Structural Engineering and Mechanics, Vol. 18, No. 6 (2004) 745-757
DOI: http://dx.doi.org/10.12989/sem.2004.18.6.745

Lateral buckling formula of stepped beams with length-to-height ratio factor

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(Received January 6, 2004, Accepted July 15, 2004)

Abstract. Lateral-torsional buckling moment resistances of I-shaped stepped beams with continuous lateral top-flange bracing under a single point load on the top flange and negative end moments were investigated. Stepped beam factors and a moment gradient correction factor suggested by Park *et al.* (2003, 2004) were used to develop new lateral buckling formula for beam designs. From the investigation of finite element analysis (FEA), new lateral buckling formula of beams with singly or doubly stepped member changes and with continuous lateral top-flange bracing subjected to a single point load on top flange and end moments were developed. The new design equation includes the length-to-height ratio factor to account for the increase of lateral-torsional buckling moment resistance as the increase of length-to-height ratio of stepped beams. The calculation examples for obtaining lateral-torsional buckling moment resistance using the new design equation indicate that engineers should easily determine the buckling capacity of the stepped beams.

Key words: stability; buckling; beams; design.

1. Introduction

Beams with stepped member changes are often efficient than beams with constant section, and are frequently used in situations where the major axis bending moment varies along the length of the beams. I-beams with stepped member changes are most efficiently used when they have sufficient lateral support so that the flange stresses are only limited by the yield stress of the material if local buckling at the flanges and webs of the beams are appropriately prevented. Common practice for fabrication of built-up weld beams favors the use of a constant web depth and flange width with increase in flange thickness to provide increased moment of inertia.

For doubly symmetric I-shaped beams, the lateral-torsional buckling (LTB) moment resistance is defined in the American Institute of Steel Construction (AISC) Load and Resistance Factor Design (LRFD) Specifications (1998) as below:

$$M_{cr} = C_b \frac{\pi}{L_b} \sqrt{E I_y G J + \left(\frac{\pi E}{L_b}\right)^2 I_y C_w} \tag{1}$$

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where C_b = moment gradient modifier; L_b = laterally unbraced length; E = modulus of elasticity of steel; I_y = moment of inertia about Y-axis; G = shear modulus of elasticity of steel; J = torsional constant of beam; and C_w = warping constant. Eq. (1) with C_b = 1 is the elastic lateral-torsional buckling resistance (M_{ocr}) for an I-shaped prismatic section under the action of constant moment in the plane of the web over the laterally unbraced length (Timoshenko and Gere 1961).

Park and Stallings (2003) suggested a following design solution to calculate the lateral-torsional buckling capacity of singly or doubly stepped beams under uniform bending.

$$M_{ost} = C_{st} M_{ocr} \tag{2}$$

in which

$$C_{st} = C_o + 6\alpha^2(\beta\gamma^{1.3} - 1) \quad \text{for doubly stepped beams} \tag{3}$$

$$C_{st} = C_o + 1.5\alpha^{1.6}(\beta\gamma^{1.2} - 1) \quad \text{for singly stepped beams} \tag{4}$$

where M_{ost} = critical LTB moment; C_{st} = stepped beam factor based on action of constant moment bending; M_{ocr} = LTB moment of an equal length prismatic beam having the smaller cross section along the entire span; C_o = constant to account for effect of moment gradient; and α, β, and γ = ratios defining the relative length and relative width and thickness of the large and small cross sections, respectively. That solution is for beams with bracing at discrete locations, and the LTB resistance of an unbraced length of beam is found by applying a multiplier to the resistance calculated for a prismatic beam. The solution presented by Park and Stallings (2003) is extended here for stepped beams with continuous lateral top-flange bracing subjected to a single point load at the top flange.

The Structural Stability Research Council Guide (Galambos 1998) provides a solution for a lateral-torsional buckling moment resistance of prismatic I-beams braced at the ends and with continuous lateral bracing of the top flange. An investigation of the accuracy of the solution was presented by Park *et al.* (2004). The solution can be reasonably applied in design or evaluation of beams having uniformly distributed load, or both a series of concentrated load and uniformly distributed load, along with end moments. However, the solution gives unconservative values for beams with a single point load and end moments. Park *et al.* (2004) presented the following equation that can be used for prismatic beams with a single point load at the top flange and having continuous lateral bracing at the top flange.

$$C_b = 2.5 - \frac{2}{3}\left(\frac{M_1}{M_0}\right) + \frac{5}{3}\frac{M_{CL}}{(M_0 + M_1)} \tag{5}$$

where M_0 = end moment that produces the largest compressive stress on the bottom flange; M_1 = the other end moment; and M_{CL} = the moment at the centerline of the segment. Positive values should be substituted into above equation for M_0 and M_1 when these moments produce compressive stress in the bottom flange. A positive value should be substituted for M_{CL} when this moment produces tensile stress in the bottom flange. For the quantity ($M_0 + M_1$) in the equation, M_1 should be taken as zero when the term M_1 is negative. Eq. (5) is easy and reasonable to use in prismatic beam design and is extended here for stepped beams subjected to a single point load at the top flange and end moments.

