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The G. D. Q. method for the harmonic dynamic analysis of rotational shell structural elements

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Abstract. This paper deals with the modal analysis of rotational shell structures by means of the numerical solution technique known as the Generalized Differential Quadrature (G. D. Q.) method. The treatment is conducted within the Reissner first order shear deformation theory (F. S. D. T.) for linearly elastic isotropic shells. Starting from a non-linear formulation, the compatibility equations via Principle of Virtual Works are obtained, for the general shell structure, given the internal equilibrium equations in terms of stress resultants and couples. These equations are subsequently linearized and specialized for the rotational geometry, expanding all problem variables in a partial Fourier series, with respect to the longitudinal coordinate. The procedure leads to the fundamental system of dynamic equilibrium equations in terms of the reference surface kinematic harmonic components. Finally, a one-dimensional problem, by means of a set of five ordinary differential equations, in which the only spatial coordinate appearing is the one along meridians, is obtained. This can be conveniently solved using an appropriate G. D. Q. method in meridional direction, yielding accurate results with an extremely low computational cost and not using the so-called "delta-point" technique.

Key words: shell of revolution; generalized differential quadrature method; modal analysis; numerical method; dynamic analysis.

1. Introduction

Shells of revolution are common structural elements and can be found in many fields of engineering technology. Their use spans over different branches of engineering such as pressure vessels, cooling towers, wheels, tires and turbine engine components. The objective of the present work is to enlighten a new efficient and accurate technique, called the G. D. Q. Method (Bellman and Casti 1972), for the solution of the dynamic problem of shells of revolution. The present work is based on the classic first order shear deformation theory for thin shells, as proposed by Reissner (1969) and takes into account both for transverse shear deformability and rotary inertias. The governing equations of motion, for the shell structure, are a set of five bi-dimensional partial differential equations with variable coefficients. These are initially expressed in terms of forces and couples per unit length of parametric lines of the middle surface. By introducing the constitutive

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equations and the kinematic relationships between strain measures and displacements, the equilibrium equations can be put in terms of generalized displacement components of the points lying on the middle surface of the shell.

The analysis of rotational shells can be expedited by the expansion of all variables of the problem into partial Fourier series in the circumferential coordinate θ. This achieves separation of the dependent variables and the initial two-dimensional problem is reduced to a series of one-dimensional problems. In the present linear formulation, the resulting governing equations are all uncoupled and can be solved separately for each harmonic. Precisely, they can now be approximated and solved with the aim of the numerical technique called G. D. Q. method so to give a set of generalized eigenvalue problems, each one being characterized by a circular harmonic number. This method, originally proposed by Bellman and Casti (1971, 1972) has recently been applied in several fields of structural mechanics (Bert and Malik 1996) and in computational shell mechanics (Lam *et al.* 2000, Wu *et al.* 2000, Li and Lam 2001, Jiang and Redekop 2001, Ng *et al.* 2003). Apparently, the interest of researchers in it is increasing, due to its great simplicity and versatility, particularly when compared with other classical numerical methods. The present G. D. Q. procedure permits to deal directly with the governing equations, while the same might not always hold for other approaches such as the Finite Element method. Several comparisons with available results on specimen cases can be done. The basic feature of the present application of this method is the prior-to-discretization Fourier decomposition with respect to the circumferential coordinate, which leads to significative computational costs reduction. Moreover, with the adopted G. D. Q. procedure and in the framework of the utilized shell engineering theory, an exact imposition of boundary conditions can be achieved. Even the relevant case of closed rotational shells (domes) is dealt with no approximation in terms of geometric assignments in correspondence of the apex with no need of introducing a little opening at the top of the shell. Finally, it is to be noted that in the present study no use of the so called delta-point technique (Bert and Malik 1996, Redekop and Xu 1999) is made, avoiding fictitious assignments in terms of boundary conditions. The G. D. Q. solution shows very good convergence and appears as precise and accurate as that obtained by FEM analyses. As a further confirm of the technique potentiality a complete series of shells modal shapes investigated is presented in Appendix A. It is worth noting that, to the authors' knowledge, this way of applying the G. D. Q method to rotational shell structures has not been presented earlier in the technical literature.

2. Dynamics of stress-resultants and stress-couples vectors

In the first part we will be referring to the so called first order shear deformation Reissner shell theory (Reissner and Wan 1967, Reissner 1969). We consider the reference surface assumed for the shell as a given surface in a vectorial form $\mathbf{r} = \mathbf{r}(\alpha_1, \alpha_2)$, where (α_1, α_2) are curvilinear coordinates that will be considered orthogonal in this context for simplicity's sake, with linear element $d\mathbf{r} \cdot d\mathbf{r} = A_1^2 d\alpha_1^2 + A_2^2 d\alpha_2^2$. We also consider a second associated surface with the equation $\boldsymbol{\rho} = \boldsymbol{\rho}(\alpha_1, \alpha_2)$. On this second surface the coordinate curves $\alpha_i = constant$ will in general not be orthogonal anymore. We assume that these two surfaces, being the reference surface of the thin shell considered, represent the same continuum, the first one being the undeformed state of the aggregate and the second one being its deformed state. Moreover, in what follows the reference surface will be taken conventionally as the middle surface of the shell in its undeformed state. The infinitesimal

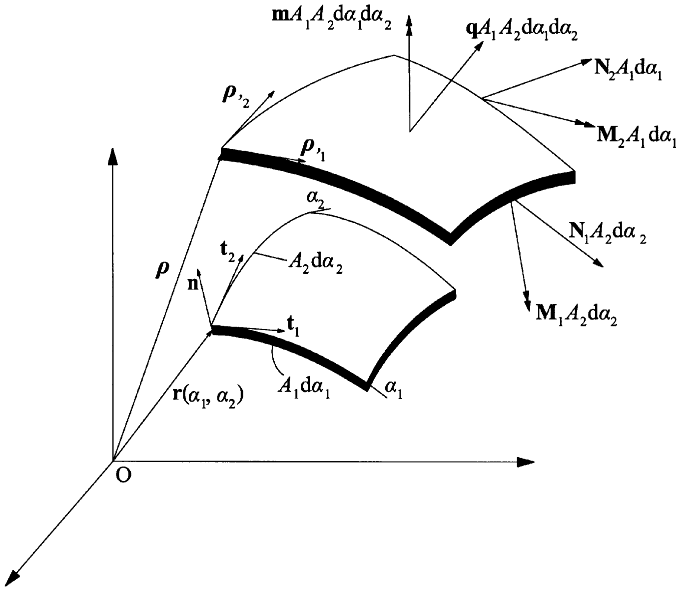

Fig. 1 Undeformed and deformed configurations for the shell infinitesimal element

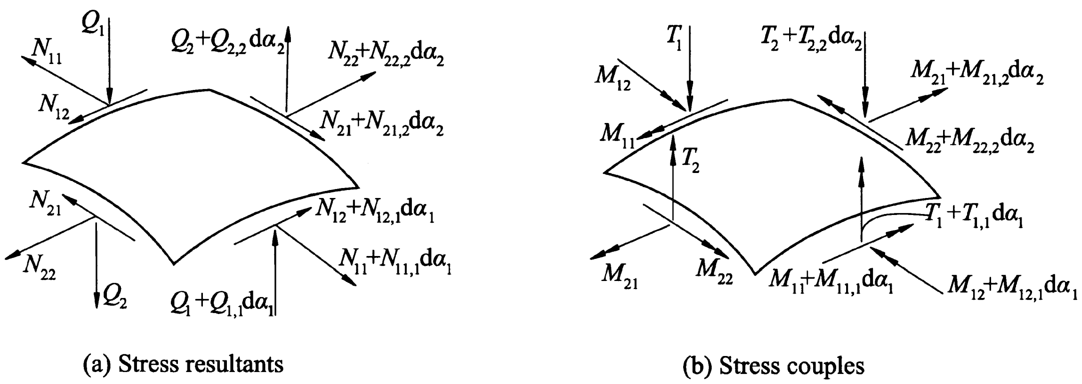

Fig. 2 Internal actions components acting on shell mid-surface element boundaries

portion of deformed surface is sided by vectors $\boldsymbol{\rho}_{,1}d\alpha_1$ and $\boldsymbol{\rho}_{,2}d\alpha_2$ (Fig. 1). For this surface element to be an element of a *shell*, we assume that its four sides are acted upon by vector forces and moments $\mathbf{N}_1A_2d\alpha_2$, $\mathbf{M}_2A_1d\alpha_1$, etc., in accordance with Fig. 1 and Fig. 2. We also assume the existence of vector surface forces and moments $\mathbf{q}A_1A_2d\alpha_1d\alpha_2$ and $\mathbf{m}A_1A_2d\alpha_1d\alpha_2$. It is worth noting that the previously defined $\mathbf{N}_i$ and $\mathbf{M}_i$ vectors are internal forces and moments acting per unit of undeformed length and that $\mathbf{q}$ and $\mathbf{m}$ are forces and moments distributed on the undeformed unitary mid-surface area as can be deduced from Fig. 1. The algebraic vectors defined above can be written as follows:

$$\mathbf{N}_i = N_{ij}\mathbf{t}_j + Q_i\mathbf{n}, \qquad \mathbf{M}_i = M_{ij}\mathbf{n}\times\mathbf{t}_j + T_i\mathbf{n} \tag{1a}$$

$$\mathbf{q} = q_i\mathbf{t}_i + p_n\mathbf{n}, \qquad \mathbf{m} = m_i\mathbf{n}\times\mathbf{t}_i + m_n\mathbf{n} \quad (i = 1, 2) \tag{1b}$$

where the first two components are directed along tangent vectors $\boldsymbol{\rho}_{,1}d\alpha_1$ and $\boldsymbol{\rho}_{,2}d\alpha_2$, while the third one is acting along the outward normal $\mathbf{n}$. It will be evident later how the previous component representation corresponds to the strain resultants and couples one. Consideration of increments in passing from sides α_i = *const*. to sides $\alpha_i + d\alpha_i$ = *const*. leads to a couple of vector dynamic equations, the first concerning the balance of forces and the other the balance of moments:

$$(A_2\mathbf{N}_1)_{,1} + (A_1\mathbf{N}_2)_{,2} + A_1A_2\mathbf{p} = \mathbf{0} \tag{2a}$$

$$(A_2\mathbf{M}_1)_{,1} + (A_1\mathbf{M}_2)_{,2} + \boldsymbol{\rho}_{,1} \times (A_2\mathbf{N}_1) + \boldsymbol{\rho}_{,2} \times (A_1\mathbf{N}_2) + A_1A_2\mathbf{m} = \mathbf{0} \tag{2b}$$

Each of the above equations is equivalent to three scalar component equations, as will presently be considered and represents an equilibrium equation for the shell infinitesimal element, written in terms of internal actions defined per unit length of coordinate lines on the reference surface (Figs. 1 and 2).

