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The standard deviations for eigenvalues of the closed-loop systems with random parameters

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Abstract. The vibration control problem of structures with random parameters is discussed, which is approximated by a deterministic one. A method for calculating the standard deviations of eigenvalues of the closed-loop systems is presented by using the random perturbation. The method presented in this paper will not require the distribution function of the random parameters of the systems other than their means and variances. Similarly, the distribution function of the random eigenvalues will not be computed other than their means and variances. The standard deviations of eigenvalues of the uncertain closed-loop systems can be used to estimate the stability robustness. The present method is applied to a vibration control system to illustrate the application. The numerical results show that the present method is effective.

Key words: uncertain systems; vibration active control; random parameters; standard deviations of eigenvalues of the closed-loop systems.

1. Introduction

The vibration control theory for systems with deterministic parameters has been well developed. For example, the standard methods for vibration control has been developed (Porter and Crossley 1972, Inman 1989, Meirovitch 1990); the modal controllability/observability and modal optimal control for defective/near defective systems with repeated/close eigenvalues were discussed (Chen *et al*. 2001).

However, in actual situations, the structural parameters are often uncertain, such as the inaccuracy of the measurement, errors in the manufacturing process, invalidity of some components, etc. The uncertainty can affect the robust stability and performance of the control systems. Therefore, the uncertain concept plays an important role in the vibration control problems of structures. Many studies have been done about the control problems of systems with uncertain parameters only from the viewpoint of mathematics. For example, the sufficient and necessary conditions of the dynamic stability for the uncertain systems were given (Mori and Kokame 1987, Argoun 1987); the robustness of control systems with uncertain parameters was discussed (Sobld *et al*. 1989, Rachid 1989); the interval analysis method was used to deal with the stability of an uncertain matrix (Juang *et al*. 1987).

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The most common methods for solving uncertain problems are to model the loads and the structural parameters as the random vectors. For example, the vibration theory of structures with random parameters was given (Chen 1992); the probabilistic eigenvalue analysis was discussed (Lyengow and Manohar 1989); the random finite element theory was given (Liu and Mani 1986, Liu *et al*. 1980, Contreras 1980). However, a few papers can be found about the control problems of systems with random parameters. Hence, it is necessary to develop an effective method to solve the control problems of systems with random parameters.

In this paper, the random model is used to deal with the vibration control problems of systems with uncertain parameters. The uncertainties of the structural parameters are described by random variables. The vibration control problems of the uncertain systems are transformed into ones of the deterministic systems. At first, by using the method of pole allocation, the state feedback gain matrix of the systems with deterministic parameters can be obtained, and then it is applied into the actual uncertain systems. By using the random model of the uncertain parameters and the perturbation method, the expressions can be developed for calculating the standard deviations of the real and imaginary parts of the eigenvalues of uncertain closed-loop systems. The present method will not require the distribution function of the random parameters of the systems other than their means and variances. Similarly, the distribution function of the random eigenvalues of the closed-loop systems will not be computed other than their means and variances. A numerical example is given to illustrate the application of the approach presented in this study.

2. The definition of the problem

Consider the linear vibration control equation in state space

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

By using the state feedback law, the input vector is

$$\mathbf{u}(t) = \mathbf{G}\mathbf{x}(t) \tag{2}$$

where $\mathbf{x}(t)$ is the $2n \times 1$ state vector, $\mathbf{u}(t)$ is an $m \times 1$ input vector, $\mathbf{A}$ is the $2n \times 2n$ state matrix, $\mathbf{B}$ is a $2n \times m$ input coefficient matrix, $\mathbf{G}$ is an $m \times 2n$ state feedback gain matrix.

The state matrix $\mathbf{A}$ and input coefficient matrix $\mathbf{B}$ of the uncertain systems can be expressed as

$$\begin{aligned} \mathbf{A} &= \mathbf{A}_0 + \Delta\mathbf{A} \\ \mathbf{B} &= \mathbf{B}_0 + \Delta\mathbf{B} \end{aligned} \tag{3}$$

where $\mathbf{A}_0$ and $\mathbf{B}_0$ are the deterministic parts of the state matrix and the input coefficient matrix, respectively; $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ are the corresponding uncertain parts, respectively. Correspondingly, the uncertain state vector $\mathbf{x}$, the uncertain gain matrix $\mathbf{G}$, and the uncertain input vector $\mathbf{u}$ are also expressed as