The LTB resistance of beams may be significantly affected by the distances of transverse loads from the shear center axis. Loading applied to concrete slabs supported by steel beams may be assumed to be applied on the top flanges of the steel beams. The top-flange loading is a more severe load case. For prismatic beams with top flange loading condition, Helwig *et al*. (1997) showed that the moment gradient correction factor, C_b, increased as the ratio of the unbraced length to the height of beams, L_b/h, increased. Park *et al*. (2004) presented that the value of M_{cr}/M_{ocr} for prismatic beams with continuous lateral top-flange bracing subjected to top-flange loading increased significantly as the length-to-height ratio increased. None provides a method that can be readily used to calculate LTB resistances of beams having the length-to-height ratio effect. Park (2002) suggested a length-to-height ratio factor for stepped beams with continuous lateral top-flange bracing subjected to top flange loading conditions. This paper presents an investigation of accuracy of the length-to-height ratio factor provided by Park (2002). Loadings considered include cases with negative bending moments at one end or at both ends, along with a single point loading on the top flange.

2. Finite element modeling

A finite-element program MSC/NASTRAN (1998) and a graphical package MSC/PATRAN (2000) were used to numerically investigate the lateral-torsional buckling behavior of stepped beams subjected to negative end moments and a single point load on the top flange. For the present investigation of the lateral-torsional buckling behavior of beams, NASTRAN was used to model the full three-dimensional configuration of the cross section.

