*Structural Engineering and Mechanics, Vol. 18, No. 4 (2004) 461-474*
DOI: http://dx.doi.org/10.12989/sem.2004.18.4.461


# A generalized adaptive incremental approach for solving inequality problems of convex nature

M. M. Hassan† and F. F. Mahmoud‡

*College of Engineering, Zagazig University, Zagazig, Egypt*

*(Received August 2002, Accepted January 2003)*

**Abstract.** A proposed incremental model for the solution of a general class of convex programming problems is introduced. The model is an extension of that developed by Mahmoud *et al.* (1993) which is limited to linear constraints having nonzero free coefficients. In the present model, this limitation is relaxed, and allowed to be zero. The model is extended to accommodate those constraints of zero free coefficients. The proposed model is applied to solve the elasto-static contact problems as a class of variation inequality problems of convex nature. A set of different physical nature verification examples is solved and discussed in this paper.

**Key words:** inequality problems; convex programming problems; contact problems; finite element.

## 1. Introduction

Many variation inequality problems could be formulated as convex programming models, where both the objective function and inequality constraints, representing the kinematic boundary conditions, are convex functions. These convex programming models could represent various problems in engineering and applied mechanics. Usually, the convex programming problems could be formulated as follows:

$$\min_{\tilde{x}} f(\tilde{x}) = \frac{1}{2}\tilde{x}^t K\tilde{x} - \tilde{p}^t\tilde{x} \tag{1}$$

Subjected to: $g_i(\tilde{x}) = \tilde{C}_i^t\tilde{x} \le b_i,\ \ i = 1, 2, \ldots, m$.
Where, $f, g_i : R^n \to R$, $\tilde{x} \in R^n$ is the vector of the design variables, $K$ is a positive definite matrix of rank $n$, $\tilde{C}_i, \tilde{P} \in R^n$, and $b_i > 0$. It should be noted that both functions $f$ and $g$ are convex ones. The objective of the present paper is to address an adaptive incremental procedure to solve a more general class of convex programming models. In the present proposed model, the linear constraint coefficient $b_i$ is allowed to be greater than or equal to zero.

In the following section, a detailed theoretical formulation of the problem is given. Then, the procedure of the proposed model is presented.

† Associate Professor
‡ Professor

## 2. Formulation of the problem

The class of convex programming problems to be considered in this paper is defined as follows:

$$\min_{\tilde{x}} f(\tilde{x}) = \frac{1}{2}\tilde{x}^t K\tilde{x} - \tilde{p}^t\tilde{x} \tag{2}$$

Subjected to: $g_i(\tilde{x}) = \tilde{C}_i^t\tilde{x} \le b_i,\ \ i = 1, 2, \ldots, m$.
where, $f, g_i$: $R^n \to R$, $\tilde{x} \in R^n$ is the vector of the design variables, $K$ is a positive definite matrix of rank $n$, $\tilde{C}_i, \tilde{P} \in R^n$, and $b_i \ge 0$. However, the problem defined by Eq. (2) looks very similar to the problem defined by Eq. (1), but it should be noticed that the coefficient $b_i$ in this model is allowed to equal zero.

The Lagrange function for the above problem is defined as follows:

$$L(\tilde{x}, \tilde{u}) = f(\tilde{x}) + \tilde{u}^t g(\tilde{x}) \tag{3}$$

where $\tilde{u} \in R^m$ is a vector of nonnegative Lagrange multipliers.

$$\tilde{g}(\tilde{x}) = [g_1(\tilde{x})\ \ g_2(\tilde{x})\ \ \ldots\ \ g_m(\tilde{x})]^t = \tilde{C}^t\tilde{x} \tag{4}$$

where
$$\tilde{C} = [\tilde{C}_1\ \ \tilde{C}_2\ \ \ldots\ \ \tilde{C}_m] \tag{5}$$

By substituting $\tilde{g}(\tilde{x})$ from (4) into (3), we obtain

$$L(\tilde{x}, \tilde{u}) = f(\tilde{x}) + \tilde{u}^t C^t \tilde{x}$$

Then, there exists a Lagrange multipliers vector $\tilde{u}^*$ such that the Lagrangian function $L(\tilde{x}, \tilde{u})$ is stationary with respect to both $\tilde{x}$ and $\tilde{u}$. As the problem defined by Eq. (2) is convex, the global minimum point defined by $\tilde{x}^*$ and $\tilde{u}^*$ should satisfy the following Kuhn-Tucker (K-T) conditions, Arora (1989):

$$K\tilde{x}^* + C\tilde{u}^* = \tilde{P}, \quad C^t\tilde{x}^* \le \tilde{b}, \quad \tilde{u}_i^*(\tilde{C}_i^t\tilde{x}^* - b_i) = 0, \quad i = 1, 2, \ldots, m. \tag{6}$$

which represent the dual form of the problem.