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_0 + \Delta\mathbf{x} \\ \mathbf{u} &= \mathbf{u}_0 + \Delta\mathbf{u} \\ \mathbf{G} &= \mathbf{G}_0 + \Delta\mathbf{G} \end{aligned} \tag{4}$$

where $\mathbf{x}_0$, $\mathbf{u}_0$ and $\mathbf{G}_0$ are the deterministic parts of the state vector, the input vector and the gain

matrix. $\Delta\mathbf{x}$, $\Delta\mathbf{u}$ and $\Delta\mathbf{G}$ are their uncertain parts, respectively.
Substituting Eqs. (3) and (4) to Eqs. (1) and (2) yields

$$\dot{\mathbf{x}}_0 + \Delta\dot{\mathbf{x}} = (\mathbf{A}_0 + \Delta\mathbf{A})(\mathbf{x}_0 + \Delta\mathbf{x}) + (\mathbf{B}_0 + \Delta\mathbf{B})(\mathbf{u}_0 + \Delta\mathbf{u}) \tag{5}$$

and

$$\mathbf{u}_0 + \Delta\mathbf{u} = (\mathbf{G}_0 + \Delta\mathbf{G})(\mathbf{x}_0 + \Delta\mathbf{x}) \tag{6}$$

Expanding Eqs. (5), (6) and comparing the same order terms of the left and the right side, we obtain

$$\begin{aligned} \dot{\mathbf{x}}_0 &= \mathbf{A}_0\mathbf{x}_0 + \mathbf{B}_0\mathbf{u}_0 \\ \mathbf{u}_0 &= \mathbf{G}_0\mathbf{x}_0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} \Delta\dot{\mathbf{x}} &= \mathbf{A}_0\Delta\mathbf{x} + \Delta\mathbf{A}\mathbf{x}_0 + \mathbf{B}_0\Delta\mathbf{u} + \Delta\mathbf{B}\mathbf{u}_0 \\ \Delta\mathbf{u} &= \mathbf{G}_0\Delta\mathbf{x} + \Delta\mathbf{G}\mathbf{x}_0 \end{aligned} \tag{8}$$

From the above discussion it can be seen that the uncertain system (1) and (2) have been separated into the deterministic part (7) and the uncertain part (8). The closed-loop system corresponding to the deterministic one (7) is

$$\dot{\mathbf{x}}_0(t) = (\mathbf{A}_0 + \mathbf{B}_0\mathbf{G}_0)\mathbf{x}_0(t) \tag{9}$$

and the corresponding eigenvalue problem is

$$\lambda_{0i}\varphi_{0i} = (\mathbf{A}_0 + \mathbf{B}_0\mathbf{G}_0)\varphi_{0i} \qquad (i = 1, 2, \ldots, n) \tag{10}$$

3. The gain matrix of the deterministic system

To guarantee asymptotic stability of the vibration control system, it is necessary to impart the eigenvalues larger negative real parts. Using the pole allocation, the closed-loop poles are selected in advance. Assume that the closed-loop eigenvalues of Eq. (9) are assigned to be $\lambda_1^*, \lambda_2^*, \ldots, \lambda_{2n}^*$, by using the pole allocation, the gain matrix $\mathbf{G}_0$ of the deterministic system (7) can be determined.

Suppose the right and the left modal matrices $\mathbf{U}_0 = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{2n}]$ and $\mathbf{V}_0 = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n}]$ have been obtained and they satisfy the following equations

$$\mathbf{V}_0^T\mathbf{A}_0\mathbf{U}_0 = \mathbf{\Lambda}_0, \qquad \mathbf{V}_0^T\mathbf{U}_0 = \mathbf{I} \tag{11}$$

where $\mathbf{\Lambda}_0 = diag(\lambda_{01}, \lambda_{02}, \ldots, \lambda_{02n})$ is the diagonal matrix of the eigenvalues of the deterministic system.

With the modal transformation

$$\mathbf{x}_0(t) = \mathbf{U}_0\boldsymbol{\xi}(t) \tag{12}$$

the Eq. (7) can be transferred into

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{\Lambda}_0\boldsymbol{\xi}(t) + \mathbf{B}_0'\mathbf{u}_0(t) \tag{13}$$

and

$$\mathbf{u}_0(t) = \mathbf{G}_0'\boldsymbol{\xi}(t) \tag{14}$$

where $\mathbf{B}_0' = \mathbf{V}_0^T\mathbf{B}_0 = (b_1', b_2', \ldots, b_{2n}')^T$, $\mathbf{G}_0' = \mathbf{G}_0\mathbf{U}_0 = (g_1', g_2', \ldots, g_{2n}')$ (15)

If the single input is used, $\mathbf{B}_0$ is a column vector, $\mathbf{G}_0$ is a row vector. Substituting Eq. (14) into Eq. (13) yields

$$\dot{\boldsymbol{\xi}}(t) = (\boldsymbol{\Lambda}_0 + \mathbf{B}_0'\mathbf{G}_0')\boldsymbol{\xi}(t) \tag{16}$$