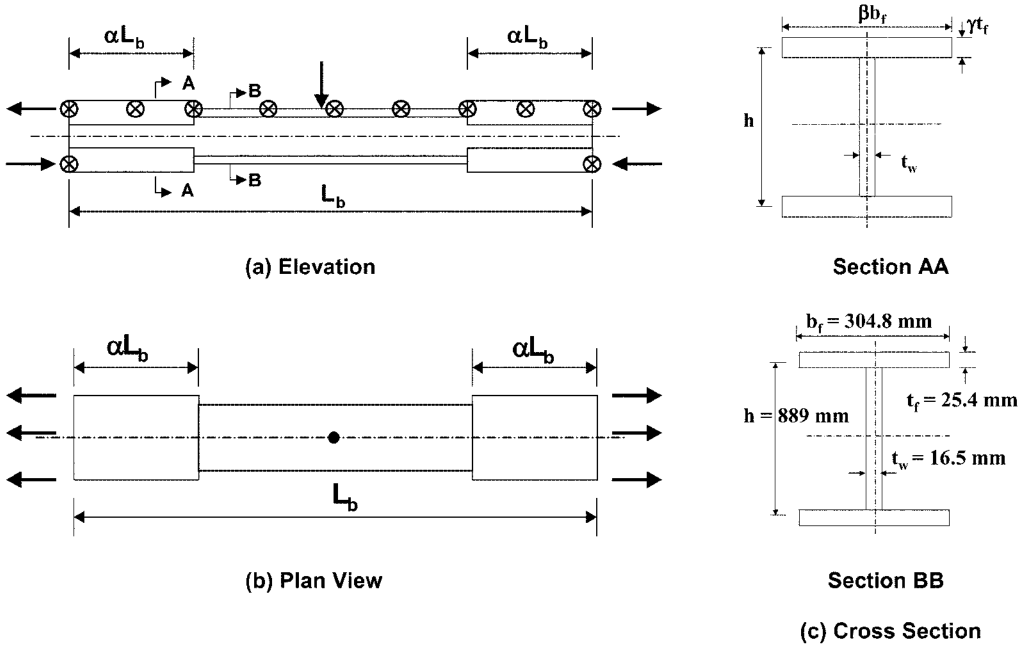

Fig. 1 Three ratios and loading in the doubly stepped beams

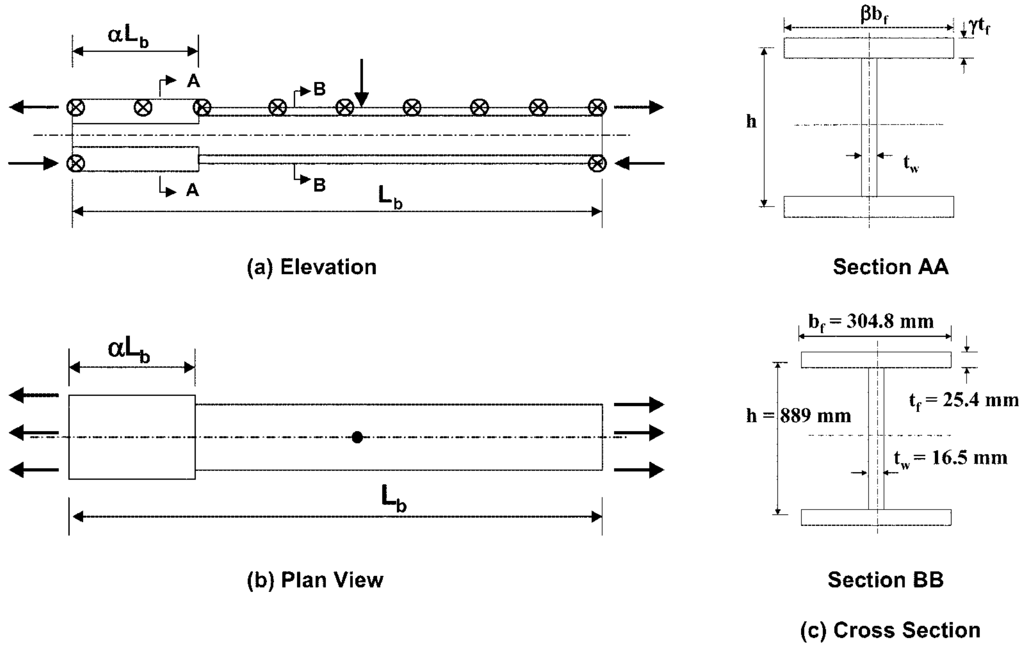

Fig. 2 Three ratios and loading in the singly stepped beams

Figs. 1 and 2 show two basic types of stepped beam considered here, and loading conditions having negative end moments and a single point load on the top flange of the center. Fig. 1 shows the doubly stepped beams used in this study. As shown in Fig. 1, the flanges of the smaller cross section were fixed at 304.8 mm × 25.4 mm while the width and/or thickness of the flanges of the larger cross section were varied at each end. The web thickness and height of beam was kept at 16.5 mm × 889.0 mm, respectively. The ratio of stepped length of beam, α, the ratio of the flange width, β, and the ratio of the flange thickness, γ, are also defined in Fig. 1. Fig. 2 shows the singly stepped beams similar to Fig. 1. For singly stepped beams, the width and/or thickness of the flanges of the larger cross section were varied at one end.

From the geometry of typical existing stepped beams, ranges of the three ratios for α, β, and γ were established, and these ranges are given in Tables 1 and 2 with L_b/h = 21. Tables 1 and 2 show 27 doubly stepped beam models and 36 singly stepped beam models having various combination of the α, β, and γ ratios, respectively. The length, L_b, in Figs. 1 and 2 represents a typical unbraced length in a beam. At the ends of the unbraced length, the beam was free to warp. Tables 3 and 4 show parameters of the α, β, and γ ratios used to investigate length-to-height ratio effect of doubly and singly stepped beams in these finite-element method (FEM) analyses.

FEM eigenvalue analyses of 126 stepped beam models with L_b/h = 21 were performed to develop a design equation for obtaining the LTB moment resistance of stepped beams. The 27-parameter combination of Table 1 was investigated for doubly stepped beams subjected to a single point load and end moments. The 36-parameter combination of Table 2 was investigated for singly stepped beams subjected a single point load and end moments. The 252 beam models of Tables 3 and 4 were analyzed to investigate an expression to account for the change in LTB moment resistance as the change of L_b/h.

Table 1 Parameters used in the FEM analyses of doubly stepped beams (L_b/h = 21)

α (1)	β (2)	γ (3)
0.167	1.0	1.2
0.167	1.2	1.0; 1.4; 1.8
0.167	1.4	1.0; 1.4; 1.8
0.25	1.0	1.4; 1.8
0.25	1.2	1.0; 1.4; 1.8
0.25	1.4	1.0; 1.4; 1.8
0.333	1.0	1.2; 1.4; 1.6; 1.8
0.333	1.2	1.0; 1.4; 1.8
0.333	1.4	1.0; 1.2; 1.4; 1.6; 1.8

Table 2 Parameters used in the FEM analyses of singly stepped beams (L_b/h = 21)

α (1)	β (2)	γ (3)
0.167	1.0	1.2; 1.4; 1.8
0.167	1.2	1.0; 1.4; 1.8
0.167	1.4	1.0; 1.4; 1.8
0.25	1.0	1.2; 1.4; 1.8
0.25	1.2	1.0; 1.4; 1.8
0.25	1.4	1.0; 1.4; 1.8
0.333	1.0	1.2; 1.4; 1.8
0.333	1.2	1.0; 1.4; 1.8
0.333	1.4	1.0; 1.4; 1.8
0.5	1.0	1.2; 1.4; 1.8
0.5	1.2	1.0; 1.4; 1.8
0.5	1.4	1.0; 1.4; 1.8

Table 3 Parameters used in the length-to-height effect investigation of doubly stepped beams

L_b/h	α	β	γ
15;20;30;40	0.167	1.0	1.2; 1.4; 1.8
	0.167	1.2	1.0; 1.4; 1.8
	0.167	1.4	1.0; 1.4; 1.8
	0.25	1.0	1.2; 1.4; 1.8
	0.25	1.2	1.0; 1.4; 1.8
	0.25	1.4	1.0; 1.4; 1.8
	0.333	1.0	1.2; 1.4; 1.8
	0.333	1.2	1.0; 1.4; 1.8
	0.333	1.4	1.0; 1.4; 1.8

Table 4 Parameters used in the length-to-height effect investigation of singly stepped beams

L_b/h	α	β	γ
15;20;30;40	0.167	1.0	1.2; 1.4; 1.8
	0.167	1.2	1.0; 1.4; 1.8
	0.167	1.4	1.0; 1.4; 1.8
	0.25	1.0	1.2; 1.4; 1.8
	0.25	1.2	1.0; 1.4; 1.8
	0.25	1.4	1.0; 1.4; 1.8
	0.333	1.0	1.2; 1.4; 1.8
	0.333	1.2	1.0; 1.4; 1.8
	0.333	1.4	1.0; 1.4; 1.8
	0.5	1.0	1.2; 1.4; 1.8
	0.5	1.2	1.0; 1.4; 1.8
	0.5	1.4	1.0; 1.4; 1.8

