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Unconditional stability for explicit pseudodynamic testing

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Abstract. In this study, a newly developed unconditionally stable explicit method is employed to solve momentum equations of motion in performing pseudodynamic tests. Due to the explicitness of each time step this pseudodynamic algorithm can be explicitly implemented, and thus its implementation is simple when compared to an implicit pseudodynamic algorithm. In addition, the unconditional stability might be the most promising property of this algorithm in performing pseudodynamic tests. Furthermore, it can have the improved properties if using momentum equations of motion instead of force equations of motion for the step-by-step integration. These characteristics are thoroughly verified analytically and/or numerically. In addition, actual pseudodynamic tests are performed to confirm the superiority of this pseudodynamic algorithm.

Key words: unconditional stability; explicit method; pseudodynamic test; error propagation.

1. Introduction

It is generally recognized that experimental testing of a nonlinear structure can provide more reliable results than those obtained from analytical methods. This is because that the accuracy of the nonlinear response highly depends upon simplified assumptions used in analytical methods, such as the discretization of the structure and the mathematical model of the load-displacement relationship. In general, shaking table tests can give realistic response simulation. However, the size and mass of a specimen are constrained due to the limitation of electro-hydraulic system since the displacement, velocity and acceleration of the shaking table are limited in frequency content and magnitude by the characteristics of the controlling electro-hydraulic system. It seems that the pseudodynamic testing method can overcome many limitations in the shaking table testing, while using the same facilities for cyclic loading testing. Therefore, the pseudodynamic testing method is promising in evaluating dynamic behaviors of structural systems.

Since a step-by-step integration method is required to perform the step-by-step integration in a pseudodynamic test, the problems of inaccuracy and/or instability (Bathe and Wilson 1973, Belytschko and Hughes 1983) for an integration method is also experienced in performing a pseudodynamic test. Consequently, many efforts have been made to improve the accuracy of test results and to overcome numerical instability for the pseudodynamic testing (Chang and Mahin 1992, Chang 1997, 2001a,b, 2002a,b, 2003, Hilber *et al.* 1977, Nakashima *et al.* 1990, Shing and

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Mahin 1987, Shing *et al.* 1991, Thewalt and Mahin 1995). In order to overcome the conditional stability for using an explicit integration method to perform a pseudodynamic test some unconditionally stable implicit pseudodynamic algorithms were proposed (Chang and Mahin 1992, Nakashima *et al.* 1990, Shing *et al.* 1991, Thewalt and Mahin 1995). However, their implementations are more complicated than that for an explicit pseudodynamic algorithm (Chang 1997, 2001a, 2002a, Chang *et al.* 1998, Shing and Mahin 1987). Thus, it will be very promising for an explicit pseudodynamic algorithm if it can have unconditional stability. An explicit pseudodynamic algorithm with unconditional stability has been implemented by Chang (2002a). The feasibility of explicit implementation and unconditional stability of this pseudodynamic algorithm were thoroughly verified through actual pseudodynamic tests. On the other hand, a time integration technique (Chang 2001a,b, 2002b, Chang *et al.* 1998) has been also proposed to improve the accuracy of a pseudodynamic test. The most important aspect of this technique is the application of a step-by-step integration method to solve the momentum equations of motion, which is the resultant of the time integration of the force equations of motion.

Another unconditionally stable explicit pseudodynamic algorithm has been further proposed. In order to have favorable characteristics arising from the time integration, this technique (Chang 2001a,b, 2002b, Chang *et al.* 1998) is adopted to implement the proposed pseudodynamic algorithm. The feasibility of the explicitness of each time step will be addressed and an analytical proof of the unconditional stability is provided. Furthermore, it is analytically proved that it has superior error propagation properties over those of the previously developed algorithm (Chang 2002a), which is referred to as Chang explicit method herein for convenience. Numerical examples and verification tests are used to confirm its superior properties for pseudodynamic tests. In this paper, it is implicitly assumed that force equations of motion are solved by the Newmark explicit method and Chang explicit method and momentum equations of motion are solved by the proposed explicit method, as the three methods are mentioned.

2. Proposed explicit method

It has been shown that there are several advantages for the use of the time integration technique. Details can be found in references (Chang 2001a,b, 2002b, Chang *et al.* 1998) and will not be elaborated herein. This technique relies upon the step-by-step solution of the momentum equations of motion instead of the force equations of motion in conducting pseudodynamic tests. The momentum equations of motion can be directly derived from the dynamic equilibrium of momentum or the time integration of the force equations of motion. In general, it can be expressed as

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{C}\mathbf{u} + \mathbf{K}\bar{\mathbf{u}} = \bar{\mathbf{f}} \tag{1}$$

where **M**, **C** and **K** are the mass, viscous damping and stiffness matrices; $\dot{\mathbf{u}}$, $\mathbf{u}$, $\bar{\mathbf{u}}$ and $\bar{\mathbf{f}}$ are the nodal vectors of velocity, displacement, time integral of displacement and time integral of external force, respectively.

In this study, the proposed explicit method is applied to solve Eq. (1) in performing a pseudodynamic test and its general formulation can be written as

$$
\begin{aligned}
\mathbf{M}\mathbf{v}_{i+1} + \mathbf{C}\mathbf{d}_{i+1} + \bar{\mathbf{r}}_{i+1} &= \bar{\mathbf{f}}_{i+1} \\
\mathbf{s}_{i+1} &= \beta_0 \mathbf{s}_i + \beta_1 (\Delta t) \mathbf{d}_i + \beta_2 (\Delta t)^2 \mathbf{v}_i \\
\mathbf{d}_{i+1} &= \mathbf{d}_i + \frac{1}{2}(\Delta t)(\mathbf{v}_i + \mathbf{v}_{i+1})
\end{aligned} \tag{2}
$$

where $\bar{\mathbf{r}}_i$ and $\bar{\mathbf{f}}_i$ are the time integrals of the restoring force vector and the external force vector at the i-th step, respectively; $\mathbf{s}_i$, $\mathbf{d}_i$ and $\mathbf{v}_i$ are approximations corresponding to $\bar{\mathbf{u}}(t_i)$, $\mathbf{u}(t_i)$ and $\dot{\mathbf{u}}(t_i)$ respectively. In addition, the coefficient matrices β_0, β_1 and β_2 are defined as

$$
\begin{aligned}
\beta_0 &= \left[\mathbf{I} + \frac{1}{2}(\Delta t)\mathbf{C}\mathbf{M}^{-1} + \frac{1}{4}(\Delta t)^2 \mathbf{K}_0 \mathbf{M}^{-1}\right]^{-1} \left[\mathbf{I} + \frac{1}{2}(\Delta t)\mathbf{C}\mathbf{M}^{-1}\right] \\
\beta_1 &= \left[\mathbf{I} + \frac{1}{2}(\Delta t)\mathbf{C}\mathbf{M}^{-1} + \frac{1}{4}(\Delta t)^2 \mathbf{K}_0 \mathbf{M}^{-1}\right]^{-1} \left[\mathbf{I} + \frac{1}{4}(\Delta t)\mathbf{C}\mathbf{M}^{-1}\right] \\
\beta_2 &= \left[\mathbf{I} + \frac{1}{2}(\Delta t)\mathbf{C}\mathbf{M}^{-1} + \frac{1}{4}(\Delta t)^2 \mathbf{K}_0 \mathbf{M}^{-1}\right]^{-1} \left(\frac{1}{4}\right)
\end{aligned} \tag{3}
$$

where $\mathbf{I}$ is an identity matrix. It is worthwhile to note that $\mathbf{K}_0$ is the initial stiffness matrix and will be employed to determine the coefficient matrices β_0, β_1 and β_2, which remain unchanged for a whole pseudodynamic test.