In Eq. (16), suppose the assigned eigenvalues are $\lambda_i^*(i = 1, 2, \ldots, 2n)$, the corresponding eigenvectors are $\mathbf{w}_i(i = 1, 2, \ldots, 2n)$, and they satisfy the following eigenproblem

$$(\boldsymbol{\Lambda}_0 + \mathbf{B}_0'\mathbf{G}_0')\mathbf{w}_i = \lambda_i^*\mathbf{w}_i \qquad (i = 1, 2, \ldots, 2n) \tag{17}$$

Because $\mathbf{w}_i \neq 0$ there exists

$$\det(\boldsymbol{\Lambda}_0 + \mathbf{B}_0'\mathbf{G}_0' - \lambda_i^*\mathbf{I}) = 0 \tag{18}$$

Solving Eq. (18), we obtain (Meirovitch 1990)

$$g_i' = \prod_{k=1}^{2n}(\lambda_i^* - \lambda_i) \Big/ \left(b_i' \prod_{\substack{k=1 \\ k \neq i}}^{2n}(\lambda_k - \lambda_i) \right) \qquad (i = 1, 2, \ldots, 2n) \tag{19}$$

thus obtaining the matrix $\mathbf{G}_0' = (g_1', g_2', \ldots, g_{2n}')$.
Considering Eq. (12), Eq. (14) becomes

$$\mathbf{u}_0(t) = \mathbf{G}_0'\boldsymbol{\xi}(t) = \mathbf{G}_0'\mathbf{V}_0^T\mathbf{x}_0(t) = \mathbf{G}_0\mathbf{x}_0(t)$$

where

$$\mathbf{G}_0 = \mathbf{G}_0'\mathbf{V}_0^T \tag{20}$$

If the deterministic gain matrix $\mathbf{G}_0$ is applied to the actual uncertain system, there must exist some errors between the closed-loop eigenvalues and the assigned eigenvalues $\lambda_i^*(i = 1, 2, \ldots, 2n)$. Using the random perturbation method, the expressions for computing the standard deviations of the closed-loop eigenvalues $\lambda_i(i = 1, 2, \ldots, 2n)$ can be developed.

4. Random eigenvalue analysis of the close-loop systems

If the gain matrix $\mathbf{G}_0$ is introduced to the actual uncertain closed-loop system we obtain

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{G}_0\mathbf{x}(t) = (\mathbf{A} + \mathbf{B}\mathbf{G}_0)\mathbf{x}(t) \tag{21}$$

The corresponding eigenvalue problem is

$$(\mathbf{A} + \mathbf{B}\mathbf{G}_0)\varphi = \lambda\varphi \tag{22}$$

Because of the randomness of the system parameters, the state matrix $\mathbf{A}$ and input coefficient matrix $\mathbf{B}$ are random, the eigensolutions λ and φ are also random. To obtain the mean values and the variances of eigenvalues, we substitute $\mathbf{A}$, $\mathbf{B}$, λ_i, φ_i for the summation of two terms. The first term is deterministic part equal to the mean value of the corresponding random variable, and the second term is random part with zero mean value. Thus, we have

$$\mathbf{A} = \mathbf{A}_0 + \varepsilon\mathbf{A}_r, \qquad \mathbf{B} = \mathbf{B}_0 + \varepsilon\mathbf{B}_r \tag{23}$$

$$\varphi_i = \varphi_{di} + \varepsilon\varphi_{ri}, \qquad \lambda_i = \lambda_{di}^* + \varepsilon\lambda_{ri} \qquad (i = 1, 2, \dots, 2n) \tag{24}$$

where ε is a small parameter. The terms with subscript "0" or "d" are deterministic, the terms with subscript "r" are random. Here we assume the coefficient matrix $\mathbf{B}$ is deterministic. That is $\mathbf{B}_r = \mathbf{0}$. Substituting Eqs. (23) and (24) into Eq. (22) we obtain

$$[(\mathbf{A}_0 + \varepsilon\mathbf{A}_r) + \mathbf{B}_0\mathbf{G}_0](\varphi_{di} + \varepsilon\varphi_{ri}) = (\lambda_{di}^* + \varepsilon\lambda_{ri})(\varphi_{di} + \varepsilon\varphi_{ri}) \qquad (i = 1, 2, \dots, 2n) \tag{25}$$