3. Finite-element method results and design recommendation

From the results of the finite element investigation, the proposed design equation for stepped beams with continuous lateral top-flange bracing subjected to a single point load on the top flange and end moments is:

$$M_{st} = F_p C_b C_{st} M_{ocr} \tag{6}$$

in which C_b should be calculated using Eq. (5); $C_{st} = 0.9 + 6\alpha^2(\beta\gamma^{1.3} - 1)$ for doubly stepped beams and $C_{st} = 0.9 + 1.5\alpha^{1.6}(\beta\gamma^{1.2} - 1)$ for singly stepped beams with a negative end moment at each end, and $C_{st} = 1.25 + 6\alpha^2(\beta\gamma^{1.3} - 1)$ for doubly stepped beams and $C_{st} = 1.25 + 1.5\alpha^{1.6}(\beta\gamma^{1.2} - 1)$ for singly stepped beams with a negative end moment at one end; M_{ocr} = lateral-torsional buckling moment of an equal length prismatic beam having the smaller cross section along the entire span; $F_p = L_b/(20h)$ for doubly stepped beams, and $F_p = L_b/(40h) + 0.5$ for singly stepped beams; and α, β, and γ = ratios defining the relative length and relative width and thickness of the large and small cross sections, respectively.

Doubly and singly stepped beam cases investigated with continuous lateral top-flange bracing include beams with fixed-ends and propped cantilever beams along with a single point load. Comparisons between FEM results and the proposed solution for stepped beams with $L_b/h = 21$ are shown in Figs. 3 and 4. In those figures, the loading conditions and moment diagrams are presented, and the ratio of FEM results to the results from Eq. (6) are plotted against the value of α, β, and γ. If the predicted value from Eq. (6) is exactly same as the FEM result, the value on the vertical axis in these figures is 1. The data below line of 1 in the vertical axis indicate unconservative estimates with respect to FEM results. These figures show that the ratio of FEM results to predicted values increases as the ratio of flange thickness increases. These figures also show that the proposed solution gives conservative values for almost cases. For the doubly stepped beam cases, the maximum difference for an unconservative estimate is 3% with $\alpha = 0.33$, $\beta = 1.0$, and $\gamma = 1.2$ of the propped cantilever beam case. The maximum difference for a conservative estimate is 46% with