Precisely, the three scalar equations coming from Eq. (2a) are translational ones along the tangent $\mathbf{t}_1$, $\mathbf{t}_2$ and normal $\mathbf{n}$ directions and the last three equations are rotational equilibrium ones, about the same directions, respectively.

In addition to the balance Eqs. (2a-b), which are valid in the interior of the shell we also have balance equations for the element ds of the edges of the shell. With $\mathbf{N}$ and $\mathbf{M}$ being edge forces and moment intensities, we have as expressions for $\mathbf{N}$ and $\mathbf{M}$ in terms of the edge values of the stress resultants:

$$\mathbf{N} = \mathbf{N}_i\cos(n, \alpha_i) \qquad \mathbf{M} = \mathbf{M}_i\cos(n, \alpha_i) \tag{3}$$

In the previous, n indicates the outward normal direction of the boundary curve, tangent to shell surface, and the summation convention for repeated subscript indices is invoked.

3. Strain-displacement relations via principle of virtual work

Defining *virtual* displacements and *virtual* strains as sets of (infinitesimally small) kinematically admissible arbitrary displacements and strains which, by proper association with corresponding sets of internal forces and moments, give rise to a quantity of virtual work in a way that the equilibrium Eqs. (2)-(3) are equivalent to an integral equation of the form (Reissner and Wan 1967):

$$\int(\mathbf{q}\cdot\delta\boldsymbol{\rho} + \mathbf{m}\cdot\delta\boldsymbol{\varphi})dS + \oint(\mathbf{N}\cdot\delta\boldsymbol{\rho} + \mathbf{M}\cdot\delta\boldsymbol{\varphi})ds = \int(\mathbf{N}_i\cdot\delta\boldsymbol{\varepsilon}_i + \mathbf{M}_i\cdot\delta\boldsymbol{\kappa}_i)dS \tag{4}$$

In this integral equation, $dS = A_1A_2d\alpha_1d\alpha_2$, $\delta\mathbf{r}$ and $\delta\boldsymbol{\varphi}$ are virtual translational and rotational displacement vectors and $\delta\boldsymbol{\varepsilon}_i$ and $\delta\boldsymbol{\kappa}_i$ are virtual strain resultant and strain-couples vectors.

The way the previous equation is usually considered is to assume that $\delta\boldsymbol{\varepsilon}_i$ and $\delta\boldsymbol{\kappa}_i$ are known functions in terms of $\delta\boldsymbol{\rho}$, $\delta\boldsymbol{\varphi}$ and suitable derivatives, thereof, by means of integration by parts, in order to eliminate derivatives of $\delta\boldsymbol{\rho}$ and $\delta\boldsymbol{\varphi}$ and by considering $\delta\boldsymbol{\rho}$ and $\delta\boldsymbol{\varphi}$ arbitrary in the interior as well as along the boundary, satisfaction of Eqs. (2a-b)-(3) comes as necessary and sufficient a condition for the validity of Eq. (4).

In this paragraph, it is assumed that Eqs. (2a-b)-(3) are given a priori. In this manner they can be utilized to eliminate $\mathbf{q}$, $\mathbf{m}$, $\mathbf{N}$ and $\mathbf{M}$ in Eq. (4), with $\mathbf{N}_i$ and $\mathbf{M}_i$ now being arbitrary functions.

Introduction of Eqs. (2a-b)-(3) into Eq. (4) gives first:

$$\iint(\mathbf{N}_i \cdot \delta\boldsymbol{\varepsilon}_i + \mathbf{M}_i \cdot \delta\boldsymbol{\kappa}_i)A_1A_2 d\alpha_1 d\alpha_2 = \oint(\mathbf{N}_i \cdot \delta\boldsymbol{\rho} + \mathbf{M}_i \cdot \delta\boldsymbol{\varphi})\cos(n, \alpha_i)ds+$$

$$-\iint\{[(A_2\mathbf{N}_1)_{,1}+(A_1\mathbf{N}_2)_{,2}]\cdot\delta\boldsymbol{\rho}+[(A_2\mathbf{M}_1)_{,1}+(A_1\mathbf{M}_2)_{,2}+\boldsymbol{\rho}_{,1}\times(A_2\mathbf{N}_1)+\boldsymbol{\rho}_{,2}\times(A_1\mathbf{N}_2)]\cdot\delta\boldsymbol{\varphi}\}d\alpha_1 d\alpha_2 \quad (5)$$

Then, integrating by parts to eliminate derivatives of $\mathbf{N}_i$ and $\mathbf{M}_i$ in Eq. (5), one obtains:

$$\int(\mathbf{N}_i \cdot \delta\boldsymbol{\varepsilon}_i + \mathbf{M}_i \cdot \delta\boldsymbol{\kappa}_i)dS = \int\{\mathbf{N}_i \cdot [(\delta\boldsymbol{\rho})_{,i} + \boldsymbol{\rho}_{,i} \times \delta\boldsymbol{\varphi}] + \mathbf{M}_i \cdot (\delta\boldsymbol{\varphi})_{,i}\}\alpha_i^{-1}dS \quad (6)$$

recalling that $(\boldsymbol{\rho}_{,i} \times \mathbf{N}_i) \cdot \delta\boldsymbol{\varphi} = -(\boldsymbol{\rho}_{,i} \times \delta\boldsymbol{\varphi}) \cdot \mathbf{N}_i$.

Considering the internal actions $\mathbf{N}_i$ and $\mathbf{M}_i$ as arbitrary vector functions in Eq. (6), the virtual strain-displacement relations are obtained:

$$\alpha_i\delta\boldsymbol{\varepsilon}_i = (\delta\boldsymbol{\rho})_{,i} + \boldsymbol{\rho}_{,i} \times \delta\boldsymbol{\varphi}; \qquad \alpha_i\delta\boldsymbol{\kappa}_i = (\delta\boldsymbol{\rho})_{,i} \quad (7)$$

It has to be noted that since by assumption there exists a space function $\boldsymbol{\rho} = \boldsymbol{\rho}(\alpha_1, \alpha_2)$, one may write $(\delta\boldsymbol{\rho})_{,i} = \delta(\boldsymbol{\rho}_{,i})$ in Eq. (7), but as long as we've not established the existence of a function $\boldsymbol{\varphi}$ corresponding to the quantity $\delta\boldsymbol{\varphi}$ in Eq. (7) we are not in a position to replace $(\delta\boldsymbol{\varphi})_{,i}$ by $\delta(\boldsymbol{\varphi}_{,i})$. In what follows and throughout the whole work some simplifications will be assumed in order to overcome some difficulties such as the ones arising from foregoing relations of nonlinear theory. First of all we will reduce to geometrically linear equations.

4. Linearization

The equilibrium and strain-displacement relations of the linear theory can be obtained by simply replacing in Eqs. (2) the general radius vector $\boldsymbol{\rho}$ to points on the deformed shell reference surface by the given vector $\mathbf{r}$ to points on the undeformed surface.

Changing $\boldsymbol{\rho}_{,i}$ into $\mathbf{r}_{,i}$ in Eqs. (2) changes $\boldsymbol{\rho}_{,i}$ into $\mathbf{r}_{,i}$ in Eq. (7) and therewith $\delta\boldsymbol{\varepsilon}_i$ and $\delta\boldsymbol{\kappa}_i$ are given as linear combinations of $(\delta\boldsymbol{\rho})_{,i} = \delta(\mathbf{r} + \mathbf{u})_{,i} = \delta\mathbf{u}_{,i}$ and $\delta\boldsymbol{\varphi}$ and, finally, $(\delta\boldsymbol{\varphi})_{,i} = \delta(\boldsymbol{\varphi}_{,i})$.