3. Explicit pseudodynamic implementation

In order to address that the proposed pseudodynamic algorithm can be explicitly implemented, its implementation details for the $(i+1)$-th time step are presented next. Unlike the conventional pseudodynamic implementation, the initial stiffness matrix of the specimen must be first determined before the pseudodynamic test so that the inverse matrix of $\left[\mathbf{I} + \frac{1}{2}(\Delta t)\mathbf{C}\mathbf{M}^{-1} + \frac{1}{4}(\Delta t)^2 \mathbf{K}_0 \mathbf{M}^{-1}\right]$ can be computed and thus the coefficient matrices β_0, β_1 and β_2 are determined. In general, the direct stiffness method is usually used to experimentally determine the initial stiffness matrix. In order to explicitly obtain the displacement vector $\mathbf{d}_{i+1}$ for the next time step the following computing procedure is adopted. After multiplying the second line of Eq. (2) by the current stiffness matrix $\mathbf{K}$, one can have

$$
\bar{\mathbf{r}}_{i+1} = \beta_0 \bar{\mathbf{r}}_i + \beta_1 (\Delta t) \mathbf{r}_i + \beta_2 (\Delta t)^2 \mathbf{K}\mathbf{v}_i \tag{4}
$$

where $\bar{\mathbf{r}}_i = \mathbf{K}\mathbf{s}_i$. It should be mentioned that the use of the current stiffness matrix $\mathbf{K}$ is just to transform the second line of Eq. (2) into Eq. (4) and it will be eliminated later since it is not determined in a pseudodynamic test. Substituting this equation and the third line of Eq. (2) into the first line of Eq. (2), one has

$$
\mathbf{v}_{i+1} = \left[\mathbf{M} + \frac{1}{2}(\Delta t)\mathbf{C}\right]^{-1} \left[\bar{\mathbf{f}}_{i+1} - \mathbf{C}\mathbf{d}_i - \frac{1}{2}(\Delta t)\mathbf{C}\mathbf{v}_i - \beta_0 \bar{\mathbf{r}}_i - \beta_1 (\Delta t)\mathbf{r}_i - \beta_2 (\Delta t)^2 \mathbf{K}\mathbf{v}_i\right] \tag{5}
$$

Further substitution of this equation into the third line of Eq. (2), the displacement vector for the next time step is

$$\mathbf{d}_{i+1} = \mathbf{d}_i + \frac{1}{2}(\Delta t)\mathbf{v}_i + \frac{1}{2}(\Delta t)\left[\mathbf{M} + \frac{1}{2}(\Delta t)\mathbf{C}\right]^{-1}\left[\bar{\mathbf{f}}_{i+1} - \mathbf{C}\mathbf{d}_i - \frac{1}{2}(\Delta t)\mathbf{C}\mathbf{v}_i - \beta_0\bar{\mathbf{r}}_i - \beta_1(\Delta t)\mathbf{r}_i - \beta_2(\Delta t)^2\mathbf{K}\mathbf{v}_i\right] \quad (6)$$

In this equation, the current stiffness $\mathbf{K}$ is required to determine the displacement vector for the next time step. Apparently, it is not feasible for a pseudodynamic test since it is very difficult to obtain accurately in each time step and thus is usually not determined during the test. As a result, an alternative is applied to overcome this difficulty. In fact, the initial stiffness matrix $\mathbf{K}_0$ is used to take place of the current stiffness matrix $\mathbf{K}$. Consequently, Eq. (6) becomes

$$\begin{aligned}\mathbf{d}_{i+1} &= \mathbf{d}_i + \frac{1}{2}(\Delta t)\mathbf{v}_i \\ &+ \frac{1}{2}(\Delta t)\left[\mathbf{M} + \frac{1}{2}(\Delta t)\mathbf{C}\right]^{-1}\left[\bar{\mathbf{f}}_{i+1} - \mathbf{C}\mathbf{d}_i - \frac{1}{2}(\Delta t)\mathbf{C}\mathbf{v}_i - \beta_0\bar{\mathbf{r}}_i - \beta_1(\Delta t)\mathbf{r}_i - \beta_2(\Delta t)^2\mathbf{K}_0\mathbf{v}_i\right]\end{aligned} \quad (7)$$

It is worth noting that the use of the initial stiffness matrix $\mathbf{K}_0$ to replace the current stiffness matrix $\mathbf{K}$ will lead to insignificant errors since it is only a high-order term when compared to the rest terms.

After obtaining the displacement vector $\mathbf{d}_{i+1}$ for the next time step, it can be imposed upon the specimen through servo hydraulic actuators. During the movement of the actuators the restoring forces are measured several times and then are integrated to obtain $\bar{\mathbf{r}}_{i+1}$, where the trapezoidal rule can be simply applied to perform the time integration. It is apparent that linearization errors caused by the variation of the resistance within the time step can be significantly reduced through the time integration of the restoring force. Finally, the velocity vector $\mathbf{v}_{i+1}$ can be obtained by substituting $\mathbf{d}_{i+1}$ and $\bar{\mathbf{r}}_{i+1}$ into the first line of Eq. (1) and is

$$\mathbf{v}_{i+1} = \mathbf{M}^{-1}(\bar{\mathbf{f}}_{i+1} - \mathbf{C}\mathbf{d}_{i+1} - \bar{\mathbf{r}}_{i+1}) \quad (8)$$

This pseudodynamic test procedure can be repeated until the desired time history is completed. It is clear that the pseudodynamic implementation of this algorithm is essentially the same as that for the Newmark explicit method.

4. Numerical characteristics

To obtain numerical characteristics of the proposed explicit method a spectral decomposition technique (Clough and Penzien 1993, Strang 1986) is often used to analyze the step-by-step solution of a linearly elastic single degree of freedom system. For this purpose, the single degree of freedom analogues of Eq. (2) are

$$\begin{aligned} m v_{i+1} + c d_{i+1} + \bar{r}_{i+1} &= \bar{f}_{i+1} \\ s_{i+1} &= \beta_0 s_i + \beta_1(\Delta t) d_i + \beta_2(\Delta t)^2 v_i \\ d_{i+1} &= d_i + \frac{1}{2}(\Delta t)(v_i + v_{i+1}) \end{aligned} \quad (9)$$

where m and c are the mass and viscous damping; $\bar{r}_i$ and $\bar{f}_i$ are the time integrals of the restoring force and the external force at the i-th step, respectively; s_i, d_i and v_i correspond to the time integral of displacement, displacement and velocity. In addition, coefficient matrices β_0, β_1 and β_2 reduce to scalars for a single degree of freedom system and are

$$\beta_0 = \frac{1+\xi\Omega}{1+\xi\Omega+\frac{1}{4}\Omega^2}, \quad \beta_1 = \frac{1+\frac{1}{2}\xi\Omega}{1+\xi\Omega+\frac{1}{4}\Omega^2}, \quad \beta_2 = \frac{\frac{1}{4}}{1+\xi\Omega+\frac{1}{4}\Omega^2} \tag{10}$$

where $\omega = \sqrt{k/m}$, $\Omega = \omega(\Delta t)$ and $\xi = c/(2m\omega)$. Since the system is assumed to be linear elastic, the stiffness k is constant and is equal to the initial stiffness k_0. Therefore, the expression of Ω can be used to take place of Ω_0, where $\Omega_0 = \omega_0(\Delta t)$ and $\omega_0 = \sqrt{k_0/m}$. It is worth noting that the time integral of the restoring force shown in Eq. (9) can be alternatively expressed as $\bar{r}_{i+1} = ks_{i+1}$ for a linear elastic system.