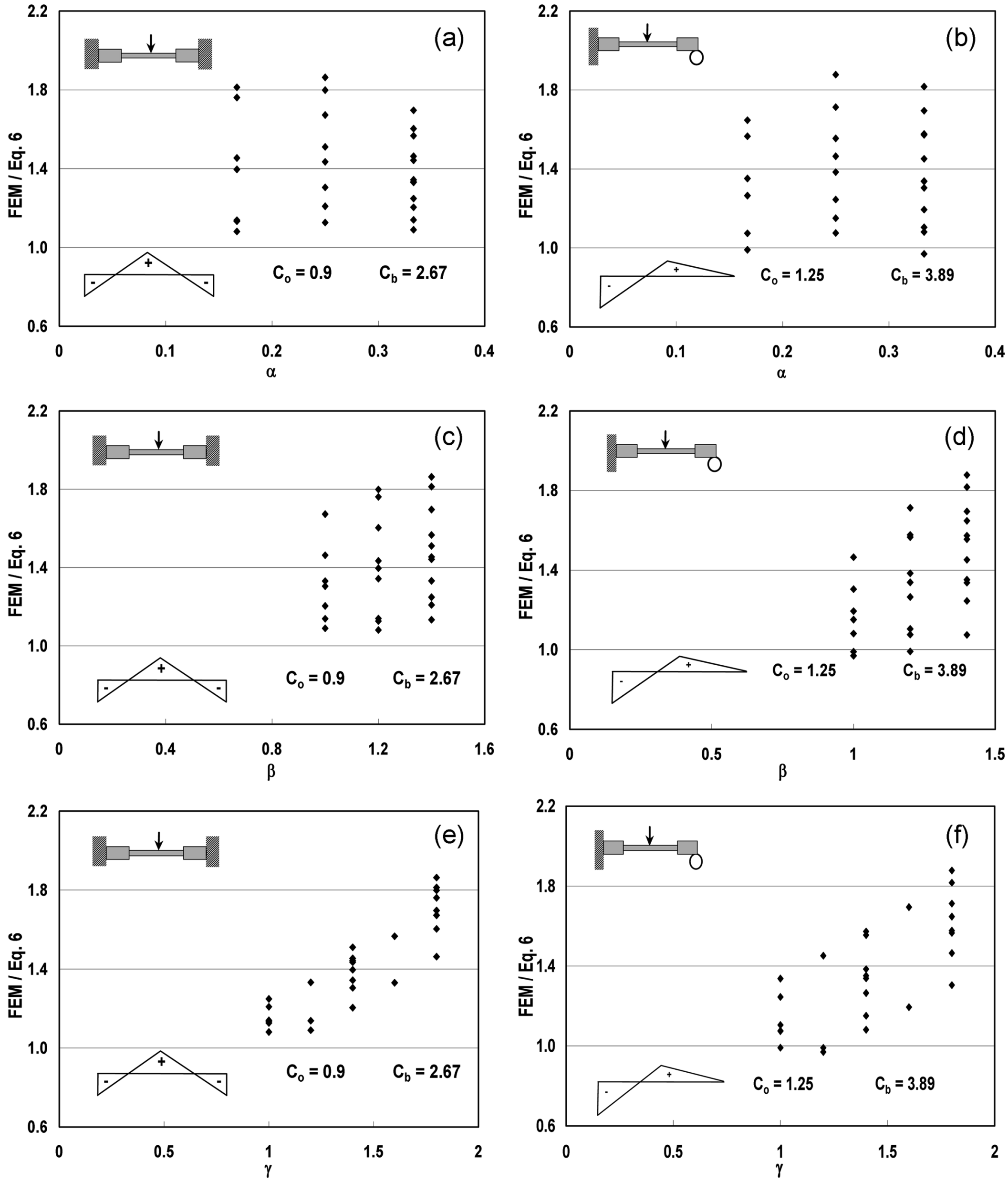

Fig. 3 Doubly stepped beams with a single point load and end moments

α = 0.25, β = 1.4, and γ = 1.8 of the fixed-ends beam case. For the singly stepped beams cases, the maximum difference for an unconservative estimate is 2% with α = 0.5, β = 1.0, and γ = 1.2 of the propped cantilever beam case. The maximum difference for a conservative estimate is 52% with α = 0.33, β = 1.4, and γ = 1.8 of the propped cantilever beam case.

Figs. 5 and 6 are graphs of L_b/h versus F_p. The values of F_p are obtained using the ratio of M_{st} from FEA results to $C_bC_{st}M_{ocr}$ from the proposed solution. Fig. 5 includes the FEA results for