In this way we can pass directly from virtual strain-displacement relations to actual strain displacement relations, in the following form:

$$A_i\boldsymbol{\varepsilon}_i = \mathbf{u}_{,i} + \mathbf{r}_{,i} \times \boldsymbol{\varphi}; \qquad A_i\boldsymbol{\kappa}_i = \boldsymbol{\varphi}_{,i} \quad (8)$$

We list in what follows the system of scalar strain-displacement relations which are equivalent to the vectorial relations (8) by writing:

$$\boldsymbol{\varepsilon}_i = \varepsilon_{ij}\mathbf{t}_j + \gamma_i\mathbf{n}; \qquad \boldsymbol{\kappa}_i = \kappa_{ij}\mathbf{n} \times \mathbf{t}_j + \lambda_i\mathbf{n} \quad (9a)$$

$$\mathbf{u}_i = u_i\mathbf{t}_i + u_n\mathbf{n}; \qquad \boldsymbol{\varphi} = \varphi_i\mathbf{n} \times \mathbf{t}_i + \omega\mathbf{n} \quad (9b)$$

where $\mathbf{r}_{,i} = A_i\mathbf{t}_i$ and $\mathbf{n} = \mathbf{t}_1 \times \mathbf{t}_2$, with the Gauss-Weingarten differentiation formulas (Reissner and Wan 1967):

$$\frac{\mathbf{t}_{1,1}}{A_1} = \frac{\mathbf{t}_2}{S_1} - \frac{\mathbf{n}}{R_{11}}; \quad \frac{\mathbf{t}_{2,1}}{A_1} = -\frac{\mathbf{t}_1}{S_1} - \frac{\mathbf{n}}{R_{12}}; \quad \frac{\mathbf{n}_{,1}}{A_1} = \frac{\mathbf{t}_1}{R_{11}} - \frac{\mathbf{t}_2}{R_{12}}; \tag{10a}$$

$$\frac{\mathbf{t}_{1,2}}{A_2} = \frac{\mathbf{t}_2}{S_2} - \frac{\mathbf{n}}{R_{21}}; \quad \frac{\mathbf{t}_{2,2}}{A_2} = -\frac{\mathbf{t}_1}{S_2} - \frac{\mathbf{n}}{R_{22}}; \quad \frac{\mathbf{n}_{,2}}{A_2} = \frac{\mathbf{t}_1}{R_{21}} - \frac{\mathbf{t}_2}{R_{22}} \tag{10b}$$

In the previous, S_1 and S_2 are in-plane radii of curvature given by $1/S_1 = -A_{1,2}/A_1A_2$ and $1/S_2 = A_{2,1}/A_1A_2$, while the $R_{ij} = R_{ji}$ are the usual out-of-plane radii of curvature of the surface (Reissner 1969). Introducing Eqs. (9)-(10) into Eq. (8) gives the expressions for in-plane strain resultants ε_{ij}, transverse (shearing) strain resultants γ_i, bending and twisting strain couples κ_{ij} and in-plane strain couples λ_i (Reissner 1969):

$$\varepsilon_{11} = \frac{u_{1,1}}{A_1} - \frac{u_2}{S_1} + \frac{w}{R_{11}}; \qquad \varepsilon_{22} = \frac{u_{2,2}}{A_2} + \frac{u_1}{S_2} + \frac{w}{R_{22}} \tag{11a}$$

$$\varepsilon_{12} = \frac{u_{2,1}}{A_1} + \frac{u_1}{S_1} + \frac{w}{R_{12}} - \omega; \qquad \varepsilon_{21} = \frac{u_{1,2}}{A_2} - \frac{u_2}{S_2} + \frac{w}{R_{21}} + \omega \tag{11b}$$

$$\kappa_{11} = \frac{\beta_{1,1}}{A_1} - \frac{\beta_2}{S_1} + \frac{\omega}{R_{12}}; \qquad \kappa_{22} = \frac{\beta_{2,2}}{A_2} + \frac{\beta_1}{S_2} - \frac{\omega}{R_{12}} \tag{12a}$$

$$\kappa_{12} = \frac{\beta_{2,1}}{A_1} + \frac{\beta_1}{S_1} - \frac{\omega}{R_{11}}; \qquad \kappa_{21} = \frac{\beta_{1,2}}{A_2} - \frac{\beta_2}{S_2} + \frac{\omega}{R_{22}} \tag{12b}$$

$$\gamma_1 = \frac{w_{,1}}{A_1} - \frac{u_1}{R_{11}} - \frac{u_2}{R_{12}} + \beta_1; \qquad \gamma_2 = \frac{w_{,2}}{A_2} - \frac{u_2}{R_{22}} - \frac{u_1}{R_{21}} + \beta_2 \tag{13}$$

$$\lambda_1 = \frac{\omega_{,1}}{A_1} - \frac{\beta_1}{R_{11}} + \frac{\beta_2}{R_{11}}; \qquad \lambda_2 = \frac{\omega_{,2}}{A_2} + \frac{\beta_2}{R_{12}} - \frac{\beta_1}{R_{22}} \tag{14}$$

Finally, it is to be noted that all six strain couple components (12a-b)-(14) are given in terms of components of rotational displacements only, while four of the six strain-resultant components are given in terms of both translational and rotational displacement components.

5. Basic simplifications of five D. O. F. shell theory

In the previous statements a fundamental concept has been ignored. In fact, no mathematical association has been established between internal stress resultants and couples and stress components acting on normal sections traced along coordinate directions of the reference surface of the shell. Defining this kind of relation is basilar in order to permit some simplification to the theory treated so far and to be able to make use of the G. D. Q. solution procedure. We admit that internal stress resultants and couples which rise from a concept of static equivalence with stress components represented in Fig. 3 are given by Gould (1999):

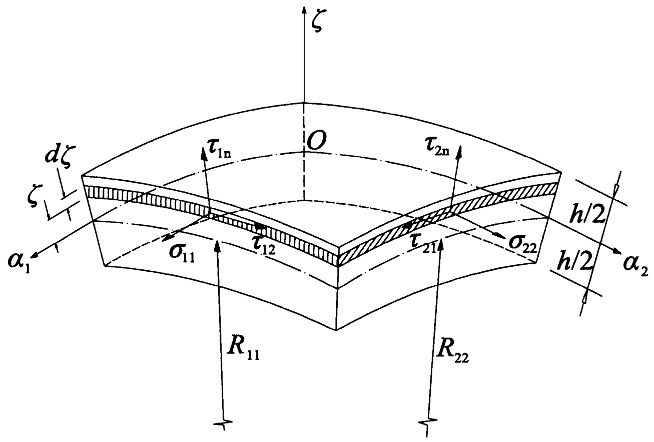

Fig. 3 Internal stress components acting on shell element coordinate sections

$$\begin{bmatrix} N_{11} \\ N_{12} \\ Q_1 \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{11} \\ \tau_{12} \\ \tau_{1n} \end{bmatrix} \left(1 + \frac{\varsigma}{R_{22}}\right) d\varsigma; \qquad \begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{11} \\ \tau_{12} \end{bmatrix} \varsigma \left(1 + \frac{\varsigma}{R_{22}}\right) d\varsigma; \tag{15a}$$

$$\begin{bmatrix} N_{22} \\ N_{21} \\ Q_2 \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{22} \\ \tau_{21} \\ \tau_{2n} \end{bmatrix} \left(1 + \frac{\varsigma}{R_{11}}\right) d\varsigma; \qquad \begin{bmatrix} M_{22} \\ M_{21} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{22} \\ \tau_{21} \end{bmatrix} \varsigma \left(1 + \frac{\varsigma}{R_{11}}\right) d\varsigma \tag{15b}$$

where h is the shell thickness and ζ is the coordinate along the normal to the reference surface (Fig. 3). In this way, assuming the symmetry of the stress tensor, the components M_{ij}, M_{ji}, N_{ij} and N_{ji} can be considered equal two by two with acceptable approximation and the sixth scalar equation coming from the system (2) is identically satisfied. The above mentioned equation is a rotational equilibrium one and, being of no use, puts in evidence how its dual degree of freedom (rotation ω about the normal-to-mid-surface axis) is of no use itself, in defining the displacement field and, therefore, can be discarded.

Accordingly, the remaining stress resultants and couples are grouped as follows:

$$\mathbf{R}(\alpha_1, \alpha_2, t) = [N_{11}\ N_{22}\ N_{12}\ M_{11}\ M_{22}\ M_{12}\ Q_1 Q_2]^T \tag{16}$$

Only five dynamic equilibrium equations remain:

$$[(A_2N_{11})_{,1} + (A_1N_{12})_{,2} + A_{1,2}N_{12} - A_{2,1}N_{22}] + Q_1((A_1A_2)/(R_{11})) + q_1A_1A_2 = \rho hA_1A_2(\partial^2 u_1/\partial t^2) \tag{17a}$$

$$[(A_2N_{12})_{,1} + (A_1N_{22})_{,2} + A_{2,1}N_{12} - A_{1,2}N_{11}] + Q_2((A_1A_2)/(R_{22})) + q_2A_1A_2 = \rho hA_1A_2(\partial^2 u_2/\partial t^2) \tag{17b}$$

$$[(A_2Q_1)_{,1} + (A_1Q_2)_{,2}] - N_{11}((A_1A_2)/(R_{11})) - N_{22}((A_1A_2)/(R_{22})) + q_nA_1A_2 = \rho hA_1A_2(\partial^2 u_n/\partial t^2) \tag{17c}$$

$$-(A_2M_{12})_{,1}-(A_1M_{22})_{,2}-A_{2,1}M_{12}+A_{1,2}M_{11}+Q_2A_1A_2-m_2A_1A_2 = (1/12)\rho h^3A_1A_2(\partial^2\beta_2/\partial t^2) \quad (17d)$$

$$(A_2M_{11})_{,1}+(A_1M_{12})_{,2}+A_{1,2}M_{12}-A_{2,1}M_{22}-Q_1A_1A_2+m_1A_1A_2 = (1/12)\rho h^3A_1A_2(\partial^2\beta_1/\partial t^2) \quad (17e)$$

and only five degrees of freedom are sufficient to fully define the motion of a point lying within the shell. On the right hand sides of Eq. (17), appropriate terms (Reddy 1984) have been introduced to account for the translational and rotary inertias. In the above system, the first three equations are equilibrium ones along tangents to coordinate lines $\mathbf{t}_1$, $\mathbf{t}_2$ and and along the normal $\mathbf{n}$ direction, respectively; while the last two are rotational equilibrium equations about coordinate tangential directions. The strain-displacements relationships simplify too, assuming that strain mixed components are approximately equal and that only $1/R_{11}$ and $1/R_{22}$ are not null. Then, only eight independent components remain:

$$\varepsilon_{11} = \frac{u_{1,1}}{A_1}+\frac{A_{1,2}}{A_1A_2}u_2+\frac{u_n}{R_{11}}; \qquad \varepsilon_{22} = \frac{u_{2,2}}{A_2}+\frac{A_{2,1}}{A_1A_2}u_1+\frac{u_n}{R_{22}} \quad (18a)$$

$$\bar{\varepsilon}_{12} = \varepsilon_{12}+\varepsilon_{21} = \frac{u_{2,1}}{A_1}-\frac{A_{1,2}}{A_1A_2}u_1+\frac{u_{1,2}}{A_2}-\frac{A_{2,1}}{A_1A_2}u_2 \quad (18b)$$

$$\kappa_{11} = \frac{\beta_{1,1}}{A_1}+\frac{A_{1,2}}{A_1A_2}\beta_2; \qquad \kappa_{22} = \frac{\beta_{2,2}}{A_2}+\frac{A_{2,1}}{A_1A_2}\beta_1 \quad (19a)$$

$$\bar{\kappa}_{12} = \kappa_{12}+\kappa_{21} = \frac{\beta_{2,1}}{A_1}-\frac{A_{1,2}}{A_1A_2}\beta_1+\frac{\beta_{1,2}}{A_2}-\frac{A_{2,1}}{A_1A_2}\beta_2 \quad (19b)$$

$$\gamma_1 = \beta_1+\frac{u_{n,1}}{A_1}-\frac{u_1}{R_{11}}; \qquad \gamma_2 = \beta_2+\frac{u_{n,2}}{A_2}-\frac{u_2}{R_{22}} \quad (20)$$

grouped in the algebraic vector $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = [\varepsilon_{11}\varepsilon_{22}\bar{\varepsilon}_{12}\kappa_{11}\kappa_{22}\bar{\kappa}_{12}\gamma_1\gamma_2]^T \quad (21)$$

When the fundamental equations are obtained, for this generally shaped shell structures, the specialized equations for the rotational case will be derived, assuming a proper coordinate system $(\alpha_1, \alpha_2, \zeta)$.

The displacement field assumption that is retained is that the three displacement components along the coordinate directions $U_1(\alpha_1, \alpha_2, \zeta)$, $U_2(\alpha_1, \alpha_2, \zeta)$, $U_n(\alpha_1, \alpha_2, \zeta)$, are locally defined by the displacements of points lying on the reference surface (Fig. 4). In fact, we have:

$$U_1(\alpha_1, \alpha_2, \zeta) = u_1+\zeta\beta_1, \qquad u_1 = u_1(\alpha_1, \alpha_2) = U_1(\alpha_1, \alpha_2, 0) \quad (22a)$$

$$U_2(\alpha_1, \alpha_2, \zeta) = u_2+\zeta\beta_2, \qquad u_2 = u_2(\alpha_1, \alpha_2) = U_2(\alpha_1, \alpha_2, 0) \quad (22b)$$

$$U_n(\alpha_1, \alpha_2, \zeta) = u_n, \qquad u_n = u_n(\alpha_1, \alpha_2) = U_n(\alpha_1, \alpha_2, 0) \quad (22c)$$

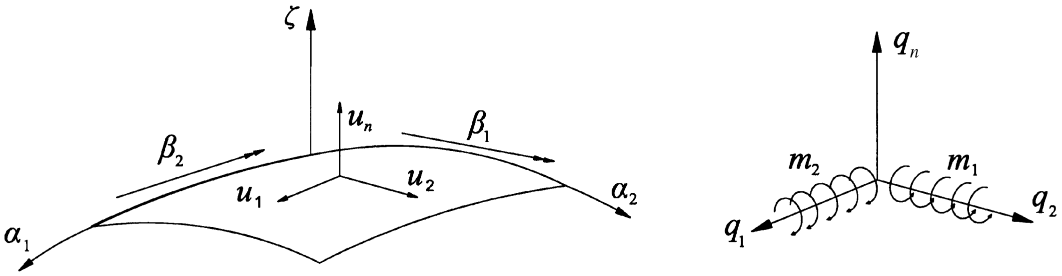

Fig. 4 Displacements, rotations and reciprocal load intensities defined upon shell element mid-surface

from which it can be seen that for the tangential displacements U_1 and U_2 is assumed a pattern linearly varying through the thickness, depending on the local normal to mid-surface rotations about the coordinate directions β_1 and β_2, while a constant pattern of the normal translation U_n along the ζ coordinate is hypothesized.

In this way, in order to determine the complete assessment of displacements throughout the shell structure, five degrees of freedom (u_1, u_2, u_n, β_1 and β_2) pertaining to the reference surface are to be obtained (Fig. 4). Then, we have the vector of the reference surface generalized displacements:

$$\mathbf{u}(\alpha_1, \alpha_2, t) = [u_1 \;\; u_2 \;\; u_n \;\; \beta_1 \;\; \beta_2]^T \tag{23}$$

and the dual algebraic vector of mid-surface distributed loads components:

$$\mathbf{q}(\alpha_1, \alpha_2, t) = [q_1 \;\; q_2 \;\; q_n \;\; m_1 \;\; m_2]^T \tag{24}$$

where q_1, q_2 and q_n are forces acting on mid-surface unit area, while m_1 and m_2 are distributed couples per unit length of parametric lines, each corresponding to β_2 and β_1 rotations respectively (Fig. 4).

6. Constitutive equations

The shell material assumed in this paper and in the numerical examples presented is a mono-laminar linearly elastic isotropic one. Accordingly, the following constitutive equations relate internal stress resultants and couples with strain components on the middle surface (Gould 1999):

$$N_{11} = E_1\varepsilon_{11} + E_2\varepsilon_{22}; \quad N_{22} = E_2\varepsilon_{11} + E_{12}\varepsilon_{22}; \quad N_{12} = E_3\bar{\varepsilon}_{12} \tag{25a}$$

$$M_{11} = E_4\kappa_{11} + E_5\kappa_{22}; \quad M_{22} = E_5\kappa_{11} + E_4\kappa_{22}; \quad M_{12} = E_6\bar{\kappa}_{12} \tag{25b}$$

$$Q_1 = E_7\gamma_1; \quad Q_2 = E_7\gamma_2 \tag{25c}$$

where E is Young modulus, ν is the Poisson ratio and $E_1 = Eh/2(1-\nu^2)$, $E_2 = \nu E_1$, $E_3 = (1-\nu)E_1/2$, $E_4 = Eh^3/[12(1-\nu^2)]$, $E_5 = \nu E_4$, $E_6 = (1-\nu)E_4/2$, $E_7 = \lambda E_3$. λ is a transverse shearing factor such that $\lambda = 5/6$.

7. Specialization to rotational geometry and fundamental system

When the shell is rotational, the shape of the reference surface is completely assigned by the cartesian equation of the meridional curve $f(Z, R) = 0$ or $R = R(Z)$ (Fig. 5(a)). In this case, a fine choice for coordinate lines on the reference surface, is to take meridians and parallel circles, respectively. The radii of curvature $R_{11} = R_1(Z) = R_\phi$ and $R_{22} = R_2(Z) = R_\theta = R(Z)/\sin\phi(Z)$ become meridian radii of curvature and grand-normal radii respectively, while $R = R(Z)$ is the parallel radius of the shell. Each of the three kind of radii of curvature depend on the Z coordinate only.

The following geometric relationships are valid (Figs. 5(a)-(b)):

$$R_{11}(Z) = R_1(Z) = \frac{[1 + (R_{,Z})^2]^{3/2}}{R_{,ZZ}} = R_\phi(Z) \tag{26a}$$

$$R_{22}(Z) = R_2(Z) = R_\theta(Z) = R/\sin\theta \tag{26b}$$

$$\cos\phi(Z) = R_{,Z}/(RA_1) \tag{26c}$$

A suitable system of curvilinear coordinates, then, is to assign $(\alpha_1, \alpha_2) \equiv (Z, \theta)$ where Z is a vertical abscissa measured from a proper origin on the axis of revolution, while θ measures the longitudinal angle in the horizontal plane (Figs. 5(a)-(b)). Using this coordinate system, the Lamè parameters take the following form:

$$A_1(Z) = [1 + (R_{,Z})^2]^{1/2} \tag{27a}$$

$$A_2(Z) = R(Z) \tag{27b}$$

and the equilibrium Eqs. (17a-e) become (Reddy 1984):

$$[(RN_{,\phi})_{,Z} + (A_1 N_{\phi\theta})_{,\theta} - R_{,Z}N_\theta] + A_1 Q_\phi(R/R_\phi) + q_1 RA_1 = \rho h A_1 R(\partial^2 u/\partial t^2) \tag{28a}$$