For an undamped single degree of freedom system, the step-by-step integration procedure as shown in Eq. (9) can be rewritten in a recursive matrix form as

$$\bar{\mathbf{X}}_{i+1} = \mathbf{A}\bar{\mathbf{X}}_i + \mathbf{L}\bar{f}_{i+1} \tag{11}$$

where $\bar{\mathbf{X}}_i = [s_i \;\; (\Delta t)d_i \;\; (\Delta t)^2 v_i]^T$. Explicit expressions of the amplification matrix $\mathbf{A}$ and the load vector $\mathbf{L}$ are found to be

$$\mathbf{A} = \frac{1}{1+\frac{1}{4}\Omega^2}\begin{bmatrix} 1 & 1 & \frac{1}{4} \\ -\frac{1}{2}\Omega^2 & 1-\frac{1}{4}\Omega^2 & \frac{1}{2} \\ -\Omega^2 & -\Omega^2 & -\frac{1}{4}\Omega^2 \end{bmatrix}, \quad \mathbf{L} = \left[\frac{(\Delta t)^2}{m}\right]\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \tag{12}$$

The characteristic equation for the amplification matrix $\mathbf{A}$ can be obtained from $|\mathbf{A}-\lambda\mathbf{I}| = 0$ and is found to be

$$\lambda\left(\lambda^2 - \frac{2-\frac{1}{2}\Omega^2}{1+\frac{1}{4}\Omega^2}\lambda + 1\right) = 0 \tag{13}$$

This characteristic equation is exactly the same as that for the constant average acceleration method and Chang explicit method. This implies that they have exactly the same numerical properties, such as unconditional stability and second-order accuracy (Bathe and Wilson 1973, Belytschko and Hughes 1983). It should be mentioned that previous studies (Chang 2001a, Chang *et al.* 1998) concluded that the basic numerical characteristics of an integration method to solve momentum equations of motion are exactly the same as those to solve force equation of motion.

The period distortion of an integration method is usually evaluated by the relative period error, which is defined as $|(\bar{T}-T)/T|$. The symbol T represents the true period of the system while $\bar{T}$

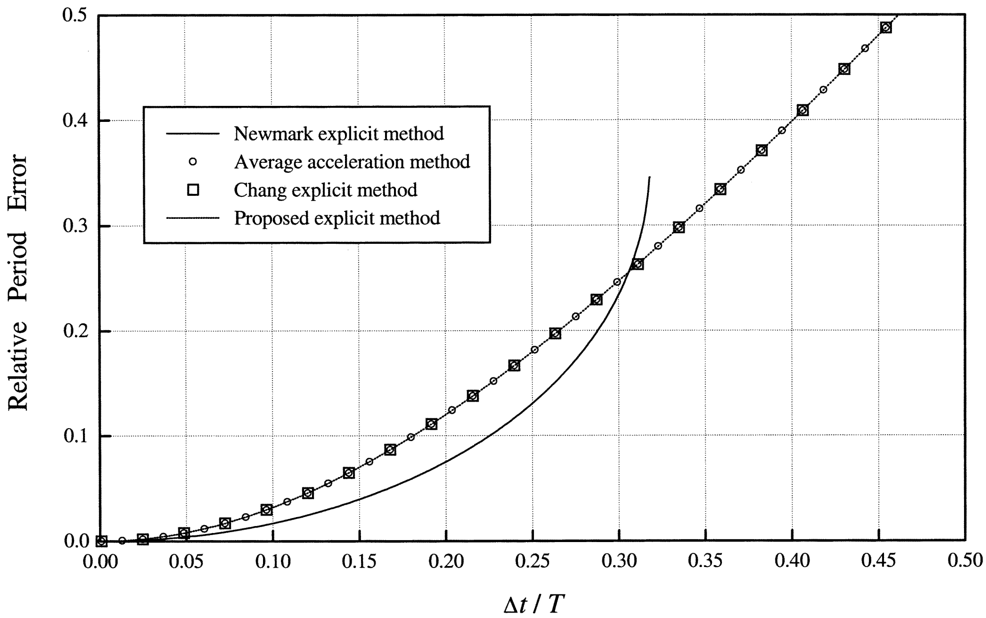

Fig. 1 Comparison of characteristics of period distortion

denotes the computed period in a numerical procedure. Variations of the relative period errors with $\Delta t/T$ for the Newmark explicit method, constant average acceleration method (Bathe and Wilson 1973, Belytschko and Hughes 1983, Newmark 1959), Chang explicit method and proposed explicit method are plotted in Fig. 1. Apparently, the relative period error for the proposed explicit method is exactly the same as that of the Chang explicit method and the constant average acceleration method. It should be mentioned that absolute values of the relative period errors are considered in this figure for the convenience of comparisons. In fact, the proposed explicit method, Chang explicit method and constant average acceleration method generally lead to period elongation while the Newmark explicit method results in period shrinkage.

5. Error propagation properties

Error propagation properties of the proposed pseudodynamic algorithm can be obtained after the error propagation analysis of the step-by-step solution of a linear elastic system (Shing and Mahin 1987, 1990). In this analysis, errors in displacement and restoring force are introduced into the step-by-step integration procedure and these errors will be carried over to the subsequent time steps. In fact, it is impossible to exactly impose computed displacements upon the test structure due to displacement control errors and thus lead to incorrect restoring forces (Shing and Mahin 1987, 1990). Furthermore, in addition to displacement control errors, measurement errors might be introduced in measuring the actually developed restoring forces. Thus, error propagation is that incorrect imposed displacements lead to incorrect restoring forces and these incorrect restoring forces result in incorrect displacements, which will be imposed upon the test specimen for the next time step. This step-by-step error propagation procedure can be formulated in a recursive matrix form and be analyzed by a spectral decomposition technique. As a result, a cumulative equation for displacement error can be achieved. The following notations are defined for the convenience of subsequent derivations and discussions.

d_i = exact numerically computed displacement at step i without errors.
d_i^e = exact displacement at step i, including the effects of errors at previous steps.
d_i^a = actual displacement at step i, including the effects of previous errors and errors introduced at the current step.
r_i = exact numerically computed restoring force at step i without errors.
r_i^e = exact restoring force at step i, including the effects of errors at previous steps.
r_i^a = actual restoring force at step i, including the effects of previous errors and errors introduced at the current step.
e_i^d = displacement error introduced at step i.
e_i^r = force error introduced at step i.

Using the above notations, it is straightforward to obtain the following relationships in performing a pseudodynamic test

$$\begin{aligned} d_{i+1}^a &= d_{i+1}^e + e_{i+1}^d \\ r_{i+1}^a &= r_{i+1}^e + e_{i+1}^r \\ \bar{r}_{i+1}^a &= \bar{r}_{i+1}^e + e_{i+1}^{\bar{r}} \end{aligned} \tag{14}$$

where the restoring force feedback error e_{i+1}^r and the time integral of the restoring force feedback error $e_{i+1}^{\bar{r}}$ can be also expressed as

$$\begin{aligned} e_{i+1}^r &= k e_{i+1}^{rd} \\ e_{i+1}^{\bar{r}} &= k(\Delta t) e_{i+1}^{rd} \end{aligned} \tag{15}$$