Expanding Eq. (25) and comparing the coefficients of the same power of ε yields

$$\varepsilon^0: \quad (\mathbf{A}_0 + \mathbf{B}_0\mathbf{G}_0)\varphi_{di} = \lambda_{di}^*\varphi_{di} \qquad (i = 1, 2, \dots, 2n) \tag{26}$$

$$\varepsilon^1: \quad (\mathbf{A}_0 + \mathbf{B}_0\mathbf{G}_0)\varphi_{ri} + \mathbf{A}_r\varphi_{di} = \lambda_{di}^*\varphi_{ri} + \lambda_{ri}\varphi_{di} \qquad (i = 1, 2, \dots, 2n) \tag{27}$$

From Section 3, we know that the eigenvalues of the matrix $(\mathbf{A}_0 + \mathbf{B}_0\mathbf{G}_0)$ are the assigned eigenvalues $\lambda_{di}^*(i = 1, 2, \dots, 2n)$, and the eigenvalue problems are

$$(\mathbf{A}_0 + \mathbf{B}_0\mathbf{G}_0)\mathbf{\Phi} = \mathbf{\Phi}\mathbf{\Lambda}^*, \qquad (\mathbf{A}_0 + \mathbf{B}_0\mathbf{G}_0)^T\mathbf{\Psi} = \mathbf{\Psi}\mathbf{\Lambda}^* \tag{28}$$

According to the perturbation theory (Chen 1992), the perturbations for the eigenvalues of the closed-loop systems can be expressed as

$$\lambda_{ri} = (\psi_i)^T\mathbf{A}_r\varphi_i \qquad (i = 1, 2, \dots, 2n) \tag{29}$$

The random parameter $b_j(j = 1, 2, \dots, m)$ of systems can be expressed as

$$b_j = b_{dj} + \varepsilon b_{rj} \qquad (j = 1, 2, \dots, m) \tag{30}$$

where b_{dj} is the mean value of b_j, b_{rj} is the random variable with zero mean value. According to the Taylor series when b_{rj} is small (compared with b_{dj}) we can expand $\mathbf{A}$ about $b_{dj}(j = 1, 2, \ldots, m)$ as

$$\mathbf{A} = \mathbf{A}(b_{d1}, b_{d2}, \ldots, b_{dm}) + \sum_{j=1}^{m}\left(\frac{\partial \mathbf{A}(b_1, b_2, \ldots, b_m)}{\partial b_j}\right)_{b_j = b_{dj}} \varepsilon b_{rj}$$

Comparing with Eq. (23) we get

$$\mathbf{A}_0 = \mathbf{A}(b_{d1}, b_{d2}, \ldots, b_{dm}), \qquad \mathbf{A}_r = \sum_{j=1}^{m}\left(\frac{\partial \mathbf{A}(b_1, b_2, \ldots, b_m)}{\partial b_j}\right)_{b_j = b_{dj}} b_{rj} \tag{31}$$

letting $[\mathbf{A}_{d,j}] = \left(\frac{\partial \mathbf{A}(b_1, b_2, \ldots, b_m)}{\partial b_j}\right)_{b_j = b_{dj}}$, Eq. (29) becomes

$$\lambda_{ri} = \sum_{j=1}^{m} (\psi_i)^T [\mathbf{A}_{d,j}] \varphi_i b_{rj} \qquad (j = 1, 2, \ldots, m) \tag{32}$$

Expanding λ_i about b_{dj} $(j = 1, 2, \ldots, m)$ and comparing with Eq. (24) we get

$$\lambda_{di}^* = \lambda_i(b_{d1}, b_{d2}, \ldots, b_{dm}), \quad \lambda_{ri} = \sum_{j=1}^{m}\left(\frac{\partial \lambda_i(b_1, b_2, \ldots, b_m)}{\partial b_j}\right)_{b_j = b_{dj}} b_{rj} \tag{33}$$

The sensitivity of the eigenvalue λ_i is

$$\lambda_{i,j} = \left(\frac{\partial \lambda_i(b_1, b_2, \ldots, b_m)}{\partial b_j}\right)_{b_j = b_{dj}} = (\psi_i)^T [\mathbf{A}_{d,j}] \varphi_i \qquad (j = 1, 2, \ldots, m) \tag{34}$$

5. The standard deviations for eigenvalues of the closed-loop systems

Because the eigenvalues of the system are complex, the real and the imaginary parts of the closed-loop eigenvalues will be discussed, respectively.

Suppose

$$\lambda_i = d_i + f_i j \qquad j = \sqrt{-1} \qquad (i = 1, 2, \ldots, 2n) \tag{35}$$

$$\lambda_{i,k} = d_{i,k} + f_{i,k} j \qquad (k = 1, 2, \ldots, m) \tag{36}$$

where the d_i is real part of λ_i, the f_i is imaginary part , the $\lambda_{i,k}$, $d_{i,k}$ and $f_{i,k}$ are sensitivities of λ_i, d_i and f_i respectively.