Assume that at the minimum point $\tilde{x}^*$, a part of the constraints set is active and the other one is inactive. Then the constraint matrix may be partitioned as Mahmoud *et al.* (1993):

$$[C] = [C_A\ \ C_N] \tag{7}$$

where $C_A$ is the submatrix corresponding to the subset of active constraints and $C_N$ is the inactive constraints submatrix. According to the definition of the active and inactive constraints, the set of inequalities shown in Eq. (6) may be reformulated as:

$$\begin{bmatrix} \tilde{C}_A^t \\ \tilde{C}_N^t \end{bmatrix} \tilde{x}^* + \begin{bmatrix} \tilde{0} \\ \tilde{s} \end{bmatrix} = \begin{bmatrix} \tilde{b}_A \\ \tilde{b}_N \end{bmatrix} \tag{8}$$

where $\tilde{s}$ is a slack vector. Also, the Lagrange multiplier vector $\tilde{u}^*$ could be partitioned as:

$$\tilde{u}^* = \begin{bmatrix} \tilde{u}_A^* \\ \tilde{u}_N^* \end{bmatrix} \tag{9}$$

where $\tilde{u}_A^*$ is a nonzero vector, corresponding to the active constraints, but $\tilde{u}_N^*$ is a zero one which corresponds to the inactive set. By substitution of Eqs. (8) and (9) into Eq. (6) and rearrangement of the results, we obtain the dual form of the problem as:

$$\begin{bmatrix} K & C_A & C_N \\ C_A^t & 0 & 0 \\ C_N^t & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}^* \\ \tilde{u}_A^* \\ \tilde{u}_N^* \end{bmatrix} = \begin{bmatrix} \tilde{P} \\ \tilde{b}_A \\ \tilde{b}_N - \tilde{s} \end{bmatrix} \tag{10}$$

## 3. The solution procedure of the proposed model

According to the incremental model proposed by Mahmoud *et al*. (1993), the original model, defined by Eq. (2), is replaced by a sequence of models of the following type, Hassan *et al.*:

$$\min_{\tilde{x}} f(\tilde{x}) = \frac{1}{2}\tilde{x}^t K\tilde{x} - \tilde{p}_j^t \tilde{x} \tag{11}$$

Subjected to $C^t\tilde{x} \le b_j$, such that:

$$\sum_{j=1}^{L} \tilde{P}_j = \tilde{P} \quad \text{and} \quad \sum_{j=1}^{L} \tilde{b}_j = \tilde{b}$$

and the global minimum $\tilde{x}^*$ is defined by:

$$\tilde{x}^* = \sum_{j=1}^{L} \tilde{x}_j^*$$

where $j$ is the number of increment and $L$ is the total number of increments.

The total number of increments $L$ and the capacity of loading vector corresponding for each step are determined according to the following adaptive procedure.

1- The algorithm starts by assuming that the set of inactive constraints $N_I$, of the first model, contains all constraints, whether $b_i$ equals zero or not. All Lagrange multipliers corresponding to these constraints should be zero and the problem turns into:

$$K\tilde{x} = \tilde{p} \tag{12}$$

$$\tilde{x} = K^{-1}\tilde{p} \tag{13}$$

Now, we can compute the violation vector $\tilde{V}_1$ such that:

$$\tilde{V}_1 = \tilde{C}_N^t \tilde{x} - \tilde{b}_N \tag{14}$$

The component $i$ of the vector $\tilde{V}_1$ can be stated as:

$$\tilde{V}_{1,\,i} \;=\; \tilde{C}^t_{N,\,i}\tilde{x} - \tilde{b}_{N,\,i} \tag{15}$$

We can notice that if $V_{1,\,i}$, is negative, zero, or positive, the constraint number $i$ is still inactive, active or violated respectively. Select the maximum positive value of $V_{1,\,i}$ associated with the constraint to be active in the next step. Designating this constraint as $\alpha_1$, the adaptive scale factor $a_1$ required to establish this new active constraint is:

$$\alpha_1 \;=\; \frac{ba_1}{\tilde{C}^t_{N,\,a_1}\tilde{x}} \tag{16}$$

Now, the cardinality of the inactive set $N_I$ is reduced by one, where one of its constraints turns into active and joins the active constraints set. Consequently, $N_2$ and $A_2$ of the next model would be stated as:

$$\begin{aligned} N_2 \;&=\; N_1 - \{a_1\} \\ A_2 \;&=\; A_1 \cup \{a_1\} \end{aligned} \tag{17}$$

The values of $\tilde{P}_1$, $\tilde{b}_1$, $\tilde{x}^*_1$, and $\tilde{u}^*_{A,\,1}$ are computed as follows:

$$\begin{gathered} \tilde{P}_1 \;=\; \alpha_1\tilde{P}, \quad \tilde{b}_1 \;=\; \alpha_1\tilde{b} \\ \tilde{x}^*_1 = \alpha_1\tilde{x}^* \quad \text{and} \quad \tilde{u}^*_{A,\,1} \;=\; \alpha_1\tilde{u}^*_A \end{gathered} \tag{18}$$