It is apparent that the term e_{i+1}^{rd} is used to represent the amount of displacement error corresponding to e_{i+1}^r for a linear elastic system. The second line of this equation indicates that the displacement error introduced in each time step is constant within the time step. If the actual restoring forces and displacements, which include the errors introduced in each time step, are used for an actual test, the recursive matrix form shown in Eq. (11) can be reformulated as

$$\bar{\mathbf{X}}_{i+1}^e = \mathbf{A}\bar{\mathbf{X}}_i^e + \mathbf{L}\bar{f}_{i+1} + \mathbf{M}(\Delta t)e_i^d - \mathbf{N}(\Delta t)e_{i+1}^{rd} \tag{16}$$

where $\bar{\mathbf{X}}_i^e = [s_i^e \ \ (\Delta t)d_i^e \ \ (\Delta t)^2 v_i^e]^T$ and the vectors **M** and **N** are found to be

$$\mathbf{M} = \left(\frac{1}{1+\frac{1}{4}\Omega^2}\right)\begin{bmatrix} 1 \\ 1-\frac{1}{4}\Omega^2 \\ -\Omega^2 \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} 0 \\ 0 \\ \Omega^2 \end{bmatrix} \tag{17}$$

for the proposed pseudodynamic algorithm. After subtracting Eq. (11) from Eq. (16), the following error vector can be obtained

$$\bar{\varepsilon}_{n+1} = \sum_{i=0}^{n} \mathbf{A}^{(n-i)} \mathbf{M}(\Delta t) e_i^d - \sum_{i=0}^{n} \mathbf{A}^{(n-i)} \mathbf{N}(\Delta t) e_{i+1}^{rd} \tag{18}$$

where $\bar{\varepsilon}_{n+1} = \overline{\mathbf{X}}_{n+1}^e - \overline{\mathbf{X}}_{n+1}$ and each element $(\bar{\varepsilon}_{n+1})_j$, where j = 1, 2 and 3, are defined as

$$\begin{aligned} (\bar{\varepsilon}_{n+1})_1 &= s_{n+1}^e - s_{n+1} \\ (\bar{\varepsilon}_{n+1})_2 &= (\Delta t)(d_{n+1}^e - d_{n+1}) = (\Delta t) e_{n+1}^d \\ (\bar{\varepsilon}_{n+1})_3 &= (\Delta t)^2 (v_{n+1}^e - v_{n+1}) \end{aligned} \tag{19}$$

In these derivations, $\varepsilon_0 = \bar{\varepsilon}_0 = \mathbf{0}$ is assumed since no errors will be introduced at the beginning of the test. The first term on the right hand side of Eq. (18) is the cumulative error due to the displacement feedback errors and the second term is that due to the force feedback errors.

Substituting the explicit expressions of **A**, **M** and **N** into Eq. (18) and using a spectral decomposition technique (Shing and Mahin 1990), the cumulative equation to describe the error propagation of the proposed explicit pseudodynamic algorithm can be derived. In fact, the displacement cumulative error for the proposed explicit method is found to be

$$e_{n+1}^d = E_d \sum_{i=0}^{n} \cos[(n-i+1)\overline{\Omega}] e_i^d - E_r \sum_{i=0}^{n-1} \cos\left[\left(n-i+\frac{1}{2}\right)\overline{\Omega}\right] e_{i+1}^{rd} \tag{20}$$

where

$$E_d = 1, \qquad E_r = \frac{\frac{1}{2}\Omega^2}{\sqrt{1+\frac{1}{4}\Omega^2}} \tag{21}$$

On the other hand, the displacement cumulative error for using the Newmark explicit method (Newmark 1959) and the Chang explicit method (2002a) to pseudodynamically solve force equations of motion has been derived and can be generally expressed as

$$e_{n+1}^d = E_d \sum_{i=0}^{n} \cos\left[\left(n-i+\frac{1}{2}\right)\overline{\Omega}\right] e_i^d - E_r \sum_{i=0}^{n-1} \sin[(n-i)\overline{\Omega}] e_{i+1}^{rd} \tag{22}$$

where

$$E_d = \frac{1}{\sqrt{1-\frac{1}{4}\Omega^2}}, \qquad E_r = \frac{\Omega}{\sqrt{1-\frac{1}{4}\Omega^2}} \tag{23}$$

for the Newmark explicit method while for the Chang explicit method they are

$$E_d = \sqrt{1+\frac{1}{4}\Omega^2}, \qquad E_r = \Omega \tag{24}$$

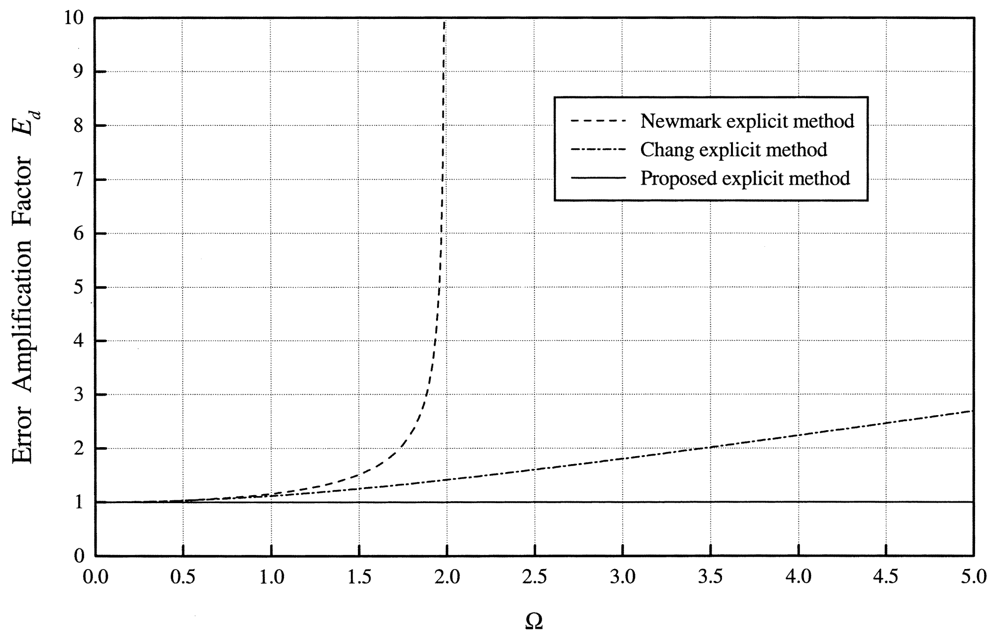

Fig. 2 Error amplification factor for displacement feedback error

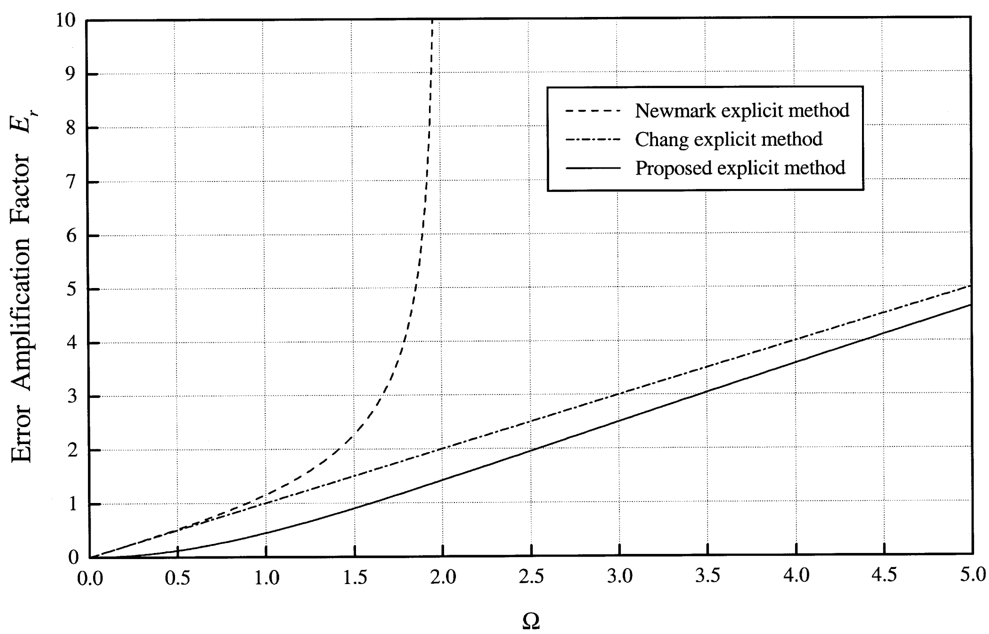

Fig. 3 Error amplification factor for restoring force feedback error

Variations of error amplification factors E_d and E_r with Ω are shown in Figs. 2 and 3 for the Newmark explicit method, Chang explicit method and proposed explicit method. It is clear from Fig. 2 that the error amplification factor E_d for the Newmark explicit method increases from 1 to ∞