To compute the standard deviation of the f_i, the f_i can be expressed as $f_i = f_{id} + \varepsilon f_{ir}$ where the f_{id} is the mean value of the f_i and the f_{ir} is the random part with zero mean value. The variance of the f_i is given by

$$Var(f_i) = E[(f_i)^2] - [E(f_i)]^2 = E[(\varepsilon f_{ir})^2] \qquad (i = 1, 2, \ldots, 2n) \tag{37}$$

where f_{ir} can be expressed as

$$f_{ir} = \sum_{j=1}^{m}\left(\frac{\partial f_i}{\partial b_j}\right)_{b_j = b_{dj}} b_{rj} = \sum_{j=1}^{m} f_{i,j} b_{rj} \qquad (i = 1, 2, \ldots, 2n) \tag{38}$$

Thus one has

$$Var(f_i) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} (\varepsilon f_{ir})^2 p(b_1, b_2, \ldots, b_m) db_1 \ldots db_m$$

$$= \varepsilon^2 \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \sum_{l=1}^{m} \sum_{k=1}^{m} (f_{i,l})(f_{i,k}) b_{rl} b_{rk} p(b_1, b_2, \ldots, b_m) db_1 \ldots db_m \quad (i = 1, 2, \ldots, 2n) \tag{39}$$

where $(f_{i,l})$, $(f_{i,k})$ are the sensitivities of f_i with respect to b_l, b_k, $p(b_1, b_2, \ldots, b_m)$ is the join probability density function for $b_1, b_2, \ldots, b_m$. From Eq. (39) we obtain

$$Var(f_i) = \sum_{l=1}^{m} \sum_{k=1}^{m} (f_{i,l})(f_{i,k}) Cov(b_l, b_k) \tag{40}$$

where $Cov(b_l, b_k)$ is the covariance between b_l and b_k given by

$$Cov(b_l, b_k) = \varepsilon^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b_{rl} b_{rk} p(b_l, b_k) db_l db_k = \rho_{lk} \sigma_{bl} \sigma_{bk}$$

$$(l = 1, 2, \ldots, m), (k = 1, 2, \ldots, m) \tag{41}$$

where $p(b_l, b_k)$ is the join probability density function for b_l and b_k, and ρ_{lk} is the correlation coefficient, and ρ_{bl} is the standard deviation for b_l. The covariance between two eigenvalues is given by

$$Cov(f_i, f_s) = \sum_{l=1}^{m} \sum_{k=1}^{m} (f_{i,l})(f_{s,k}) Cov(b_l, b_k) \quad (i = 1, 2, \ldots, 2n)(s = 1, 2, \ldots, 2n) \tag{42}$$

So the covariance matrix of $f_i (i = 1, 2, \ldots, 2n)$ of the closed-loop systems can be defined as

$$\left[\sum^{f}\right] = \begin{bmatrix} Var(f_1) & & & \\ Cov(f_2, f_1) & Var(f_2) & sym & \\ \ldots\ldots & & & \\ Cov(f_{2n}, f_1) & Cov(f_{2n}, f_2) & \ldots & Var(f_{2n}) \end{bmatrix} \tag{43}$$

From Eq. (40), Eq. (42) and Eq. (43) one gets

$$\left[\sum^{f}\right] = \left[\frac{\partial f}{\partial b}\right]\left[\sum^{b}\right]\left[\frac{\partial f}{\partial b}\right]^T \tag{44}$$

where $\left[\frac{\partial f}{\partial b}\right]$ is the sensitivity matrix for the imaginary parts of eigenvalues and $\left[\sum^{b}\right]$ is the covariance matrix of the random structure parameters. By using Eq. (41), the $\left[\sum^{b}\right]$ can be expressed as

$$\left[\sum^{b}\right] = [\sigma_b][\rho][\sigma_b] \tag{45}$$

where $[\sigma_b]$ is the standard deviation matrix and $[\rho]$ is the correlation coefficient matrix, given by

$$[\sigma_b] = diag[\sigma_{b1} \;\; \sigma_{b2} \;\; \dots \;\; \sigma_{bm}] \qquad [\rho] = \begin{bmatrix} 1 & & & \\ \rho_{21} & 1 & sym & \\ \dots\dots & & & \\ \rho_{m1} & \rho_{m2} & \dots & 1 \end{bmatrix} \tag{46}$$

Substituting Eq. (45) into Eq. (44) yields

$$\left[\sum^{f}\right] = \left[\frac{\partial f}{\partial b}\right][\sigma_b][\rho][\sigma_b]\left[\frac{\partial f}{\partial b}\right]^T \tag{47}$$