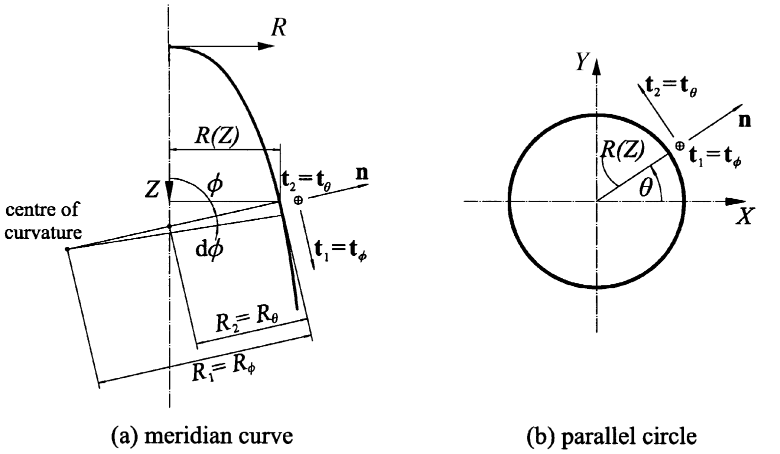

Fig. 5 Shell of revolution. (a) meridian section (b) parallel section

$$[(RN_{\phi\theta})_{,Z} + (A_1N_\theta)_{,\theta} - R_{,Z}N_{\phi\theta}] + Q_\theta(A_1\sin\phi) + q_2RA_1 = \rho hA_1R(\partial^2 v/\partial t^2) \tag{28b}$$

$$[(RQ_\phi)_{,Z} + (A_1Q_\theta)_{,\theta}] - A_1N_\phi(R/R_\phi) - N_\theta(A_1\sin\phi) + q_nRA_1 = \rho hA_1R(\partial^2 w/\partial t^2) \tag{28c}$$

$$-(RM_{\phi\theta})_{,Z} - (A_1M_\theta)_{,\theta} - R_{,Z}M_{\theta Z} + A_{1,2}M_\phi + RA_1Q_\theta - RA_1m_\theta = (1/12)\rho h^3A_1R(\partial^2\beta_\theta/\partial t^2) \tag{28d}$$

$$(RM_\phi)_{,Z} + (A_1M_{\phi\theta})_{,\theta} - R_{,Z}M_\theta - RA_1Q_\phi + RA_1m_\phi = (1/12)\rho h^3A_1R(\partial^2\beta_\phi/\partial t^2) \tag{28e}$$

with the following substitution:

$$\mathbf{q}(Z,\theta,t) = [q_\phi \;\; q_\theta \;\; q_n \;\; m_\phi \;\; m_\theta]^T = [q_1 \;\; q_2 \;\; q_n \;\; m_1 \;\; m_2]^T \tag{29}$$

and

$$\mathbf{R}(Z,\theta,t) = [N_\phi \;\; N_\theta \;\; N_{\phi\theta} \;\; M_\phi \;\; M_\theta \;\; M_{\phi\theta} \;\; Q_\phi \;\; Q_\theta]^T = [N_{11} \;\; N_{22} \;\; N_{12} \;\; M_{11} \;\; M_{22} \;\; M_{12} \;\; Q_1 \;\; Q_2]^T \tag{30}$$

The strain-displacement relationships reduce to:

$$\varepsilon_\phi = (1/A_1)(u_{,Z} + w/R_\phi) \tag{31a}$$

$$\varepsilon_\theta = (1/R)(v_{,\theta} + \cos\phi u + \sin\phi w) \tag{31b}$$

$$\bar{\varepsilon}_{\phi\theta} = (1/A_1)(v_{,Z}) + (1/R)(u_{,\theta} - \cos\phi v) \tag{31c}$$

$$\kappa_\phi = (1/A_1)(\beta_{\phi,Z}) \tag{31d}$$

$$\kappa_\theta = (1/R)\beta_{\theta,\theta} + (\cos\phi/R)\beta_\phi \tag{31e}$$

$$\bar{\kappa}_{\phi\theta} = (1/A_1)\beta_{\theta,Z} + (1/R)(\beta_{\phi,\theta} - \cos\phi\beta_\theta) \tag{31f}$$

$$\gamma_\phi = \beta_\phi + w_{,Z}/A_1 - u/R_\phi \tag{31g}$$

$$\gamma_\theta = \beta_\theta + (1/R)(w_{,\theta} - \sin\phi v) \tag{31h}$$

with

$$\mathbf{u}(Z,\theta,t) = [u \;\; v \;\; w \;\; \beta_\phi \;\; \beta_\theta]^T = [u_1 \;\; u_2 \;\; u_n \;\; \beta_1 \;\; \beta_2]^T \tag{32}$$

and

$$\boldsymbol{\varepsilon}(Z,\theta) = [\varepsilon_\phi \;\; \varepsilon_\theta \;\; \bar{\varepsilon}_{\phi\theta} \;\; \kappa_\phi \;\; \kappa_\theta \;\; \bar{\kappa}_{\phi\theta} \;\; \gamma_\phi \;\; \gamma_\theta]^T = [\varepsilon_{11} \;\; \varepsilon_{22} \;\; \bar{\varepsilon}_{12} \;\; \kappa_{11} \;\; \kappa_{22} \;\; \bar{\kappa}_{12} \;\; \gamma_1 \;\; \gamma_2]^T \tag{33}$$

The preceding sets of Eqs. (28a-e) and (31a-h) can be further specialized if the shell of revolution has a particular meridian shape. In fact one has:

- circular plates specializations: $1/R_\phi = 1/R_\theta = 0$, $Z \equiv R(Z)$
- conical shells specializations : $1/R_\phi = 0$, $R(Z) = R_u - Z\mathrm{tg}\alpha$ (α: opening angle, R_u base radius)
- spherical shells specializations: $R_\phi = R_\theta = const. = a$, $R(Z) = [a^2 - (a - Z)^2]^{1/2}$

Introducing the kinematic Eqs. (31a-h) into the constitutive relations (25a-c) and then substituting the obtained modified constitutive equations into the set of equilibrium Eqs. (28a-e) leads to the following new set of equations of motion, expressed in terms of displacements components of points lying on the reference surface:

$$\mathsf{L}_{11}u + \mathsf{L}_{12}v + \mathsf{L}_{13}w + \mathsf{L}_{14}\beta_\phi + \mathsf{L}_{15}\beta_\theta = \rho hR\ddot{u} - q_\phi R \tag{34a}$$

$$\mathsf{L}_{21}u + \mathsf{L}_{22}v + \mathsf{L}_{23}w + \mathsf{L}_{24}\beta_\phi + \mathsf{L}_{25}\beta_\theta = \rho hR\ddot{v} - q_\theta R \tag{34b}$$

$$\mathsf{L}_{31}u + \mathsf{L}_{32}v + \mathsf{L}_{33}w + \mathsf{L}_{34}\beta_\phi + \mathsf{L}_{35}\beta_\theta = \rho hR\ddot{w} - q_n R \tag{34c}$$

$$\mathsf{L}_{41}u + \mathsf{L}_{42}v + \mathsf{L}_{43}w + \mathsf{L}_{44}\beta_\phi + \mathsf{L}_{45}\beta_\theta = (\rho h^3/12)R\ddot{\beta}_\theta - m_\theta R \tag{34d}$$

$$\mathsf{L}_{51}u + \mathsf{L}_{52}v + \mathsf{L}_{53}w + \mathsf{L}_{54}\beta_\phi + \mathsf{L}_{55}\beta_\theta = (\rho h^3/12)R\ddot{\beta}_\phi - m_\phi R \tag{34e}$$

or else in matrix notation:

$$\mathsf{L}\mathbf{u} + \mathbf{q} = \mathbf{M}\ddot{\mathbf{u}} \tag{35}$$

The differential operator of elasticity $\mathbf{L}$ and the matrix $\mathbf{M}$ can be found in detail in Artioli and Viola (2003), for the shell types treated in this paper.

Last, a proper set of boundary conditions is to necessary, in order to get a well posed problem. The following are the types of boundary edges presented in the numerical examples examined in the following:

- free edge: $N_\phi = N_{\phi\theta} = Q_\phi = M_\phi = M_{\phi\theta} = 0$ (36a)
- simply supported edge : $u = v = w = M_\phi = \beta_\theta = 0$ (36b)
- clamped edge: $u = v = w = \beta_\phi = \beta_\theta = 0$ (36c)

and can easily be re-written in terms of displacement components only, as we have done for the equilibrium equations.

8. G. D. Q. technique fundamentals

The Differential Quadrature Method (D. Q. M.), originated by Bellman and Casti (1971, 1972) is an efficient numerical method for the rapid solution of linear and nonlinear partial differential equations. As an approximate technique, the G. D. Q. method is based on the concept that the values of the derivatives of a function can be approximated by weighted linear sums of the function values at all sampling points within the domain under consideration.

Shu and Richards (1992) slightly modified the original procedure by a recurrence relationship useful in finding the various weighting coefficients for any order derivatives. Their method rids the original D. Q. method of ill conditioning which had plagued the previous method. Since then, there have been many publications on both the theoretical development and the engineering application of the method. An excellent review paper contributed by Bert and Malik (1996) summarized a detailed reference list on D. Q. M. applications to several fields of engineering numerical analysis.

In what follows it is meant to give a short resume of basic concept of a convenient formulation of the technique in argument which will be applied to the modal analysis of shells of revolution.

The fundamental concepts underlying the G. D. Q. method are that a continuous function can be approximated by a higher-order polynomial in the overall domain and that a derivative of a function with respect to a variable at any discrete point can be put in the form of a weighted linear sum of function values at all discrete points chosen in the overall domain of that variable.

Assuming a 2-D function, on a regular rectangular domain, the derivatives of the generic function $f(Z, \theta)$ at a point (Z_i, θ_i) have the following representation:

$$\left.\frac{\partial^r f(Z, \theta)}{\partial Z^r}\right|_{(Z_i, \theta_j)} = \sum_{k=1}^{N_1} A_{ik}^{(r)} f_{kj} \tag{37a}$$

$$\left.\frac{\partial^s f(Z, \theta)}{\partial \theta^s}\right|_{(Z_i, \theta_j)} = \sum_{l=1}^{N_2} B_{jl}^{(s)} f_{ij} \tag{37b}$$

$$\left.\frac{\partial^{r+s} f(Z, \theta)}{\partial Z^r \partial \theta^s}\right|_{(Z_i, \theta_j)} = \sum_{k=1}^{N_1}\sum_{l=1}^{N_2} A_{ik}^{(r)} B_{jl}^{(s)} f_{kl} \tag{37c}$$

where $f_{ij} = f(Z_i, \theta_j)$ and $f(Z, \theta)$ can represent a general unknown function of the problem.