as Ω is increased from 0 to its upper stability limit 2 while for the Chang explicit method it is gradually increased from 1 to ∞ as Ω increases from 0 to ∞. However, for the proposed explicit method, the amplification factor E_d is always equal to 1 for any value of Ω. On the other hand, Fig. 3 shows that the error amplification factor E_r for the Newmark explicit method increases from 0 to ∞ with the increasing value of Ω from 0 to its upper stability limit 2. However, for the Chang explicit method and proposed explicit method, it gradually increases from 0 to ∞ as Ω increases from 0 to ∞. It is also found that the error amplification factor E_r for the proposed explicit method is less than that for the Chang explicit method for any value of Ω. As a summary, error amplification factors, both E_d and E_r, for the proposed explicit method are not only less than those for the Newmark explicit method for any value of Ω but also for the Chang explicit method.

6. Numerical examples

Analytical results from previous investigations conclude that the proposed explicit method to solve momentum equations of motion is unconditionally stable, and its error propagation properties are superior to those for the Newmark explicit method and Chang explicit method in the solution of force equations of motion pseudodynamically. In order to confirm these results, some numerical examples are investigated next.

6.1 Unconditional stability

Numerical solutions of a 3-story shear-beam type structure subject to a ground excitation of sin(5t) are used to confirm the unconditional stability of the proposed explicit method. The lumped mass for each story is assumed to be $m_1 = m_2 = m_3 = 1$ kg while the stiffness from the bottom story to the top story is taken as 10^2, 10^4 and 10^6 N/m. As a result, the natural frequencies of this structure are found to be 5.8, 123 and 1416 rad/sec. Apparently, this structure is intentionally designed to have a very high third mode so that the unconditional stability of the proposed explicit method can be indicated.

Both the Newmark explicit method and proposed explicit method are used to obtain responses to the ground excitation, and numerical solutions are plotted in Fig. 4. Results obtained from the Newmark explicit method with a time step of $\Delta t = 0.001$ sec are considered as "**exact**" solutions for comparisons. Apparently, numerical explosions occur very early for the use of $\Delta t = 0.002$ sec if using the Newmark explicit method while the proposed explicit method can still provide stable solutions even though the time step is as large as $\Delta t = 0.05$ sec. This is because that the Newmark explicit method is only conditionally stable and its upper stability limit 2 requires the time step to be less than 0.0014 sec, which is determined from the third mode. On the other hand, unconditional stability of the proposed explicit method is strongly indicated by the stable computations since the value of Ω corresponding to the time step of $\Delta t = 0.05$ sec is as large as 70.8 for the third mode.

6.2 Error propagation

Error propagation properties of the proposed explicit pseudodynamic algorithm can be revealed through numerical simulations. In this example, the displacement error introduced in the imposed displacement in each time step is considered as a random variable, whose distribution is assumed to

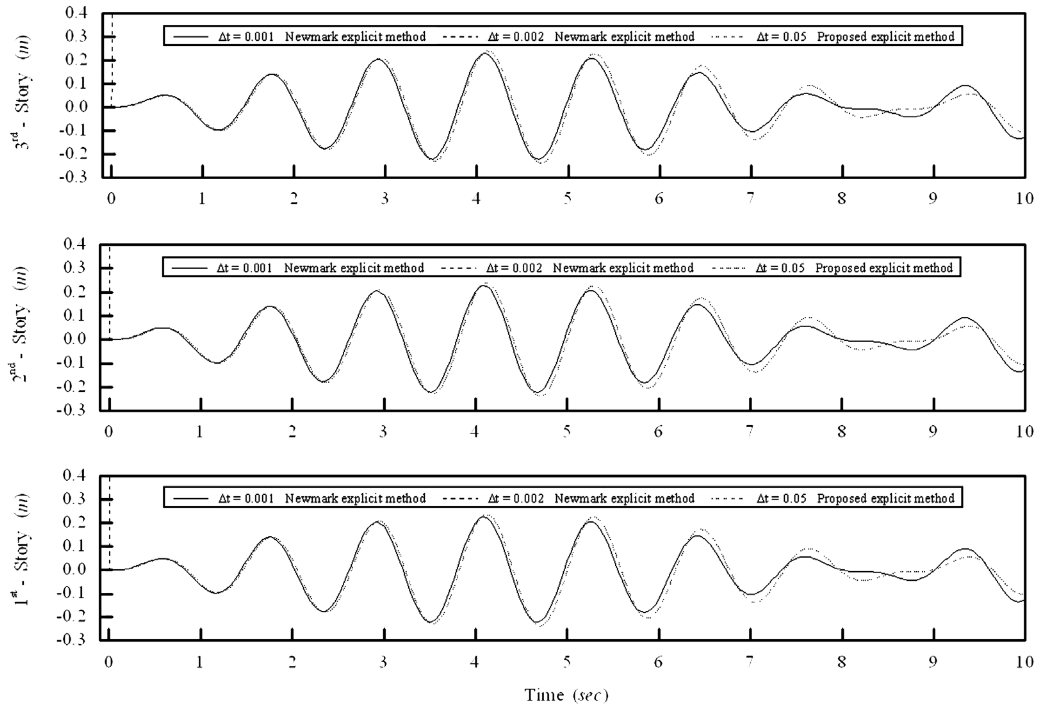

Fig. 4 Forced vibration response to ground acceleration of sin(5*t*)

be a truncated normal distribution. Simulation details can be found in (Chang 2002a) and will not be elaborated. To simulate a properly adjusted pseudodynamic test the mean value of the truncated normal distribution is taken as 0. In addition, its standard deviation is assumed to be one-third of the tolerance limit, which is taken to be 0.1 mm. A 2-story shear-beam type structure is used to confirm the error propagation properties of the proposed explicit pseudodynamic algorithm. Lumped masses of the structure are $m_1 = m_2 = 1$ kg; and the stiffness for the top story is 200 N/m while that for the bottom story is 9000 N/m. Consequently, the natural frequencies of the system are found to be 13.98 and 95.94 rad/sec. This structure is assumed to be linear elastic, and is subjected to the 1952 Taft earthquake with a peak ground acceleration of 0.3 g. A time step of $\Delta t = 0.02$ sec is used in all the computations.