From Eq. (47) the standard deviations for imaginary parts of eigenvalues σ_{f_i} can be obtained

$$\sigma_{f_i} = (\mathrm{var}(f_i))^{1/2} \qquad (i = 1, 2, \dots, 2n) \tag{48}$$

The similar expressions for real parts of eigenvalues can be derived

$$\left[\sum^{d}\right] = \left[\frac{\partial d}{\partial b}\right][\sigma_b][\rho][\sigma_b]\left[\frac{\partial d}{\partial b}\right]^T \tag{49}$$

$$\sigma_{d_i} = (\mathrm{var}(d_i))^{1/2} \qquad (i = 1, 2, \dots, 2n) \tag{50}$$

The standard deviations, σ_{d_i}, can be used to estimate the stability robustness of the uncertain control system.

6. Numerical example

In order to illustrate the application of the present method, a numerical example is given as follows.

Consider a vibration control system of frame structure shown in Fig. 1. Assume mass (kg), stiffness (N/m) and damping (N/m.s^{-1}) are given as follow:

$$m_1 = 29, \quad m_2 = 26, \quad m_3 = 26, \quad m_4 = 24, \quad m_5 = 17$$
$$k_{10} = 2000, \quad k_{20} = 1800, \quad k_{30} = 1600, \quad k_{40} = 1400, \quad k_{50} = 1200$$
$$c_{10} = 40, \quad c_{20} = 40, \quad c_{30} = 60, \quad c_{40} = 80, \quad c_{50} = 80$$
$$k_1 = k_{10} + b_{k_1}, \quad k_2 = k_{20} + b_{k_2}, \quad k_3 = k_{30} + b_{k_3}, \quad k_4 = k_{40} + b_{k_4}, \quad k_5 = k_{50} + b_{k_5}$$
$$c_1 = c_{10} + b_{c_1}, \quad c_2 = c_{20} + b_{c_2}, \quad c_3 = c_{30} + b_{c_3}, \quad c_4 = c_{40} + b_{c_4}, \quad c_5 = c_{50} + b_{c_5}$$

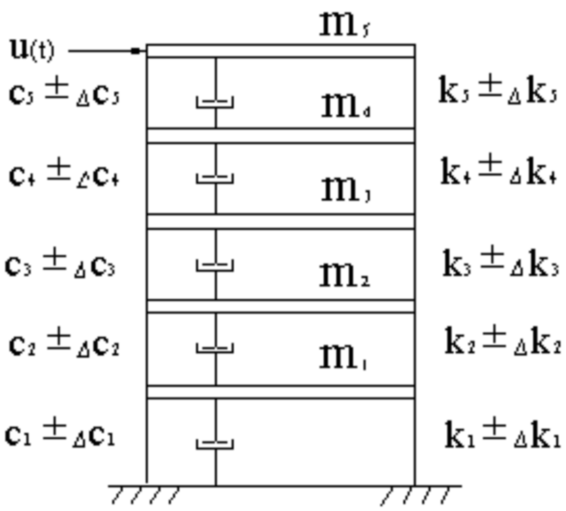

Fig. 1 The frame structure

where the mass parameters are assumed to be deterministic; stiffness parameters, k_{10}, k_{20}, k_{30}, k_{40} and k_{50} are deterministic, and b_{k_1}, b_{k_2}, b_{k_3}, b_{k_4} and b_{k_5} are random with zero means; damping parameters, c_{10}, c_{20}, c_{30}, c_{40} and c_{50} are deterministic, and b_{c_1}, b_{c_2}, b_{c_3}, b_{c_4} and b_{c_5} are random with zero means. Assume that a control force, $\mathbf{u}(t)$, is input to m_5. The mass matrix is

$$\mathbf{M} = diag[m_1 \;\; m_2 \;\; m_3 \;\; m_4 \;\; m_5] = diag[29 \;\; 26 \;\; 26 \;\; 24 \;\; 17]$$

The stiffness matrix of the system with random parameters is $\mathbf{K} = \mathbf{K}_d + \mathbf{K}_r$ where $\mathbf{K}_d$ is the deterministic part, $\mathbf{K}_r$ the random part. The damping matrix of the system with random parameters is $\mathbf{C} = \mathbf{C}_d + \mathbf{C}_r$ where $\mathbf{C}_d$ is the deterministic part, $\mathbf{C}_r$ the random part. Suppose the state vector is

$$\mathbf{x}(t) = [q_1(t) \;\; q_2(t) \;\; q_3(t) \;\; q_4(t) \;\; q_5(t) \;\; \dot{q}_1(t) \;\; \dot{q}_2(t) \;\; \dot{q}_3(t) \;\; \dot{q}_4(t) \;\; \dot{q}_5(t)]^T$$