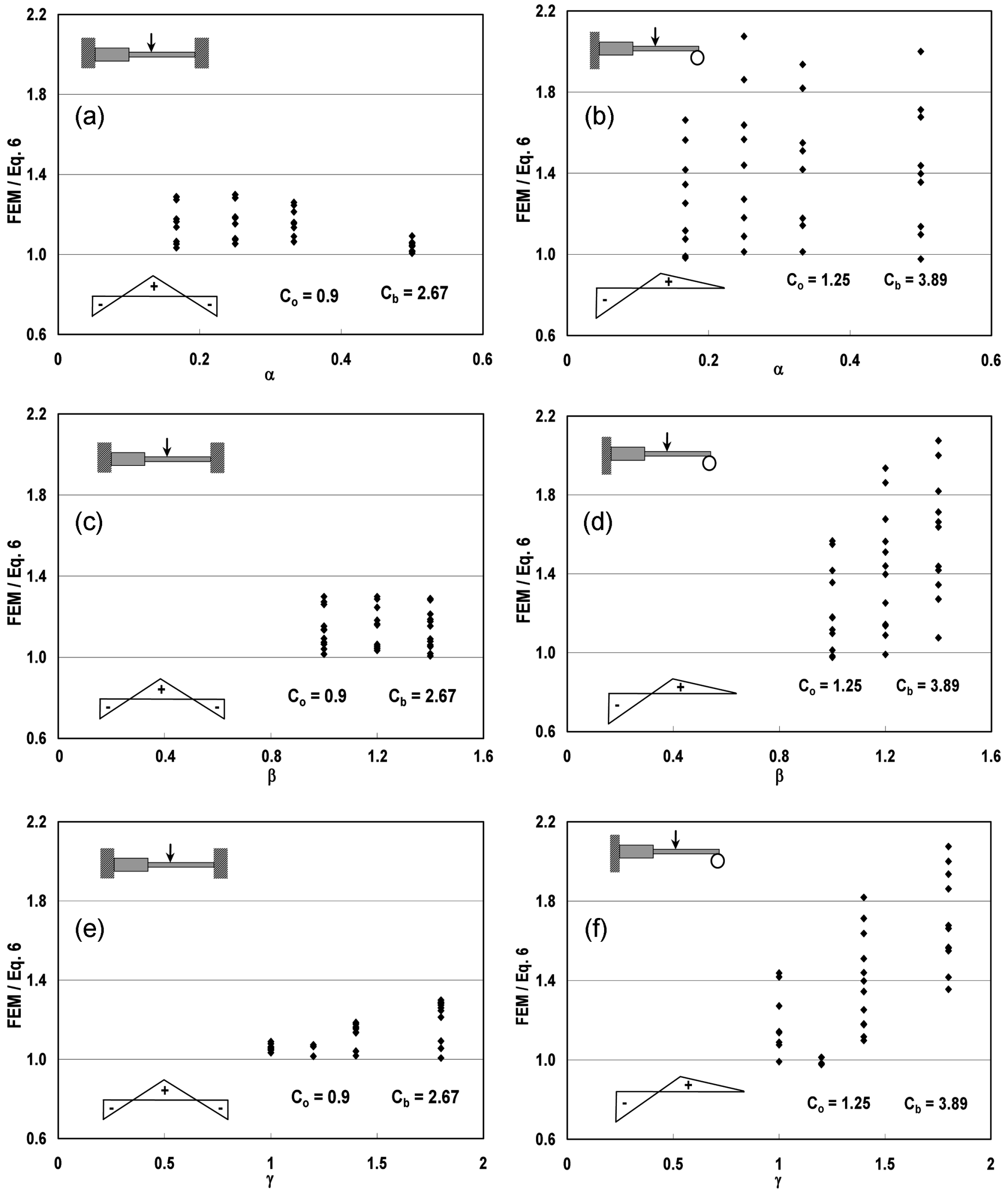

Fig. 4 Singly stepped beams with a single point load and end moments

doubly stepped beam models having the combination of α, β, and γ as shown in Table 3. Fig. 6 includes the FEA results for singly stepped beam models having the combination of α, β, and γ as shown in Table 4. Fig. 6(d) shows a comparison between the proposed solution and all FEM results from Fig. 6 (a, b and c) along with values for α = 0.50. These FEA results indicate that the proposed solutions give somewhat unconservative values for some stepped beams having α = 0.17,

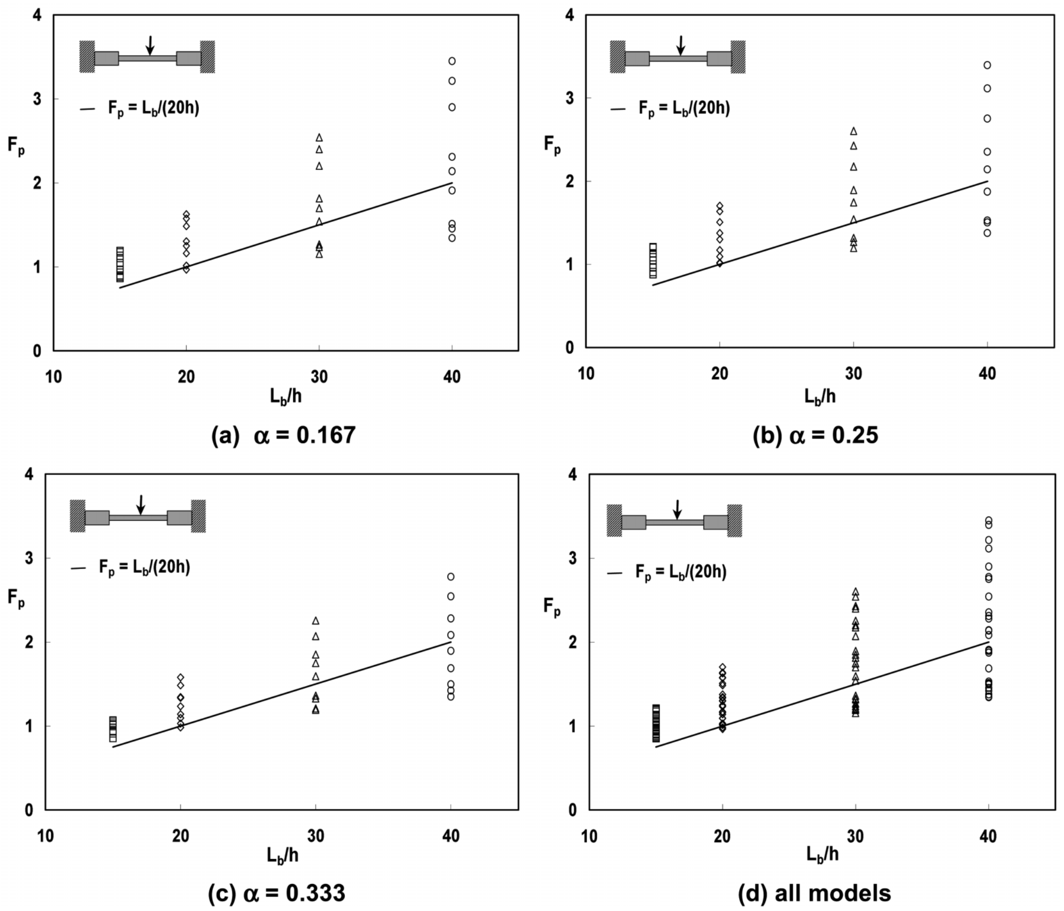

Fig. 5 Comparison of FEM results with length-to-height factor for doubly stepped beams

0.25, 0.33, or 0.5, β = 1.0, 1.2, or 1.4 and γ = 1.0 with L_b/h = 30 or 40. However, the proposed equations give reasonably accurate results that are conservative for almost all the cases having increased ratios of stepped flange thickness.

4. Applications

Existing continuous multi-span beam shown in Fig. 7 was considered to illustrate the calculation procedure for obtaining the lateral-torsional buckling moment strength of beams using the Eq. (6). Fig. 7 shows beam details, applied loading, bending moment diagram, and analytical models. The center span model of Fig. 7 is a doubly stepped beam with a negative end moment at each end and a single point load on the top flange. The end span model is a singly stepped beam with a negative end moment at one end and a single point load on the top flange. Bracing is initially assumed to be provided at the supports and continuous lateral top-flange bracing provided by slab or metal deck form is applied to the top flange of the models. The analytical beam models are free to warp at the