Simulation results are plotted in Figs. 5 to 8. Fig. 5 shows the bottom story responses of the structure while those for the top story are shown in Fig. 6. Meanwhile, the cumulative errors for the bottom story response are plotted in Fig. 7 and those for the top story response are depicted in Fig. 8. Each figure contains 3 plots, where numerical results for the Newmark explicit method, Chang explicit method and proposed explicit method are shown from the top plot to the bottom plot, respectively. It is apparent in Fig. 5 that simulation results obtained from the Newmark explicit method are entirely destroyed by the simulation errors while those obtained from the Chang explicit method and the proposed explicit method are slightly contaminated by these errors. This concluding result is also evident from Fig. 7. Meanwhile, Fig. 6 shows that simulation results obtained from the three methods are almost unaffected by the simulation errors and Fig. 8 reveals that the three methods lead to commensurate cumulative errors. These phenomena can be thoroughly explained by

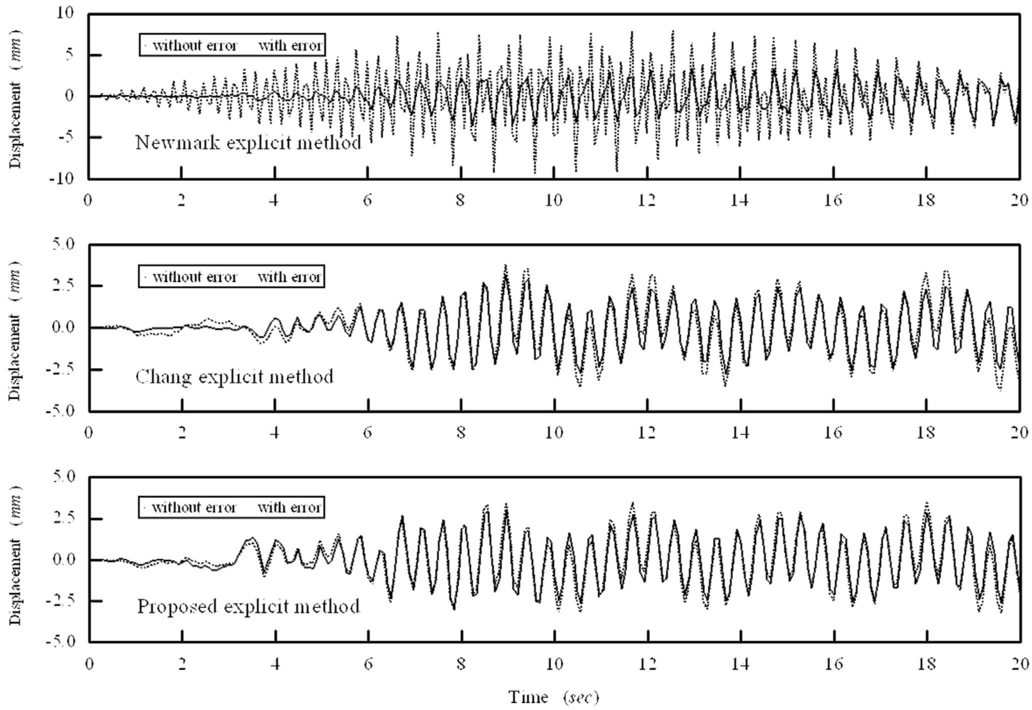

Fig. 5 Displacement responses at bottom story for simulating the pseudodynamic test of a 2-story frame

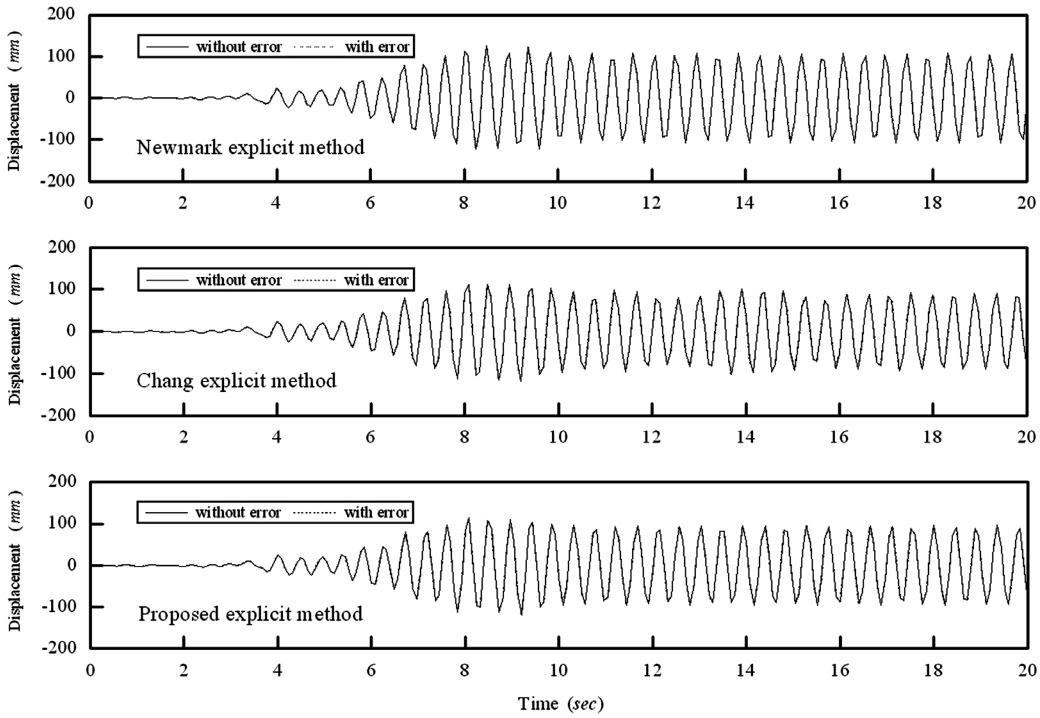

Fig. 6 Displacement responses at top story for simulating the pseudodynamic test of a 2-story frame

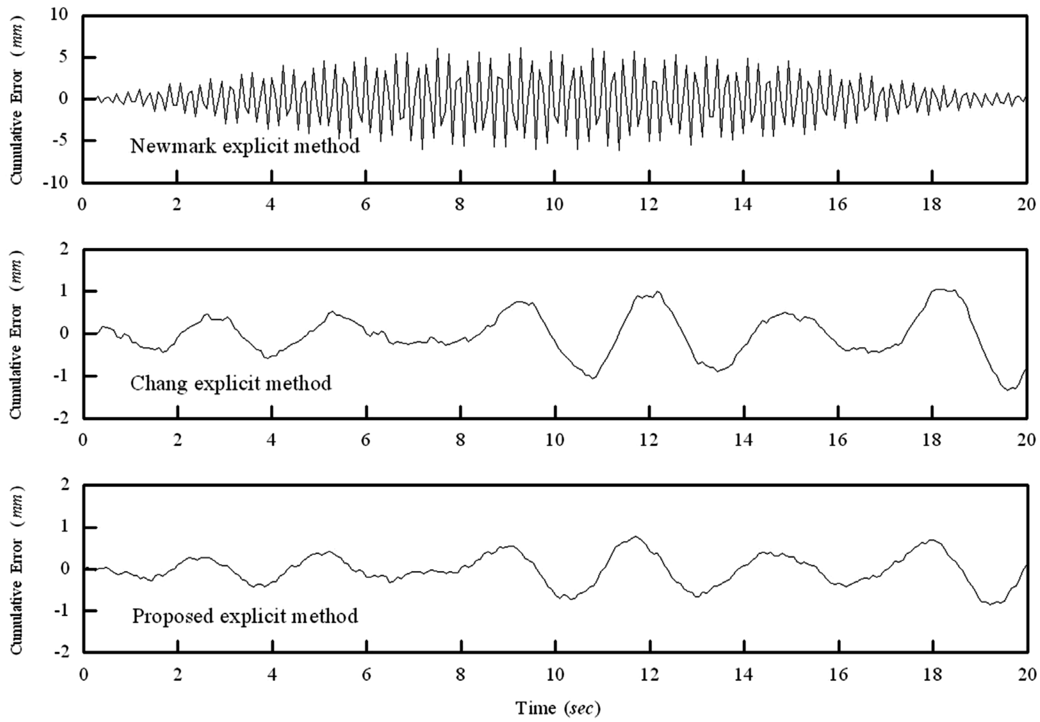

Fig. 7 Cumulative errors at bottom story for simulating the pseudodynamic test of a 2-story frame