Therefore, $\tilde{p}$ and $\tilde{b}$ should be updated as

$$\tilde{p} \leftarrow \tilde{p} - \tilde{p}_1 \quad \text{and} \quad \tilde{b} \leftarrow \tilde{b} - \tilde{b}_1 \tag{19}$$

2- Set $j = 2$, where $j$ identifies the step number which is equivalent to the model number.
3- Set up the next model, according to Eq. (10) as:

$$\begin{bmatrix} K & \tilde{C}_{A,\,j} \\ \tilde{C}^t_{A,\,j} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{u}_{A,\,i} \end{bmatrix} \;=\; \begin{bmatrix} \tilde{p} \\ \tilde{b}_{A,\,j} \end{bmatrix} \tag{20}$$

4- Solve Eq. (20) by any stationary iterative technique for $\tilde{x}$ and $\tilde{u}_A$, assuming that the initial trial values are:

$$\tilde{x}^o_j \;=\; \tilde{x}^*_{j-1}(1-\alpha_{j-1}) \;\text{ and }\; \tilde{u}^o_{A,\,j} = \tilde{u}^*_{A,\,j-1}(1-\alpha_{j-1}) \tag{21}$$

5- It is important to notice that the incremental value of $u_{A,\,i\,,\,j}$ of the constrain $i$ may be positive ornegative according to the proposed model. If the value of $\tilde{u}_{A,\,j}$ is negative, the active constraint corresponding to the negative value may be switched to an inactive one in the next step. To check this condition, we can carry out the following criterion: for $u_{A,\,i,\,j} < 0$, then if $|u_{A,\,i,\,j}| > u^*_{A,\,i,\,j-1}$, the active constraint tends to become an inactive one. Among the set of constraints that have negative incremental Lagrange multipliers, that one with maximum ratio of $|u_{A,i,j}|/|u^*_{A,i,j-1}|$ is selected as the next active constraint that would turn into an inactive one. Designating this constraint as $R_j$, the adaptive scale factor $\alpha_{Rj}$ required to establish this new inactive constraint is:

$$\alpha_{Rj} = \left|\frac{u^*_{A,Rj,j-1}}{u_{A,Rj,j}}\right|$$

Compute the violation components $V_{j,i}$ for each inactive constraint $i$ as:

$$V_{j,i} = \tilde{C}^t_{N,i} \cdot \tilde{x} - b_i, \qquad i = 1, 2, \ldots, m_a$$

where $m_a$ is the cardinality of the inactive set $N_j$ and $\tilde{C}_{N,j,i}$ is the coefficient vector of the inactive constraint $i$. Designating the inactive constraint $a_j$, candidate to be active in the next step, according to the following condition:

$$a_j \in N_{j-1}, \ V_{j,a_j} = \max_{i=1}^{m_a}(V_{j,i})$$

Compute the adaptive scale factor $\alpha_{a_j}$ required to establish a new active constraint:

$$\alpha_{a_j} = \frac{b_{a_j}}{\tilde{C}_{N,a_j}\tilde{x}}$$

6- From the above two scale factors, $\alpha_{R_j}, \alpha_{a_j}$ the adaptive scale factor, $\alpha$, for the next step is computed as

$$\alpha_j = \min(\alpha_{R_j}, \alpha_{a_j})$$

7- Compute $\tilde{x}^*_j = \alpha_j\tilde{x}$ and $u^*_{A,j} = \alpha_j\tilde{u}_{A,j}$

8- Update the intensities of the two vectors $\tilde{p}$ and $\tilde{b}$:

$$\tilde{p} \leftarrow \tilde{p}(1-\alpha_j) \quad \text{and} \quad \tilde{b} \leftarrow \tilde{b} - \alpha_j(\tilde{x}_j - \tilde{x}_{j-1})$$

9- Update the step number $j \leftarrow j + 1$
10- Repeat steps (5-9) until the intensity of the vector $\tilde{p}$ becomes zero.

## 4. Formulation of the contact problem as a convex programming model

Frictionless contact problems represent an important class of variation inequality problems (Duvaut and Lions 1976, Ciarlet 1978, Necas and Hlavacek 1981, Glowinski 1984, Panagiotopoulos 1985, Kikuchi and Oden 1988). The contact area, and consequently the kinematic boundary conditions along that area, are not known apriori. Furthermore, contact states depend basically on the capacity of loads, geometry and relative material compliance. Therefore, by changing loading capacity, a few boundary conditions may be relaxed and others would be added. Accordingly, those types of problems are highly non-linear ones having inequality type of boundary conditions.