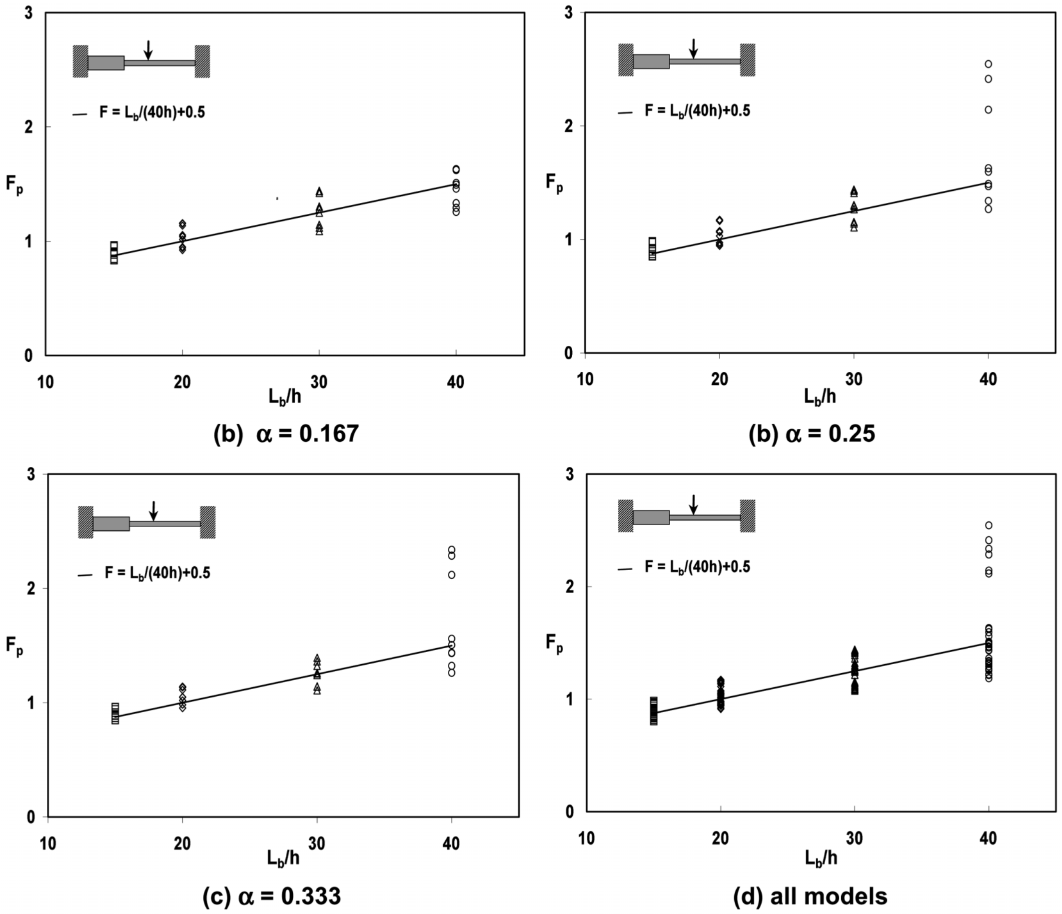

Fig. 6 Comparison of FEM results with length-to-height factor for singly stepped beams

ends of the unbraced length. All beams shown in Fig. 7 are of A36 (F_y = 2.5 MPa) steel (M_p = 2365 kN-m for W36 × 150 > M_{max} = 451 kN-m). Cross-section properties and material properties of W36 × 150 I-shaped steel beam used in the example are depth (d) = 910.6 mm, web thickness (t_w) = 15.9 mm, height (h) = $d - t_f$ = 894.7 mm, flange width (b_f) = 304.8 cm, flange thickness (t_f) = 23.9 mm, modulus of elasticity (E) = 200,000 MPa, Poisson's ratio (ν) = 0.3, and shear modulus of elasticity (G) = E/2(1 + ν) = 77,000 MPa.

4.1 Center span

$M_0 = M_1$ = 451 kN-m, M_{CL} = 417 kN-m, α = 3.05/24.38 = 0.13, β = 1.00,
γ = (23.9 + 19.1)/23.9 = 1.80, L_b (Unbraced length) = 24.38 m, L_b/h = 27.25,

$$M_{ocr} = \frac{\pi}{L_b}\sqrt{EI_yGJ + \left(\frac{\pi E}{L_b}\right)^2 I_y C_w} = 376 \text{ kN-m for W36} \times 150 \text{ with } L_b = 24.38 \text{ m},$$