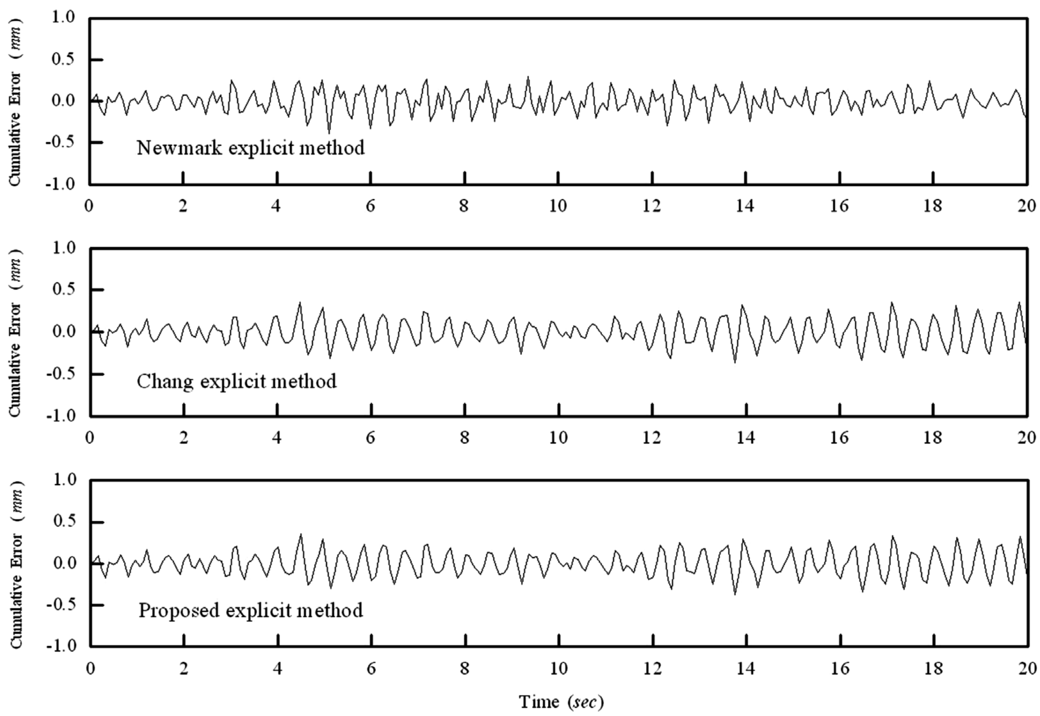

Fig. 8 Cumulative errors at top story for simulating the pseudodynamic test of a 2-story frame

error amplification factors and the modal contribution of each mode. It is clear from Fig. 5 that the response contribution from the second mode to the bottom story response is considerable while Fig. 6 reveals that it is insignificant to the top story response. This implies that the bottom story response might be contaminated by simulation errors while the top story response is not strongly affected. On the other hand, it is found from Figs. 2 and 3 that error amplification factors E_d and E_r for the first mode of Ω = 0.28 are commensurate and very small for the three methods. However, for the second mode of Ω = 1.92, E_d = 3.57 and E_r = 6.86 for the Newmark explicit method; E_d = 1.39 and E_r = 1.92 for the Chang explicit method; and E_d = 1 and E_r = 1.33 for the proposed explicit method. As a result, the large contribution from the second mode response to the bottom story response and large error amplification factors are responsible for the inaccurate responses of the bottom story obtained from the Newmark explicit method. Apparently, the slight contamination of the bottom story responses for the simulation results obtained from the Chang explicit method and proposed explicit method is due to the small error amplification factors. In Fig. 7, it is also found after comparing the middle plot to the bottom plot that the cumulative error for the result obtained from the proposed explicit method is less than that obtained from the Chang explicit method. This is due to the fact that error amplification factors, E_d = 1 and E_r = 1.33, for the proposed explicit method are less than those E_d = 1.39 and E_r = 1.92 for the Chang explicit method.

7. Actual pseudodynamic tests

A series of pseudodynamic tests were conducted to confirm the unconditional stability of using the proposed explicit method to solve the momentum equations of motion. Meanwhile, improved characteristics to capture rapid changes of dynamic loading and to eliminate adverse linearization errors if using the momentum equations of motion are also illustrated. For this purpose, a simple cantilever beam is used to simulate a 3-degree of freedom system. In fact, steel members of the

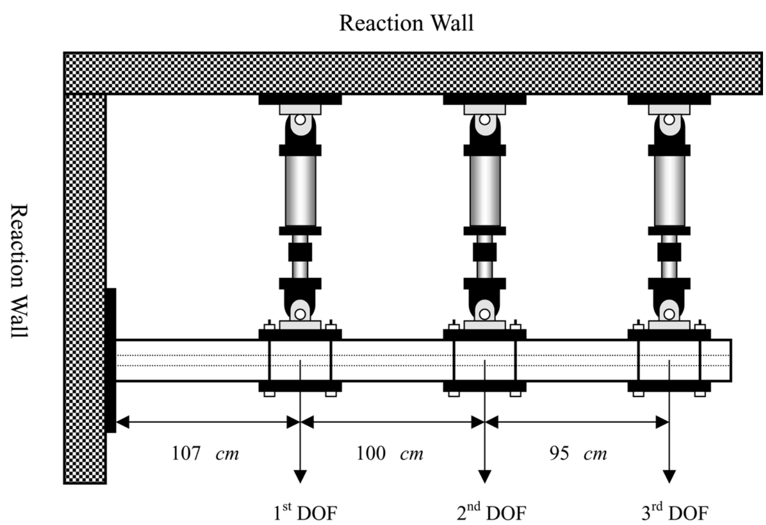

Fig. 9 Test setup for 3-DOF pseudodynamic tests

shape of **H** 200 × 200 × 8 × 12 are adopted for these tests. Each member has 3.2 meters long and is loaded in its minor axis by three servo hydraulic actuators with the spacing about 1 meter. The test setup is shown in Fig. 9. The Newmark explicit method and Chang explicit method are used to solve force equations of motion while momentum equations of motion are solved by the proposed explicit method. The initial stiffness matrix of each specimen can be experimentally measured and is found to be

$$\mathbf{K}_0 = \begin{bmatrix} 47.86 & -25.32 & 6.60 \\ -25.32 & 26.67 & -10.57 \\ 6.60 & -10.57 & 5.10 \end{bmatrix} \tag{25}$$

where the unit for each element is in kN/mm. This matrix is employed to compute the coefficient matrices for the proposed explicit method.

7.1 Unconditional stability

In this example, the lumped masses of the system are taken to be 10^5, 10^1 and 10^4 kg for the first, second and third degree of freedom. Hence, the natural frequencies of the system are found to be 5.78, 17.20 and 1633 rad/sec. This system is subjected to 1971 San Fernando earthquake with a