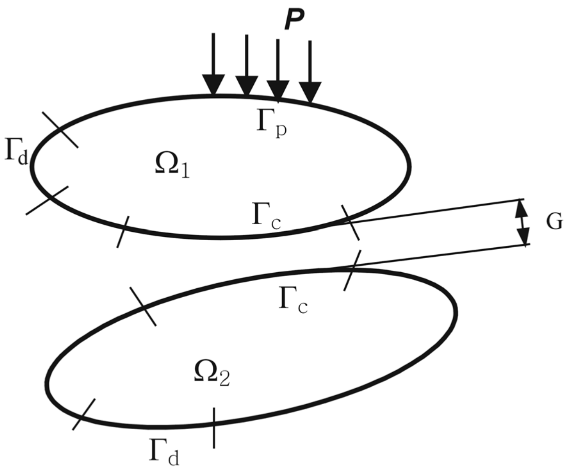



Fig. 1 Contact of two linear elastic bodies

Consider two elastic bodies, shown in Fig. 1, being subjected to static loads. The boundary $\Gamma$ of each body is assumed to consist of three disjoint parts, $\Gamma_D$, $\Gamma_p$ and $\Gamma_c$. $\Gamma_D$ and $\Gamma_p$ are the portions of the boundary on which the displacements and traction are prescribed, respectively. $\Gamma_c$ is the candidate contact area; i.e., $\Gamma_c$ is a portion of the boundary that contains the adjacent surfaces which may come into contact upon the application of loads. It should be noted that the boundary $\Gamma_c$ consists of two parts, $\Gamma_{c_1}$ where advancing contact is prescribed and $\Gamma_{c_2}$ where receding contact is prescribed.

The convex programming model is given by:

Minimize the potential energy $F_k(\{x\}) = \frac{1}{2}\{x\}_k^t[K]\{x\}_k - \{x\}_k^t\{P\}_k$

Subjected to the following non-interpenetration constraints:

$$x_{in}^k - x_{jn}^k - G_{ij}^k < 0 \quad \text{on inactive } \Gamma_c$$

$$x_{in}^k - x_{jn}^k - G_{ij}^k \geq 0 \quad \text{on active } \Gamma_c$$

$$i, j = 1, 2, 3, \ldots NC$$

where

$$\{x\}^t = \{x_1, x_2, x_3 \ldots x_n\}$$
$$\{P\}^t = \{p_1, p_2 \ldots P_{n1}\}$$
$$\{G\} = \{g_1, g_2, g_3 \ldots g_m\}$$

$[K]$ is the overall stiffness matrix of domain, which is symmetric, $x_{in}$ and $x_{jn}$ are the normal components of the displacements of node pair $i$ and $j$, and $NC$ is the number of candidate contact pairs. $\{G\}$ is the gap vector between the two contacting surfaces. $\{p\}$ is the applied external force vector.

## 5. Numerical examples

Several examples of different natures are presented and solved using the incremental convex programming models. The first problem represents the beam resting on elastic foundation with an initial gap. The second problem represents the pin joints truss on misaligned supports. Both problems are belonging to a class of non-conformal contact. The third problem detects the history of contact behavior of beam on elastic foundation. The last one represents a more complicated one of beam on gap foundation with upper barrier. Those last couple of problems represent a more realistic problem having both conformal and non-conformal contact type.

### *5.1 Example 1: Beam resting on elastic foundation*

A typical example of beam resting on elastic foundation shown in Fig. 2 is solved to represent the validity of the proposed model to accommodate of constraints of zero free coefficient. Table 1 shows the data of beam resting on Winkler springs.

The formulation of the problem

$$\min_x f(\tilde{x}) = \frac{1}{2}\tilde{x}^t K\tilde{x} - \tilde{p}^t\tilde{x}$$

Subjected to

$$x_1 - x_2 \le 0, \quad x_3 - x_4 \le 0 \quad \text{and} \quad x_5 - x_6 \le 1$$

The load $p$ is applied vertically at node 7 and, equal to 10 (F), the global solution of the problem is given in Table 2.