A_{ik} and B_{jl} are the so-called weighting coefficients of the partial derivatives, with respect to the Z and θ coordinates, respectively. The values assumed by the weighted coefficients (w.c.) are chosen so to give exact values of derivatives for a particular class of functions, called test or trial functions. Therefore, the values A_{ik} and B_{jl} are dependent on the kind of trial functions as well as on the distribution of sampling points on the domain.

The method of determining the coefficients is available from some references in the literature and can be summarized as follows.

Let N be the number of points chosen on the domain, then the Lagrange polynomials can be put in the following form (Redekop and Xu 1999):

$$g(Z) = \frac{M(Z)}{(Z - Z_i)M(Z_i)_{,Z}} \tag{38}$$

where $i = 1, 2, \ldots, N$ and, $M(Z) = \prod_{j=1}^{N}(Z - Z_j)$

Substitution of Eq. (38) into Eqs. (37a-c) gives the general form for the "first-order" w.c.:

$$A_{ik}^{(1)} = \frac{M(Z_i)_{,Z}}{(Z_i - Z_k)M(Z_k)_{,Z}} \qquad (i \neq k) \tag{39a}$$

$$A_{ii}^{(1)} = \frac{M(Z_i)_{,ZZ}}{2M(Z_i)_{,Z}} \tag{39b}$$

where $i, k = 1, 2, \ldots, N$.

A recurrence formula for these coefficients has been obtained for the general $A_{ik}^{(r)}$ coefficients, by Shu and Richards (1992) and reads as follows:

$$A_{ik}^{(r)} = r\left[A_{ik}^{(r-1)}A_{ik}^{(1)} - \frac{A_{ik}^{(r-1)}}{(Z_i - Z_k)}\right] \qquad (i \neq k) \tag{40a}$$

$$A_{ii}^{(r)} = -\sum_{\substack{k=1\\k\neq i}}^{N_1} A_{ik}^{(r)} \tag{40b}$$

where i, $k = 1, 2, \ldots, N$.

Finally, in the present paper, the following two sampling-points-choosing rules are presented:

$$Z_i = \frac{1-\cos[(i-1)\pi/(N-1)]}{2}(b-a), \qquad i = 1, 2, \ldots, N \tag{41}$$

which gives a so-called cosine distribution of the points along the domain $Z \in [a, b]$, and a uniform distribution onto the domain $[a, b]$:

$$Z_i = \frac{(i-1)}{(N-1)}(b-a), \qquad i = 1, 2, \ldots, N \tag{42}$$

9. Fourier expansion of problem variables

A useful technique for separation of variables is offered by Fourier series (Gould 1985). Dealing with shells of revolution, every dependent variable of the problem can be interpreted as a 2π periodic continuous function with respect to the θ coordinate. Accordingly, only a partial Fourier series expansion is sufficient:

$$\mathbf{u}(Z, \theta, t) = \sum_{n=0}^{\infty} \mathbf{F}_1^n(\theta)\mathbf{u}^n(Z, t) \quad \text{where} \quad \mathbf{u}^n(Z, t) = [u^n \ v^n \ w^n \ \beta_\phi^n \ \beta_\theta^n]^T \tag{43a}$$

$$\mathbf{R}(Z, \theta, t) = \sum_{n=0}^{\infty} \mathbf{F}_2^n(\theta)\mathbf{R}^n(Z, t) \quad \text{where} \quad \mathbf{R}^n(Z, t) = [N_\phi^n \ N_\theta^n \ N_{\phi\theta}^n \ M_\phi^n \ M_\theta^n \ M_{\phi\theta}^n \ Q_\phi^n \ Q_\theta^n]^T \tag{43b}$$

$$\boldsymbol{\varepsilon}(Z, \theta, t) = \sum_{n=0}^{\infty} \mathbf{F}_2^n(\theta)\boldsymbol{\varepsilon}^n(Z, t) \quad \text{where} \quad \boldsymbol{\varepsilon}^n(Z, t) = [\varepsilon_\phi^n \ \varepsilon_\theta^n \ \varepsilon_{\phi\theta}^n \ \kappa_\phi^n \ \kappa_\theta^n \ \kappa_{\phi\theta}^n \ \gamma_\phi^n \ \gamma_\theta^n]^T \tag{43c}$$

$$\mathbf{q}(Z, \theta, t) = \sum_{n=0}^{\infty} \mathbf{F}_1^n(\theta)\mathbf{q}^n(Z, t) \quad \text{where} \quad \mathbf{q}^n(Z, t) = [q_\phi^n \ q_v^n \ q_w^n \ m_\phi^n \ m_\theta^n]^T \tag{43d}$$

In the above, $\mathbf{F}_1^n$ and $\mathbf{F}_2^n$ are diagonal matrices of the following form:

$$\mathbf{F}_1^n = diag(\cos n\theta \ \sin n\theta \ \cos n\theta \ \sin n\theta \ \cos n\theta) \tag{44a}$$

$$\mathbf{F}_2^n = diag(\cos n\theta \ \cos n\theta \ \sin n\theta \ \cos n\theta \ \cos n\theta \ \sin n\theta \ \cos n\theta \ \sin n\theta) \tag{44b}$$

and the time-dependency of problem variables has not been specified still.

The technique hereby exposed is integrally taken by the shells of revolution ring finite element analysis and has a very powerful application to the problem in argument, where it's applied a priori to the formulation of the fundamental equations of equilibrium in terms of generalized displacements.

The static and kinetic variables of the problem, now, can be conveniently looked at both as simple functions or as a truncated sum of harmonic terms. As will be seen later, substitution of the series expansions within fundamental sets of equilibrium, kinematic and constitutive equations leads to a simplification of the problem.

In fact, as a series of terms can be deduced from (43a-d), equally a series of dynamic equilibrium equations can be derived, each characterized by an harmonic number n and each to be solved separately.

Omitting the various mathematical calculations and introducing the expansions (43a-d) into the equilibrium (28a-e), kinematic (31a-h) and constitutive (25a-c) equations too, one obtains a series of fundamental systems (34a-e), in terms of internal reference surface displacements n-harmonic components only (Artioli and Viola 2003):

$$K_{11}^n u^n + K_{12}^n v^n + K_{13}^n w^n + K_{14}^n \beta_\phi^n + K_{15}^n \beta_\theta^n = \rho h R \ddot{u}^n - q_\phi^n R \tag{45a}$$

$$K_{21}^n u^n + K_{22}^n v^n + K_{23}^n w^n + K_{24}^n \beta_\phi^n + K_{25}^n \beta_\theta^n = \rho h R \ddot{v}^n - q_\theta^n R \tag{45b}$$

$$K_{31}^n u^n + K_{32}^n v^n + K_{33}^n w^n + K_{34}^n \beta_\phi^n + K_{35}^n \beta_\theta^n = \rho h R \ddot{w}^n - q_n^n R \tag{45c}$$

$$K_{41}^n u^n + K_{42}^n v^n + K_{43}^n w^n + K_{44}^n \beta_\phi^n + K_{45}^n \beta_\theta^n = (\rho h^3 / 12) R \ddot{\beta}_\theta^n - m_\theta^n R \tag{45d}$$

$$K_{51}^n u^n + K_{52}^n v^n + K_{53}^n w^n + K_{54}^n \beta_\phi^n + K_{55}^n \beta_\theta^n = (\rho h^3 / 12) R \ddot{\beta}_\phi^n - m_\phi^n R \tag{45e}$$

or, in matrix notation:

$$\mathbf{K}^n \mathbf{u}^n + \mathbf{q}^n = \mathbf{M}^n \ddot{\mathbf{u}}^n \tag{46}$$

Obviously, an analogous treatment can be applied to the boundary conditions (36a-c) and a series of harmonic boundary conditions sets are obtained, each one characterized by the harmonic number n.

Finally, in order to apply the G. D. Q. solution procedure to the harmonic equations of motions we first delete all the distributed forces terms in order to obtain a free vibration system and assume an harmonic time dependency of the mid-surface displacement harmonic circular components with frequencies ω:

$$\mathbf{u}^n(Z, t) = [u^n \;\; v^n \;\; w^n \;\; \beta_\phi^n \;\; \beta_\theta^n]^T = \bar{\mathbf{u}}^n(Z) e^{i\omega^n t} = [\bar{u}^n \;\; \bar{v}^n \;\; \bar{w}^n \;\; \bar{\beta}_\phi^n \;\; \bar{\beta}_\theta^n]^T e^{i\omega^n t} \tag{47}$$

being i the imaginary unit.

The G. D. Q. technique is applied to the above mentioned system of equilibrium equations, keeping in mind that Eqs. (45a-e) and corresponding sets of boundary conditions contain derivatives with respect to the longitudinal coordinate Z only.

