. Thus, the vibration reduction effect will be small. If the Young's modulus of the constraining layer is equal to that of the original beam, the system loss factor reaches the maximum value. It seems that the increase of E_3 will increase the

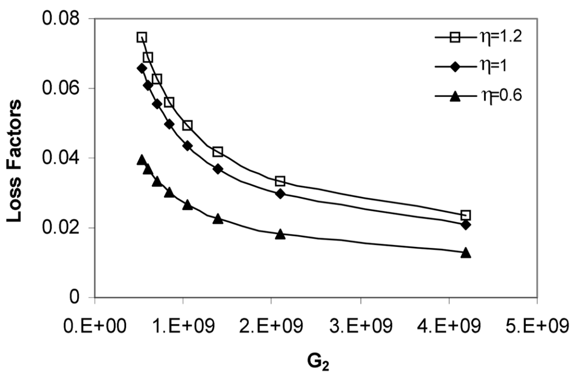

Fig. 12 The effects of shear modulus on the system loss factor for $h_2 = h_1$

shear deformation and the value of η to a certain point. Generally, the increase of the core loss factor will increase the system loss factor, as depicted in Fig. 12. The same results were obtained for the fully covered damped sandwich-type beam in the literature. The effects of shear modulus of the damping layer on the system are observed in Fig. 12 that higher G_2's require lower shear deformations for the same force, and lower shear deformations indicate lower system loss factors. Conversely, lower G_2's require higher shear deformations for the same force, which means higher system loss factors, which is illustrated in Fig. 12. Generally, if $G_2 < 1 * 10^9$, the system loss factor is found to be large. The above results obtained by the finite element method in the present paper represent a good agreement with the previous works (Levy and Chen 1994, Lall *et al*. 1998), which are carried out using an analytical approch.

5. Conclusions

In this study, the Finite Element method was applied to a partially covered cantilever beam for vibration and damping analysis. The variations of the real frequency and loss factor of the system for different parameters were also discussed. The results were obtained numerically, and the following conclusions were obtained for this study.

First, an optimum damping coverage length and constraining layer thickness was obtained. For instance, the system loss factor is observed when constrained layer length or damping layer length reaches 50% of the original beam. It was observed that the optimal values vary for different materials parameters. Thus, optimum values can be chosen to obtain the best vibration reduction effects for such a system.

Second, for the constraining layer, the same material or another material with almost the same Young's modulus as the original beam can be chosen. The system loss factor will decrease with an increase of G_2 for the core shear modulus. For the parameters chosen, the system loss factor becomes optimum if G_2 is less than 10^9.

The results obtained in the present paper is in a good agreement with that of in the literature.

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