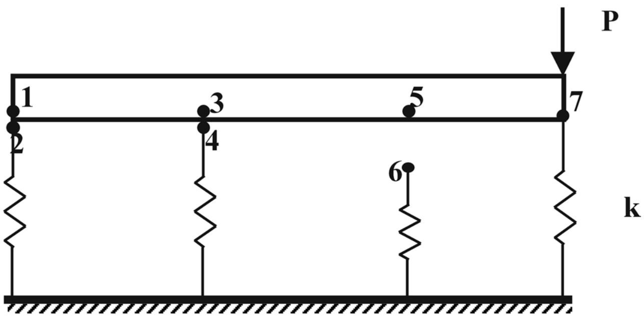



Fig. 2 Beam resting on elastic foundation

Table 1 Data of beam/spring model

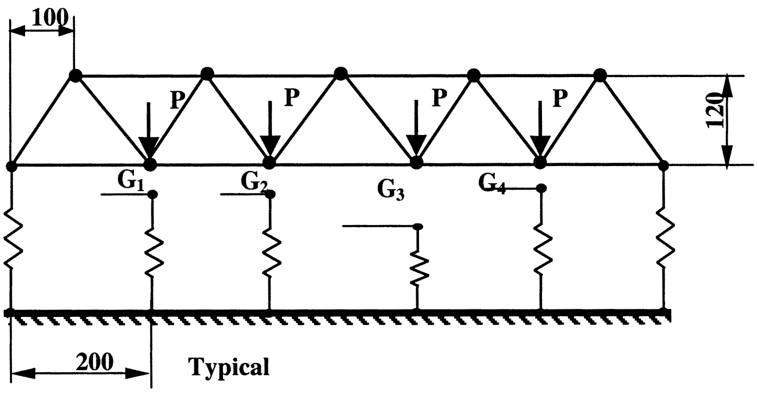
| | |
|---|---|
| Beam: Young's modulus | = 1 (F/$L^2$) |
| Moment of inertia | = 1 ($L^4$) |
| Half length | = 3 (L) |
| Node spacing | = 1 (L) |
| Spring: Stiffness | = 1 (F/m) |

Table 2 Results of beam on Winkler springs

| Candidate pair | Gap | First node vertical disp. | Second node vertical disp. | Contact | Contact force (Lagrange multiplier) | (% age) Load |
|---|---|---|---|---|---|---|
| 1 | 0 | 2.4802 ▲ 1 | 0 2 | No | 0 | 0 |
| 2 | 0 | –0.9075 ▼ 3 | –0.9075 ▼ 4 | Yes | 0.9075 | 0 |
| 3 | 1 | –4.1438 ▼ 5 | –3.1438 ▼ 6 | Yes | 3.1438 | 16.47 |

## 5.2 *Example 2: Truss on misaligned supports*

A pin jointed truss on multiple spring supports is loaded by concentrated loads as shown in Fig. 3, Mahmoud *et al*. (1982) and Mahmoud (1984). The interior springs have an elastic stiffness of $1.0 \times 10^6$ (F/L). The truss members are steel and have a section area of $10(L^2)$. The contact is initially assumed at only the two end supports. Table 3 illustrates the contact status after the loading and displacement of pair of nodes of every candidate pair. Also the table shows the contact force (Lagrange multiplier) of the contact pairs and the percent of the total load distribution at which the contact event occurs.




$G_1$ = 0.8, $G_2$ = 0.9, $G_3$ = 1.2, $G_4$ = 0.5 and $P$ = 4000 (F)

Fig. 3 Truss on misaligned supports

Table 3 Results for truss on misaligned supports

| Candidate pair | Gap | First node vertical disp. | Second node vertical disp. | Contact | Contact force (Lagrange multiplier) | (% age) Load |
|---|---|---|---|---|---|---|
| 1 | 0.8 | –0.599 | 0.0 | No | - | - |
| 2 | 0.9 | –0.9192 | –0.0192 | Yes | 19200 | 67.2 |
| 3 | 1.2 | –0.9072 | 0.0 | No | - | - |
| 4 | 0.5 | –0.5362 | –0.0362 | Yes | 36200 | 59.2 |

### *5.3 Example 3: Beam on elastic foundation*

The model is used to detect the response of a beam on elastic foundation. Subgrade reaction is simulated by a typical Winkler springs of stiffness $k$ = 15000 (F/L). The initial settlement is 0.005 (L) and extent a distance of 4 (L) about the symmetrical line Fig. 4, Mahmoud (1983). The load is increased monotonically up to 4000 (F), keeping track of the response of the system in both region of contact and advancing. Fig. 5 illustrates the corresponding finite element model. Table 4 shows the history of contact behavior vs load (F).