The resulting governing systems, for the dynamic case, respectively, take the following form:

$$\begin{bmatrix} \underset{10\times 10}{\mathbf{K}_{bb}^n} & \underset{10\times 5N-10}{\mathbf{K}_{bd}^n} \\ \underset{5N-10\times 10}{\mathbf{K}_{db}^n} & \underset{5N-10\times 5N-10}{\mathbf{K}_{dd}^n} \end{bmatrix} \begin{bmatrix} \underset{10\times 1}{\bar{\mathbf{u}}_b^n} \\ \underset{5N-10}{\bar{\mathbf{u}}_d^n} \end{bmatrix} - (\omega^n)^2 \begin{bmatrix} \underset{10\times 10}{\mathbf{M}_{bb}^n} & \underset{10\times 5N-10}{\mathbf{M}_{bd}^n} \\ \underset{5N-10\times 10}{\mathbf{M}_{db}^n} & \underset{5N-10\times 5N-10}{\mathbf{M}_{dd}^n} \end{bmatrix} \begin{bmatrix} \underset{10\times 1}{\bar{\mathbf{u}}_b^n} \\ \underset{5N-10}{\bar{\mathbf{u}}_d^n} \end{bmatrix} = \mathbf{0} \tag{48}$$

where the subscripts b and d stand for boundary and domain, respectively, in a way that b-equations represent the discretized boundary conditions which are valid only for the points lying on constrained edge of the shell; while d-equations are proper equilibrium equations discretized onto points of the interior of the domain. Hence, in Eq. (48), only submatrix $\mathbf{M}_{dd}$ is not made of zeroes.

The equilibrium equations, in a such a way, show to become a generalized eigenvalue problem when modelled with the aim of the G. D. Q. technique. The solution procedure takes advance now of the kinematic condensation, in order to obtain a system of equations in terms of domain d.o.f. only:

$$\overline{\mathbf{K}}^n_{dd}\overline{\mathbf{u}}^n_d - (\omega^n)^2\overline{\mathbf{M}}^n_{dd}\overline{\mathbf{u}}^n_d = \mathbf{0} \tag{49}$$

where subscripted matrices are, in the order, stiffness and mass condensed matrices of the discretized system.

Once the eigenfrequencies are calculated and the interior harmonic displacements components are obtained, one can trace the pattern of generalized displacements along meridional direction by simply applying a suitable interpolation scheme (i.e., Lagrange interpolation) and then obtain the approximated bidimensional displacement field $\tilde{\mathbf{u}}(Z, \theta)$ recalling the expansions (43a). Last, applying strain-displacements relationships and constitutive equations one can recover the approximated state of stress in terms of internal actions $\tilde{\mathbf{R}}(Z, \theta)$.

9.1 Assembling of discretized boundary conditions

A remarkable aspect of the exposed solution procedure is the ability of implementing appropriate sets of boundary conditions, without any approximation, following Eqs. (36a-c). This is due to the specific first order shear deformation shell theory adopted, in a way that it permits to deal with "paired" sets of conditions. In fact, for those points lying in the interior of the domain five harmonic dynamic equilibrium equations hold and the same number of boundary assignments need to be stated for the end edges of the shell. This feature pertaining to the present five-degrees-of-freedom formulation permits to avoid the so called delta-point technique for boundary conditions, often appearing in some recent papers regarding shear-undeformable shells G. D. Q. analyses (Bert and Malik 1996, Redekop and Xu 1999).

Another interesting particular case is the closed-apex rotational shell or dome, in which the material continuity requirements in correspondence of the top often imposes some approximation to the numerical analysis of the problem. In fact, in many finite element shell formulation the continuity conditions need to be replace with a little fictitious circular hole, trying not to modify too much the shape of the structure. In this way, a proper set of assignments for harmonic displacements components, depending on the harmonic number considered, needs to be imposed equivalently to a set of boundary conditions (Gould 1985). Precisely, for some harmonic numbers, these apex conditions are less than five and in such cases the "missing" ones must be replaced with appropriate equilibrium ones so to get a correct total of five discretized equations, holding at the dome vertex. For further details on this very topic the interested reader is referred to Artioli and Viola (2003).

10. Numerical results and discussion

10.1 Freely vibrating simply supported circular plate

To show the accuracy and precision of the present method, we first consider a "flat" axisymmetric shell in free vibration (Fig. 6). The boundary parallel transverse fibres are free to rotate in the radial plane. The computed non-dimensional eigenparameters $\lambda = [12(1-\nu^2)\rho a^4\omega^2/(Eh^2)]^{1/2}$ are plotted in Table 1, where in brackets are reported those available from a spline finite element solution by Luah and Fan (1989). For this analysis, a few (11-13) cosine-spaced sampling points are sufficient to yield good precision for the first 3 modes; while to get similarly good approximation even for higher modes a little more points (15-17) need to be fixed onto the domain.

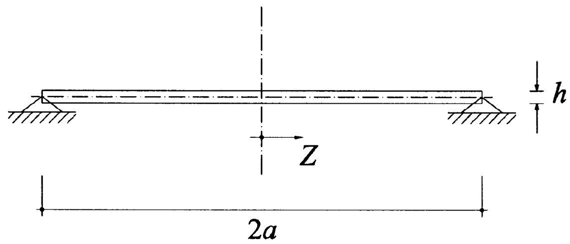

Fig. 6 A circular flat plate simply supported along the boundary middle parallel

Table 1 Frequency parameter $\lambda = [12(1-\nu^2)\rho a^4\omega^2/(Eh^2)]^{1/2}$ for the simply supported circular plate shown in Fig. 6

Harmonic number n	Mode number m					
	1	2	3	4	5	6
0	4.93515 [4.93515]	12.3899893 [29.7201]	74.1553 [74.1576]	138.324 [138.329]	222.164 [224.264]	324.896 [326.013]
1	13.6586 [13.8982]	49.9285 [48.4791]	102.909 [102.776]	178.803 [176.816]	270.489 [270.630]	384.078 [384.279]
2	25.6132 [25.6128]	70.1163 [70.1150]	134.295 [134.296]	218.195 [218.213]	321.826 [321.909]	445.187 [445.467]
3	39.9570 [39.9573]	94.5476 [95.5496]	168.670 [168.681]	262.474 [262.519]	375.991 [376.142]	509.231 [509.661]
4	56.8411 [56.8416]	121.700 [121.703]	205.845 [205.859]	309.593 [309.649]	433.021 [433.205]	576.155 [576.673]
5	76.2022 [76.2031]	151.514 [151.519]	245.769 [245.787]	359.513 [359.581]	492.884 [493.104]	645.932 [646.548]

10.2 Clamped free truncated conical shell in free vibration

Fig. 7 shows the geometry of a truncated conical shell with one clamped edge, the other one free. Results are plotted in Fig. 8 and Table 2, in terms of cyclic frequencies $f = \omega/2\pi$ for the first three

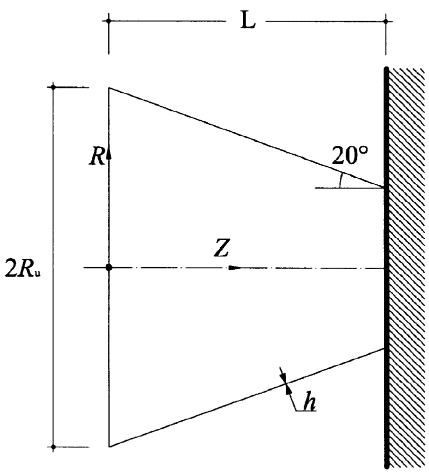

Fig. 7 A clamped-free conical shell with uniform thickness (L = 19.71 mm, R_u/L = 0.6384, h = 1.016 mm, E = 2.069 × 10^{11} N/m^2, ρ = 7868 Kg/m^3)

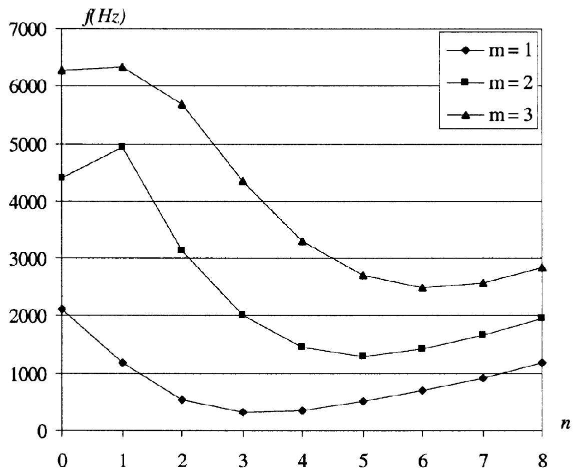

Fig. 8 Cyclic frequencies f of the first three modes for the conical shell shown in Fig. 7

Table 2 Cyclic frequencies $f(Hz)$ for the clamped free conical shell shown in Fig. 7

	Sen & Gould	Present	Sen & Gould	Present	FEMLAB	Present
n	m = 1		m = 2		m = 3	
0	/	2128.6	/	4410.4	6291	6278.3
1	/	1192.5	/	4937.8	6333	6328.5
2	544.8	542.32	3140	3127.2	5688	5684.9
3	335.3	333.2	2008	2000.1	4340	4345
4	361.2	360.9	1455	1450.2	3312	3308.2
5	505.9	506.1	1299	1296	2714	2702
6	696.4	696.5	1412	1410	2493	2486.2
7	919.8	919.9	1654	1652.5	2580	2574.7
8	1177	1176.3	1954	1951.6	2829	2835.1

modes. Good results, comparing the present method to a displacement-based F.E. solution by Sen and Gould (1974) and to another one obtained with the numerical package FEMLAB are observed.

In the whole modal analysis, the frequencies calculated needed no more that 17 grid points to be obtained, the F.E. solutions adopted as benchmarks involved much larger solving eigenvalue systems. In the case of FEMLAB solution almost 3’ on a 1.5 GHz Pentium PC was the CPU-time to solve the final eigenproblem, while the G. D. Q. solution never involved more that 4”. Another striking testimony of the great ability of the G. D. Q. method to solve structural mechanics problems with a low computational cost due to its great simplicity of implementation.