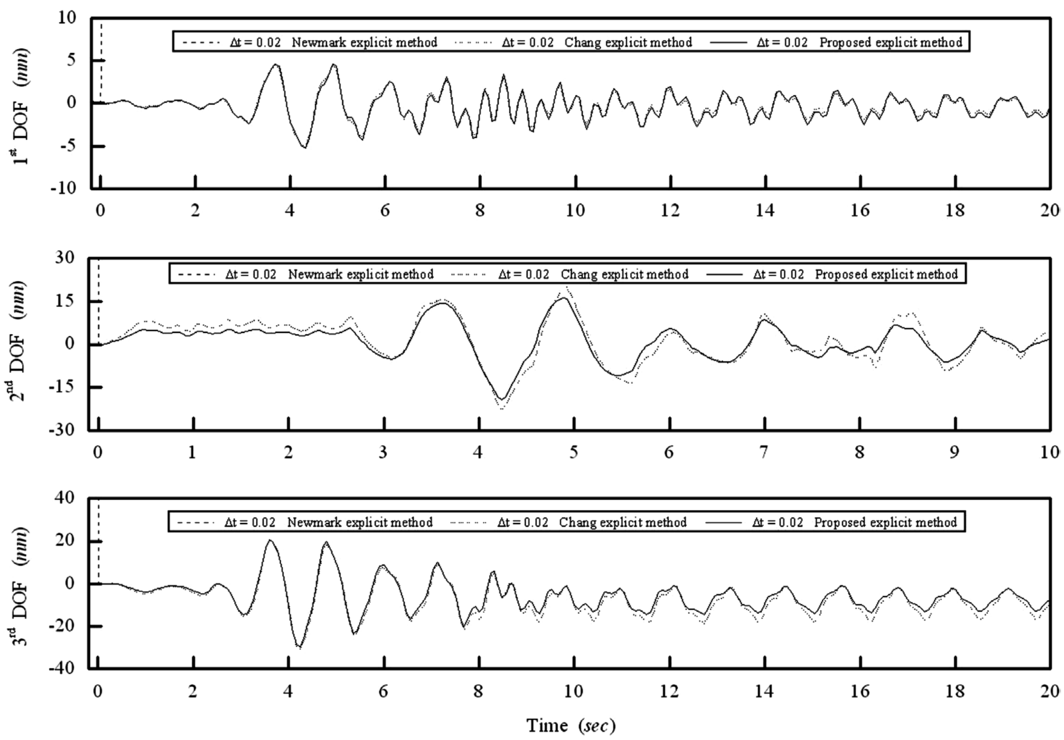

Fig. 10 Pseudodynamic responses for the structure subject to 0.03 g San Fernando earthquake

peak ground acceleration of 0.03 g and an integration time step of Δt = 0.02 sec is used for the step-by-step integration. All the pseudodynamic test results are plotted in Fig. 10. It is apparent that numerical explosions occur very early in the pseudodynamic responses obtained from the Newmark explicit method, while those obtained from the proposed explicit method are in good agreement with those obtained from the Chang explicit method and both remain stable. Thus, it is strongly indicated by this example that both pseudodynamic algorithms implemented by using the Chang explicit method and the proposed explicit method are unconditionally stable since numerical solutions remain stable even if the vale of Ω is as large as 32.66. In the middle plot of this figure, the displacement time histories for the 2^{nd} degree of freedom are plotted only for the first 10 seconds. This is intended to clearly observe the rapid fluctuation of the response, which might be caused by the significant error propagation of the third mode. This indicated that numerical dissipation might be needed to suppress the spurious growth of this high frequency response (Chang 1997). Further study of this subject is under way and will be published later.

7.2 Improved characteristics for time integration

Previously published works (Chang 2001a, Chang *et al.* 1998) have shown that the use of momentum equations of motion can effectively capture the rapid changes of dynamic loading and automatically eliminate the averse linearization errors in performing a pseudodynamic test. In order to experimentally confirm that the proposed explicit pseudodynamic algorithm also has these improved characteristics a 3-degree of freedom system is chosen for this study. This system is similar to the one used above. However, the lumped masses corresponding to the first to the third degree of freedom are specified to be 5×10^5, 3×10^5 and 1×10^5 kg respectively. As a result, the natural frequencies of the system are found to be 2.86, 6.84 and 15.75 rad/sec. The system is excited by Kobe earthquake with a peak ground acceleration of 0.1 g. The step sizes of 0.02 and 0.06 sec are used to perform the pseudodynamic tests if using the Newmark explicit method while for the proposed explicit method only the step size of Δt = 0.06 sec is tested. It should be mentioned that the trapezoidal rule is used to perform the time integration of the external force and the restoring force with a step size of 0.02 sec.

All pseudodynamic results are plotted in Fig. 11 and the results obtained from the Newmark explicit method with a time step of Δt = 0.02 sec is considered as "**correct**" solutions. It is clear that the displacement responses obtained from the Newmark explicit method with Δt = 0.06 sec are significantly deviate from the correct solutions. This is because that the high frequency content of the accelerogram and the variation of restoring force within each time step cannot be realistically captured by using a large time step of Δt = 0.06 sec to pseudodynamically solve force equations of motion. However, pseudodynamic responses obtained from the proposed explicit method with the same time step of Δt = 0.06 sec are still reliable. This drastic difference in pseudodynamic results is mainly caused by the use of momentum equations of motion instead of force equations of motion. This is because the use of momentum equations of motion engenders the use of the time integration of external force and restoring force. Thus, the high frequency content of the accelerogram can be effectively captured through the time integration of external force and the linearization errors can be drastically reduced or even entirely eliminated through the time integration of restoring force.

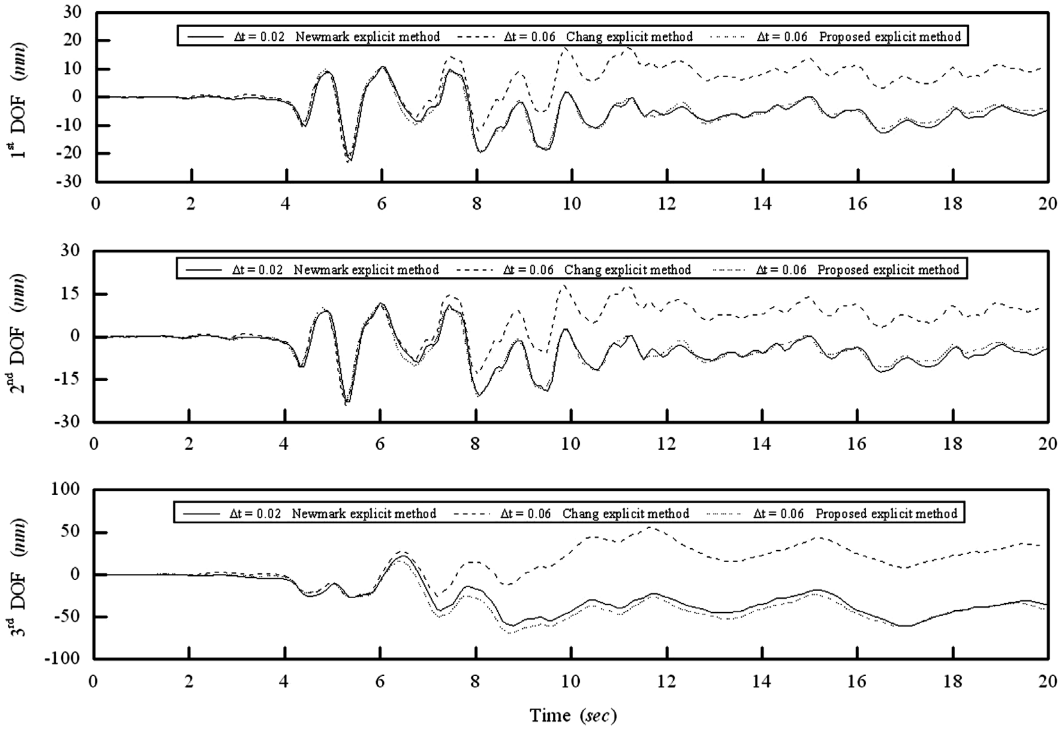

Fig. 11 Pseudodynamic responses for the structure subject to 0.1 g Kobe earthquake

8. Conclusions

An explicit integration method with unconditional stability is implemented to solve momentum equations of motion in performing a pseudodynamic test. Due to the explicitness of each time step it can be simply implemented as the common implementation for Newmark explicit method where no iteration procedure or extra hardware is needed as that for implicit pseudodynamic algorithms. Its unconditional stability enables it to perform the pseudodynamic testing of a specimen having high frequency modes. Meanwhile, the use of the momentum equations of motion can effectively capture the rapid changes of dynamic loading and automatically eliminate the adverse linearization errors due to the time integration of the external force and the restoring force.

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