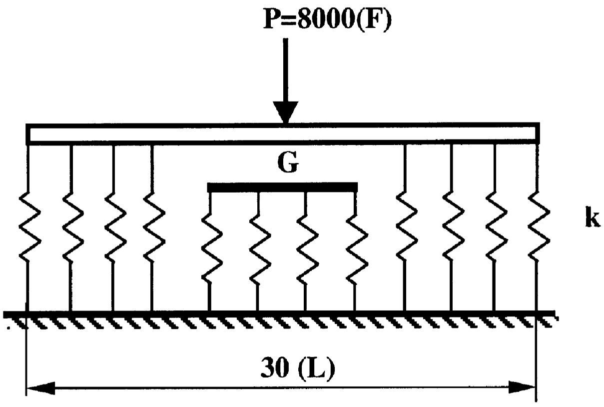



Fig. 4 Two elastic strata in contact with initial gap

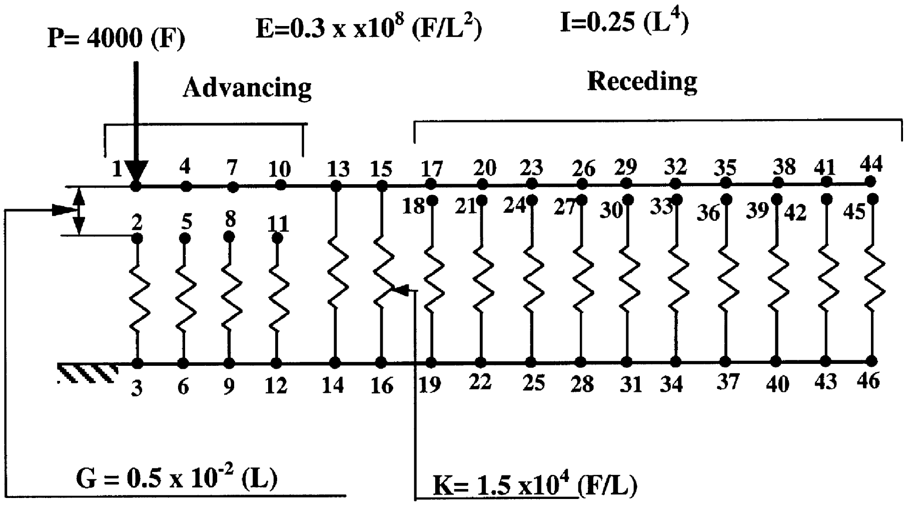



Fig. 5 Finite element model for the beam on elastic foundation

Table 4 History of contact for a beam on elastic foundation

| Load (F) | Pair | Contact type | Release/Contact |
|---|---|---|---|
| 0 | 17-18 | R | C |
| 0 | 20-21 | R | C |
| 0 | 23-24 | R | C |
| 870 | 1-2 | A | C |
| 936 | 4.5 | A | C |
| 936 | 23-24 | R | S |
| 1318 | 7-8 | A | C |
| 1318 | 20-21 | R | S |
| 2369 | 10-11 | A | C |

R = Receding contact, A = Advancing contact, S = Release, C = Contact

### *5.4 Example 4: Beam on gap foundation with upper barrier*

Several problem configurations involving a beam have been solved and Fig. 6 illustrates the most complicated one, Mahmoud *et al.* (1986). The constraints, consisting at most of a foundation and upper barrier with gaps, react unilaterally against the beam. The load $P$ is increased monotonically, causing the beam to travel through a sequence of contact states or scenarios. The solution is generated by the general adaptive incremental approach. The load $P$ is applied to conventional beam elements which are appropriately constrained by spring elements (the Winkler model). A symmetric half of the configuration in Fig. 6 is modeled in Fig. 7 with the undeflected beam resting in its initial state, $P$ = 0. Pertinent data of the problem is provided in Table 5 except that the stiffness of those springs located at end-regions of the model is halved to better approximate a continuous foundation. The beam is weightless.

The problem is begun with at most two springs connected and the center slope set to zero. In what follows, $c$ denotes the total contact length of noted portions of the beam, while the total contact length $\overline{c}$ of the beam on a level foundation equal to 0.31992 m. The center deflection $\overline{D}$ of

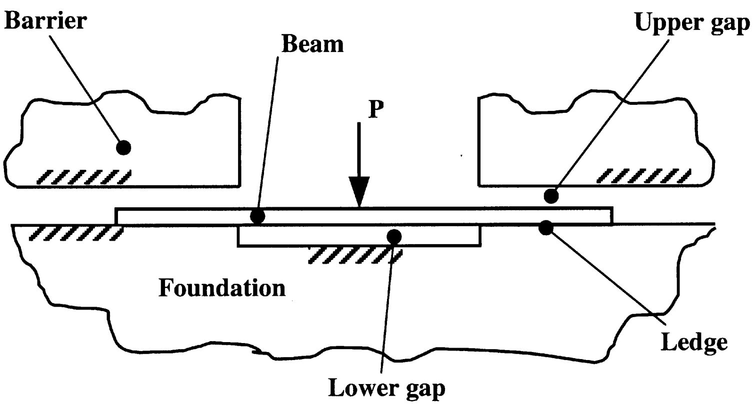



Fig. 6 Beam on foundation with gap and upper barriers

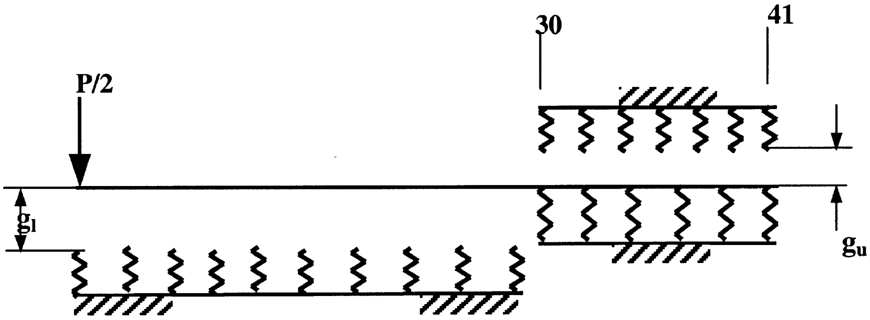



Fig. 7 The discrete beam/spring model

Table 5 Data of beam/spring model

| | | |
|---|---|---|
| Beam: | Young's modulus | $= 0.710 \times 10^{11}$ N/m$^2$ |
| | Moment of inertia | $= 1.041 \times 10^{-7}$ m$^4$ |
| | Half length | = 0.381 m |
| | Node spacing | $= 9.525 \times 10^{-3}$ m |
| Spring: | Stiffness | $= 2.627 \times 10^6$ N/m |
| | $g_l = g_u$ | = 0.002 m |