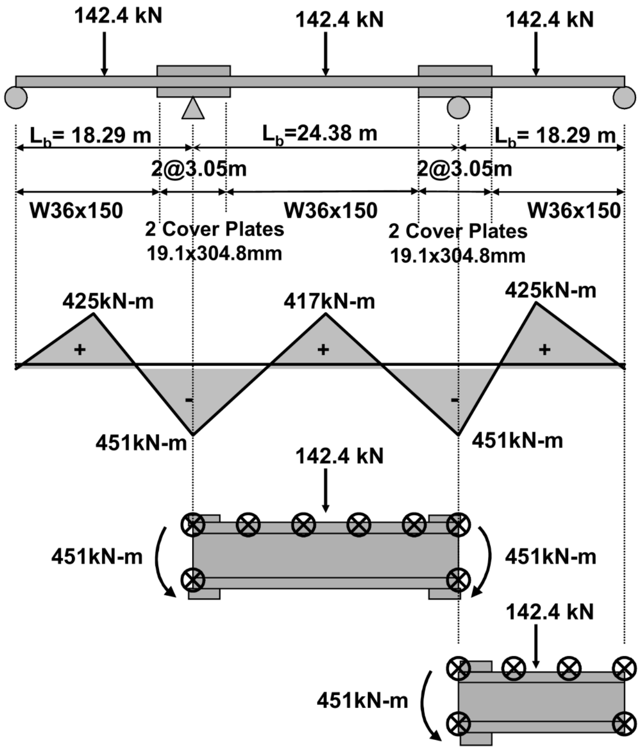

Fig. 7 Three-span continuous beam for example

$$C_b = 2.5 - \frac{2}{3}\left(\frac{M_1}{M_0}\right) + \frac{5}{3}\frac{M_{CL}}{(M_0 + M_1)} = 2.5 - \frac{2}{3}\left(\frac{451}{451}\right) + \frac{5}{3}\frac{417}{(451 + 451)} = 2.60,$$

$$C_{st} = 0.9 + 1.5\alpha^{1.6}(\beta\gamma^{1.2} - 1) = 1.00, \text{ and } F_p = \frac{L_b}{20h} = 1.36,$$

Therefore, $M_{st} = F\, C_b\, C_{st}\, M_{ocr}$ = (1.36) (2.60) (1.00) (376) = 1330 kN-m > $M_{\max}$ = 451 kN-m $\therefore$ OK
M_{st} from finite-element analysis = 2270 kN-m.

4.2 End span

M_0 = 451 kN-m, M_{CL} = 425 kN-m, M_1 = 0 kN-m, α = 3.05/18.29 = 0.17, β = 1.00, γ = 1.80,
L_b (Unbraced length) = 18.29 m, L_b/h = 20.44,

$$M_{ocr} = \frac{\pi}{L_b}\sqrt{EI_yGJ + \left(\frac{\pi E}{L_b}\right)^2 I_y C_w} = 538 \text{ kN-m for W36} \times 150 \text{ with } L_b = 18.29 \text{ m},$$

$$C_b = 2.5 - \frac{2}{3}\left(\frac{M_1}{M_0}\right) + \frac{5}{3}\frac{M_{CL}}{(M_0 + M_1)} = 2.5 - \frac{2}{3}\left(\frac{0}{451}\right) + \frac{5}{3}\frac{425}{(451 + 0)} = 4.07,$$

$$C_{st} = 1.25 + 1.5\alpha^{1.6}(\beta\gamma^{1.2} - 1) = 1.34, \text{ and } F_p = \frac{L_b}{40h} + 0.5 = 1.01,$$

Therefore, $M_{st} = F\, C_b\, C_{st}\, M_{ocr} = (1.01)\ (4.07)\ (1.34)\ (538) = 2963$ kN-m > $M_{\max} = 451$ kN-m $\therefore$ OK
M_{st} from finite-element analysis = 4780 kN-m.

5. Conclusions

Lateral-torsional buckling moment resistances of stepped beams with continuous lateral top-flange bracing under a single point load and negative end moments were investigated. The stepped beam factors, Eqs. (3) and (4), and the moment gradient correction factor, Eq. (5), suggested by Park e*t al.* (2003, 2004), were used to develop new design equations. From the investigation of the finite element analysis, new lateral buckling formula, Eq. (6), for beams with singly or doubly stepped member changes and with continuous lateral top-flange bracing subjected to a single point load on top flange and end moments were developed.

The Eq. (6) includes the length-to-height ratio factor, F_p, to account for the increase of lateral-torsional buckling moment resistance as the increase of length-to-height ratio of stepped beams. There are two kinds of F_p factor depending upon number of stepped members in the span of beams: singly stepped beams and doubly stepped beams. The F_p for doubly stepped beams is $L_b/(20h)$, and F_p for singly stepped beams is $L_b/(40h) + 0.5$. These expressions for F_p are valid for values of L_b/h from 15 to 40. The calculation examples for obtaining lateral-torsional buckling moment resistance using the new design equation, Eq. (6), indicate that engineers should easily determine the buckling capacity of the stepped beams.

If the example beams are subjected to other loading conditions such as uniformly distributed load or a series of point loads with negative end moments, the Eq. (6) would not be used to calculate lateral-torsional buckling strength. Further research is needed to develop a procedure for calculating the moment resistance when several loading conditions are applied to stepped beams.

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Notation

The following symbols are used in this paper:

C_b : moment gradient modifier;
C_{st} : stepped beam factor;
C_w : warping constant of beam;
E : modulus of elasticity of steel;
F_p : length-to-height ratio factor;
F_y : yield stress of steel;
G : shear modulus of elasticity of steel;
I_y : moment of inertia of beam about Y-axis;
J : St. Venant torsional constant for beam;
L_b : laterally unbraced length;
M_0 : end moment that produce the largest compressive stress on bottom flange;
M_1 : smaller end moment of beam;
M_{CL} : moment at centerline of segment;
$M_{\max}$: maximum moment in unbraced beam segment;
M_{ocr} : lateral-torsional buckling strength of prismatic beam under constant moment;
M_p : plastic bending moment;
M_{st} : lateral-torsional buckling strength of stepped beam;
α : ratio of stepped length along span;
β : ratio for defining the relative flange width of large and small cross section; and
γ : ratio for defining the relative flange thickness of large and small cross section.