Then the state matrix of the system is

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathrm{C} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}_d & -\mathbf{M}^{-1}\mathrm{C}_d \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{K}_r & -\mathbf{M}^{-1}\mathrm{C}_r \end{bmatrix} = \mathbf{A}_0 + \mathbf{A}_r$$

where $\mathbf{A}_0$ is the state matrix with deterministic parameters and $\mathbf{A}_r$ is the state matrix with random parameters. The input coefficient matrix is $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_r$ where $\mathbf{B}_0$ is input coefficient matrix with deterministic parameters, $\mathbf{B}_0 = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1/17]$; $\mathbf{B}_r = \mathbf{0}$.

The eigenvalues of $\mathbf{A}_0$ are

$$\begin{array}{ll} \lambda_{01} = -0.0963 + 2.5275i & \lambda_{02} = -0.0963 - 2.5275i \\ \lambda_{03} = -1.0267 + 6.7292i & \lambda_{04} = -1.0267 - 6.7292i \\ \lambda_{05} = -2.3488 + 10.2718i & \lambda_{06} = -2.3488 - 10.2718i \\ \lambda_{07} = -5.0397 + 12.5008i & \lambda_{08} = -5.0397 - 12.5008i \\ \lambda_{09} = -3.1695 + 13.8673i & \lambda_{010} = -3.1695 - 13.8673i \end{array}$$

To guarantee the stability of the control system, it is only necessary to impart the eigenvalues larger negative real parts, it is not necessary to alter the frequencies. To this end, if modal damping ratio is assumed to be 0.4 and the corresponding real parts of eigenvalues can be assigned as follows

$$\begin{array}{ll}
\lambda_1^* = -1.0110 + 2.5275i & \lambda_2^* = -1.0110 - 2.5275i \\
\lambda_3^* = -2.6917 + 6.7292i & \lambda_4^* = -2.6917 - 6.7292i \\
\lambda_5^* = -4.1087 + 10.2718i & \lambda_6^* = -4.1087 - 10.2718i \\
\lambda_7^* = -5.0003 + 12.5008i & \lambda_8^* = -5.0003 - 12.5008i \\
\lambda_9^* = -5.5469 + 13.8673i & \lambda_{10}^* = -5.5469 - 13.8673i
\end{array}$$

Using Eq. (20), the state feedback gain matrix for the system with deterministic parameters can be obtained

$$\mathbf{G}_0 = [4382.7060 \quad -5104.3220 \quad 1720.9740 \quad 1991.4040 \quad -1566.5190 \\ 177.6021 \quad -316.7099 \quad 384.1323 \quad -107.0294 \quad -227.0397]$$

If $\mathbf{G}_0$, the feedback gain matrix, is applied to the actual system with uncertain parameters, the closed-loop eigenvalues will have some perturbations.

The derivatives of $\mathbf{A}$ with respect to k_i, c_i $(i = 1, 2, \ldots, 5)$ are calculated. Using Eq. (34), the sensitivity matrix of the real and imaginary parts of the eigenvalues of the closed-loop system with uncertain parameters are obtained, respectively.

Assume the standard deviations of the k_i and the c_i are $\sigma_{k_i} = 0.01 \times k_i$ and $\sigma_{c_i} = 0.01 \times c_i$, the correlation coefficients of the k_i and the c_i are $\rho_{k_i k_j} = 0.5$, $\rho_{c_i c_j} = 0.5$ and $\rho_{k_i c_j} = 0$ $(i = 1, 2, \ldots, 5)$ $(j = 1, 2, \ldots, 5)$. That is, the stiffness coefficients of the k_i are correlative and damping coefficients of the c_i are also correlative, but the k_i and the c_i are statistically independent. Now using Eq. (47) and Eq. (49), we obtain the covariance matrices for the imaginary and real parts of eigenvalues. The standard deviations for the imaginary parts of eigenvalues are $\sigma_{f_i} = (\mathrm{var}(f_i))^{1/2}$. The standard deviations for the real parts of eigenvalues are $\sigma_{d_i} = (\mathrm{var}(d_i))^{1/2}$. The sensitivities for the real and imaginary part of eigenvalues are listed in Table 1. The standard deviations of eigenvalues are listed in Table 2. The data of the random design variables are listed in Table 3. In the Tables, $d_{i,j}$ and $f_{i,j}$ $(i = 1, 3, 5, 7, 9)$, $(j = 1, 2, \ldots, 10)$ denote sensitivities for the real and imaginary part of eigenvalues, respectively where the i denotes the ith eigenvalue and the j the jth structural parameter,