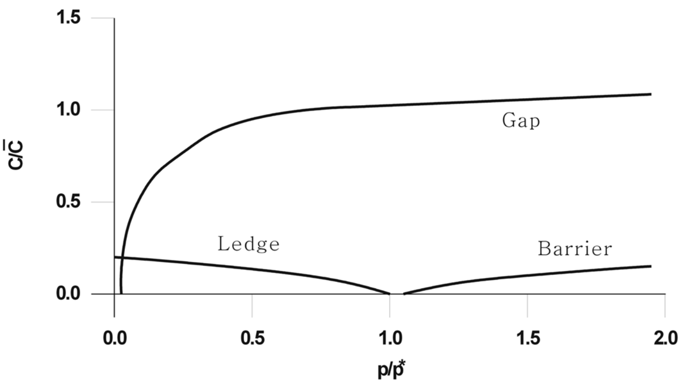



Fig. 8 Contact/load history for beam on foundation with gap and upper barriers

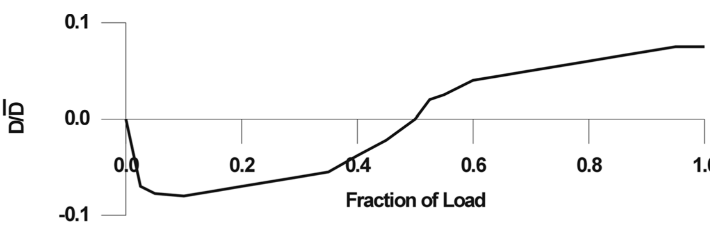



Fig. 9 Beam deflection at station 30 versus fraction of load

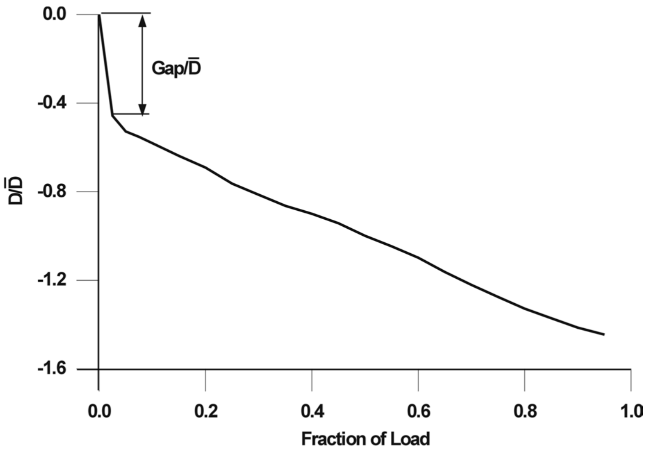



Fig. 10 Center deflection of beam versus fraction of load

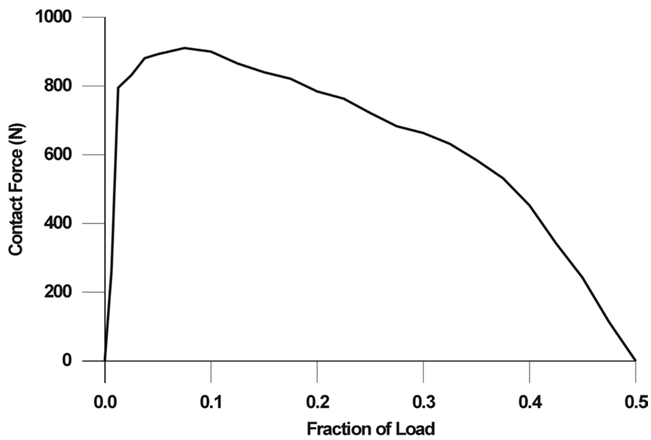



Fig. 11 Beam contact force at station 30 versus fraction of load

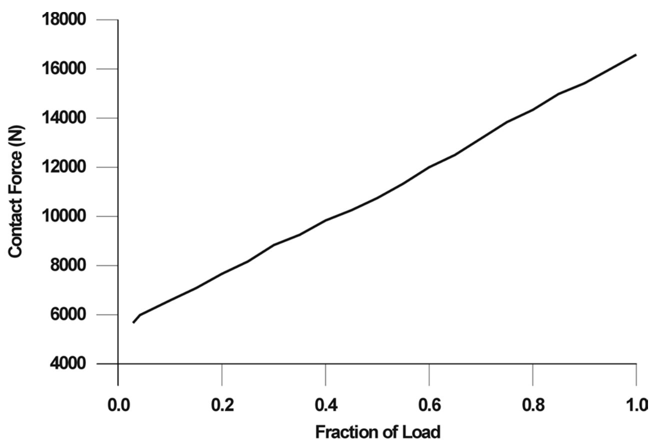



Fig. 12 Center contact force of beam versus fraction of load

0.31992 m is due to a load $P$ of $2.224 \times 10^5$ N. Fig. 8 shows the history of contact length versus load where $c/\overline{c}$ is plotted against $p/p^*$, $p^*$ the load required to just lift the beam off the ledge.

It is interesting to trace the deflections of the beam, denoted by $D$, as a function of load and also the contact forces. In Figs. 9 and 10, the paths followed by a point on the beam at station 30 under the load are plotted. On the other hand, Figs. 11 and 12 show the contact forces versus fraction of load at station 30 under the load.

## 6. Conclusions

An adaptive incremental procedure is presented to solve a general class of variational inequality problems of convex nature. The procedure could accommodate linear constraints of zero or non-zero free coefficients.

The suggested model is applied to contact problems that represent an important class of variational inequality problems. Several problems of conformal, non-conformal and multiphase contact are solved precisely. Both displacements and contact forces, which have been represented by Lagrange multipliers, were computed through this procedure.

The history of evolution of contact and separation states versus the monotonic variation of loading capacity is monitored easily.

## References

Arora, S. (1989), *Introduction to Optimum Design*, McGrow-Hill.

Ciarlet, P.G. (1978), *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam.

Duvaut, G. and Lions, J.L. (1976), *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin.

Glowinski, R. (1984), *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York.

Hassan, M.M. and Mahmoud, F.F., "An adaptive incremental approach for the solution of general convex programming models", To Appear.

Kikuchi, N. and Oden, J.T. (1988), "Contact problems in elasticity: A study of variational inequalities and finite element methods", SIAM Stud. Appl. Math. 8, SIAM, Philadelphia.

Mahmoud, F.F. (1983), "Analysis of receding and advancing contact by the automated direct method", *2nd Pedac' 83 International Conference*: Dept. of Production Eng. Univ. of Alexandria, Egypt, December.

Mahmoud, F.F. (1984), "Analysis of unbonded frictionless contact problems by means of penalty method", *Fifth Engineering Division Specialty Conference*, EM Division ASCE Laramee, Wyoming, August 1-3.

Mahmoud, F.F., Alsaffar, A.K. and Hassan, K.A. (1993), "An adaptive incremental approach for the solution of convex programming models", *Math. Comput. Simul.*, **35**, 501-508.

Mahmoud, F.F., Salamon, N.J. and Marks, W.R. (1982), "A direct automated procedure for frictionless contact problems", *Int. J. Num. Meth. Engng.*, **18**, 245-257.

Mahmoud, F.F., Salamon, N.J. and Pawlak, T.P. (1986), "Simulation of structural elements in receding/advancing contact", *Comput. Struct.*, **22**, 629-635.

Necas, J. and Hlavacek, I. (1981), "Mathematical theory of elastic and elastoplastic bodies", An Introduction, Elsevier, Amsterdam.

Panagiotopoulos, P.D. (1985), *Inequality Problems in Mechanics and Applications*, Birkhauser, Basel.

## Notation

| | |
|---|---|
| $a$ | : The inactive constraint candidate to be active in the next step |
| $C_A$ | : Set of active constraints |
| $C_N$ | : Set of inactive constraints |
| $G$ | : Gap vector |
| $K$ | : Positive difinite matrix of rank $n$ |
| $\mathrm{L}(\tilde{x}, \tilde{u})$ | : Lagrange function |
| $m_a$ | : The cardinality of the inactive set |
| $\tilde{P}$ | : Load vector |
| $\tilde{s}$ | : Slack vector |
| $\tilde{u}^*$ | : Lagrange multipliers vector |
| $\tilde{u}_A^*$ | : Nonzero vector, corresponding to the active constraints |
| $\tilde{u}_N^*$ | : Zero vector, corresponding to the inactive constraints |
| $V$ | : Violation vector |
| $\tilde{x}$ | : The vector of the design variables |
| $\Gamma$ | : Boundary of the domain |
| $\Gamma_D$ | : Region of prescribed displacement |
| $\Gamma_P$ | : Region of prescribed traction |
| $\Gamma_C$ | : Candidate contact area |
| $\alpha$ | : Scale factor |
| $\alpha_{aj}$ | : Scale factor required to establish a new active constraint |
| $\alpha_{Rj}$ | : Scale factor required to establish a new inactive constraint |