Table 1 The eigenvalue sensitivities with respect to the structure parameters

	k_1	k_2	k_3	k_4	k_5	c_1	c_2	c_3	c_4	c_5
$d_{1,j}$	–.0001	–.0002	.0002	.0005	–.0007	–.0016	.0023	–.0029	–.0008	.0013
$d_{3,j}$	–.0002	.0007	–.0023	.0016	–.0006	.0007	–.0072	.0053	–.0230	.0093
$d_{5,j}$	–.0010	–.0034	.0067	–.0060	.0021	–.0090	.0363	–.0421	–.0104	.0076
$d_{7,j}$	.0014	.0013	–.0105	.0046	.0045	–.0144	–.0966	.0657	.1106	–.0641
$d_{9,j}$	–.0002	.0016	.0059	–.0007	–.0054	.0070	.0287	–.0645	–.1165	–.0043
$f_{1,j}$	.0007	–.0008	.0011	.0001	–.0002	–.0010	.0004	–.0006	.0012	–.0014
$f_{3,j}$	.0001	.0008	.0001	.0028	–.0012	–.0011	.0028	–.0157	.0033	–.0008
$f_{5,j}$	.0013	–.0022	.0014	.0034	–.0016	–.0150	–.0263	.0627	–.0754	.0282
$f_{7,j}$	.0006	.0072	–.0010	–.0107	.0033	.0146	–.0201	–.1264	.1111	.0397
$f_{9,j}$	–.0004	–.0027	.0023	.0087	.0025	.0003	.0374	.0698	–.0586	–.0883

Table 2 Standard deviations of eigenvalues of the closed-loop system

	MV	SD	P(|SD/MV|*100%)
$R(\lambda_1)$	−1.010991	.008878	.878183
$R(\lambda_3)$	−2.691697	.036936	1.372217
$R(\lambda_5)$	−4.108713	.112961	2.749308
$R(\lambda_7)$	−5.000292	.160277	3.205353
$R(\lambda_9)$	−5.546924	.143650	2.589731
$I(\lambda_1)$	2.527472	.021161	.837236
$I(\lambda_3)$	6.729238	.042796	.635978
$I(\lambda_5)$	10.271760	.079731	.776216
$I(\lambda_7)$	12.500760	.170464	1.363625
$I(\lambda_9)$	13.867290	.156912	1.131527

Table 3 Standard deviations of structure parameters

DV	MV	SD	P(|SD/MV|*100%)
k_1	2000.0	20.0	1.0
k_2	1800.0	18.0	1.0
k_3	1600.0	16.0	1.0
k_4	1400.0	14.0	1.0
k_5	1200.0	12.0	1.0
c_1	40.0	0.4	1.0
c_2	40.0	0.4	1.0
c_3	60.0	0.6	1.0
c_4	80.0	0.8	1.0
c_5	80.0	0.8	1.0

$R(\lambda_i)$ and $I(\lambda_i)$ $(i = 1, 3, 5, 7, 9)$ the real and imaginary parts of eigenvalues of the system, respectively, MV denotes the mean value, SD the standard deviation, DV the design variable, and P is given by P = (|SD/MV|)*100%. From Tables 1, 2, it can be shown that the standard deviation of the $R(\lambda_7)$ is 0.160277 which is 3.205353 percent of 5.000292 and standard deviation of the $R(\lambda_1)$ is 0.008878 which is 0.878183 percent of 1.010991. The reason is that the $d_{7,j}$ is larger then the $d_{1,j}$. Similarly, standard deviation of the $I(\lambda_7)$ is 0.170464 which is 1.363625 percent of 12.500760 and standard deviation of the $I(\lambda_3)$ is 0.042796 which is 0.635978 percent of 6.729238. The reason is that the $f_{7,j}$ is larger then the $f_{3,j}$.

7. Conclusions

The vibration control problems of systems with uncertain parameters are discussed in this paper, which is approximated with the corresponding deterministic one. The uncertain parameters are modeled to be random variables. The formulas for calculating the standard deviations of eigenvalues of the closed-loop systems are derived with the random model and the random perturbation. To estimate the standard deviations of eigenvalues of the closed-loop systems with random parameters, the mean values and standard deviations of the random parameters are required, and their distribution function is not required. This makes the present method easier to implement for the complex structural control problems in the sense that it does not require the probabilistic distribution of the random parameters which is often difficult to obtain accurately. The results of the numerical example of the vibration system show that how the uncertain parameters of systems affect the stability robustness and that the method is effective for dealing with the vibration control of the uncertain systems.

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