10.3 Hemispherical dome

The clamped hemispherical shell shown in Fig. 9 is studied to evaluate the G. D. Q. method

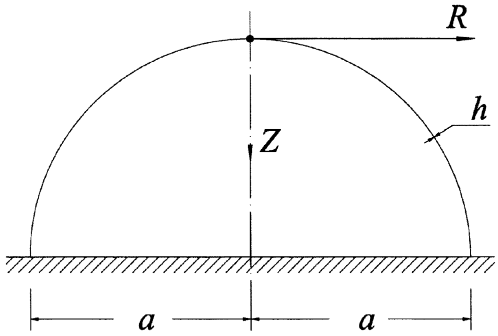

Fig. 9 A clamped hemispherical dome with uniform thickness (a/h = 100, ν = 0.3)

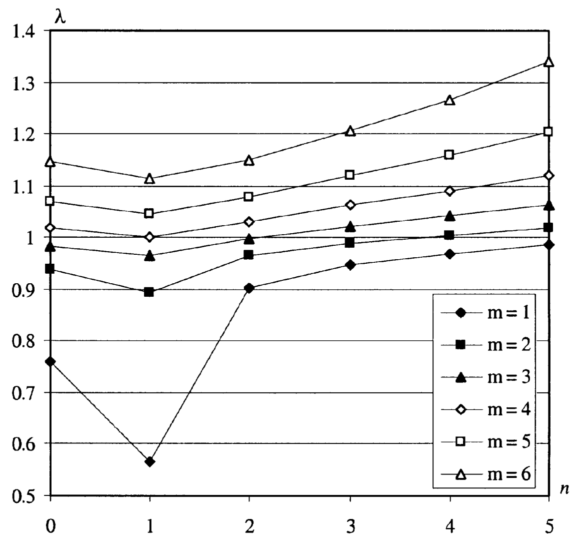

Fig. 10 Frequency parameter $\lambda = a\omega(\rho/E)^{1/2}$ for the first six modes in the vibration of the clamped emispherical shell shown in Fig. 7

Table 3 Non-dimensional frequency parameter $\lambda = a\omega(\rho/E)^{1/2}$ for the clamped emispherical dome shown in Fig. 9

Harmonic number n	Mode number m					
	1	2	3	4	5	6
0	0.760 (0.761) [0.760]	0.938 (0.938) [0.938]	0.984 (0.984) [0.984]	1.020 (1.021) [1.020]	1.070 (1.072) [1.071]	1.146 [1.144]
1	0.567 (0.567)	0.893 (0.894)	0.965 (0.966)	1.002 (1.002)	1.045 (1.046)	1.115 {1.106}
2	0.901 (0.901)	0.966 (0.966)	0.997 (0.998)	1.029 (1.031)	1.077 (1.081)	1.150 {1.153}
3	0.947 (0.948)	0.988 (0.990)	1.022 (1.024)	1.063 (1.066)	1.119 (1.124)	1.206 {1.207}
4	0.969 (0.969)	1.004 (1.005)	1.042 (1.045)	1.091 (1.095)	1.159 (1.167)	1.266 {1.264}
5	0.985 (0.985)	1.019 (1.021)	1.062 (1.066)	1.120 (1.127)	1.204 (1.213)	1.341 {1.327}

Values in [ ] are exact one for the axisymmetric modes, obtained by Kunieda (1984).
Values in ( ) are from Kim (1998).
Values in { }are from Luah and Fan (1989).

reliability on a membrane-dominated (Kunieda 1984) shell problem. The non-dimensional frequency parameter considered in this example is defined as:

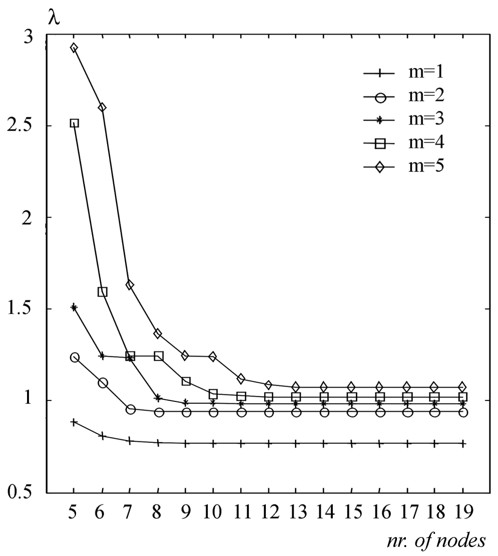

Fig. 11 Convergence characteristics of the present analysis using a cosine grid points distribution, in predicting the frequency parameter λ in the axisymmetric vibration of the clamped hemispherical shell shown in Fig. 9

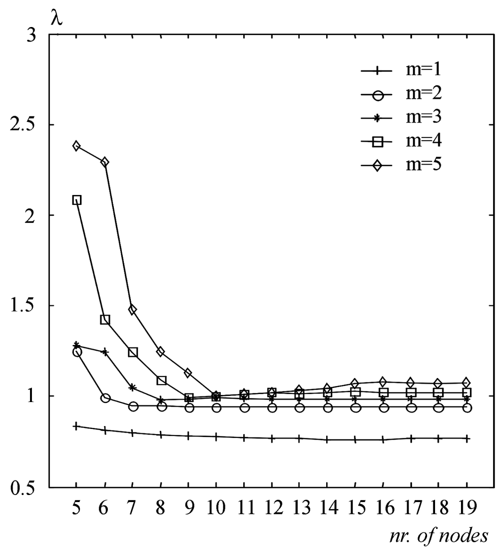

Fig. 12 Convergence characteristics of the present analysis using a uniform grid points distribution, in predicting the frequency parameter λ in the axisymmetric vibration of the clamped hemispherical shell shown in Fig. 9

$$\lambda = a\omega\left(\frac{\rho}{E}\right)^{1/2}$$

where a is the sphere radius. The reference values are the exact ones obtained by Kunieda (1984) for the axisymmetric modes (n = 0), and those by two kinds of F.E. modelling by Kim (1998) and Luah and Fan (1989).

Regarding this case, it is worth noting that the method in argument fully permits the dealing of the closed apex compatibility conditions Gould (1999), in a way that it is not necessary to model a little “hole” on the vertex to avoid the singularity of null parallel radius. The results in Table 3 are obtained from a cosine-spacing sampling rule and show again very good accuracy, the maximum percentage difference with respect to reference values being lower than 0.05%. Figs. 11 and 12 show the patterns of the first five circular frequencies, for n = 0, versus the number of sampling points forming the grid in the case of cosine spaced (Fig. 11) and uniformly spaced (Fig. 12).

It is evident how the non-uniform grid is faster in converging and shows a monotonic tendency to the desired roots.

11. Conclusions

A G. D. Q. solution procedure for the dynamic analysis of rotational shells has been presented. Starting from general shell equations a proper set of equilibrium equations in terms of circular harmonic components of generalized displacements has been obtained and equally a complete set of boundary conditions. These 2nd order ordinary differential equations in the axial coordinate only are easily discretized by means of the technique in argument, to yield a common generalized eigenvalue problem for the cyclic frequencies of the system. It is worth noting how, with the presented discretized formulation, no "delta-point" technique has been used in this study, thus avoiding further approximation in the boundary conditions assignments. Several comparisons with available results confirm how this simple numerical method provides accurate and computationally low cost results.

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Notation

α_1, α_2	: curvilinear coordinates on the reference surface
ς	: normal coordinate along shell thickness
$\mathbf{r}$	: vector radius to undeformed reference surface
$\boldsymbol{\rho}$	: vector radius to deformed reference surface
t_1, t_2, $\mathbf{n}$	: tangent and outward normal vectors to the reference surface
R_{11}, R_{22}	: normal radii of curvature to the reference surface
A_1, A_2	: Lamè parameters of the reference surface
$\mathbf{N}_i$, $\mathbf{M}_i$	: stress resultants and couples
$\mathbf{q}$, $\mathbf{m}$	: external distributed loads intensities
$\mathbf{u}(\alpha_1, \alpha_2, t)$	: generalized displacements vector
$\boldsymbol{\varphi}(\alpha_1, \alpha_2, t)$	: rotations vector
$\boldsymbol{\varepsilon}_i$	: resultant strains vector
$\boldsymbol{\kappa}_i$	: couple strains vector
$\mathbf{R}(\alpha_1, \alpha_2, t)$	: stress resultants and couples vector
$\boldsymbol{\varepsilon}(\alpha_1, \alpha_2, t)$	: strain resultants and couples vector
λ	: shearing correction factor
$R(Z)$	: parallel radius
$R_\phi(Z)$	: meridian radius
$R_\theta(Z)$	: grand normal radius
$\mathbf{L}$	: fundamental system equilibrium operator
$\mathbf{M}$	: fundamental system mass matrix
$A_{ik}^{(r)}$, $B_{jl}^{(s)}$	: weighting coefficients for the G. D. Q. approximations of derivatives
$\mathbf{F}_1^n$, $\mathbf{F}_2^n$	: diagonal matrices containing harmonic functions
$\underline{\mathbf{K}}^n$, $\underline{\mathbf{M}}^n$	: stiffness and mass matrices of discretized structure
$\overline{\mathbf{K}}_{dd}^n$, $\overline{\mathbf{M}}_{dd}^n$	: condensed stiffness and mass matrices of discretized structure

Appendix A - Modal shapes for the shells studied

A complete review of the modal shapes obtained by the eigensolutions for the three kinds of shells studied in this paper is presented. It is to be noted that the number of grid points utilized to plot the shapes is quite bigger than that needed to calculate the frequencies during the modal analysis and this is due only to the necessity of having a good graphic rendering.

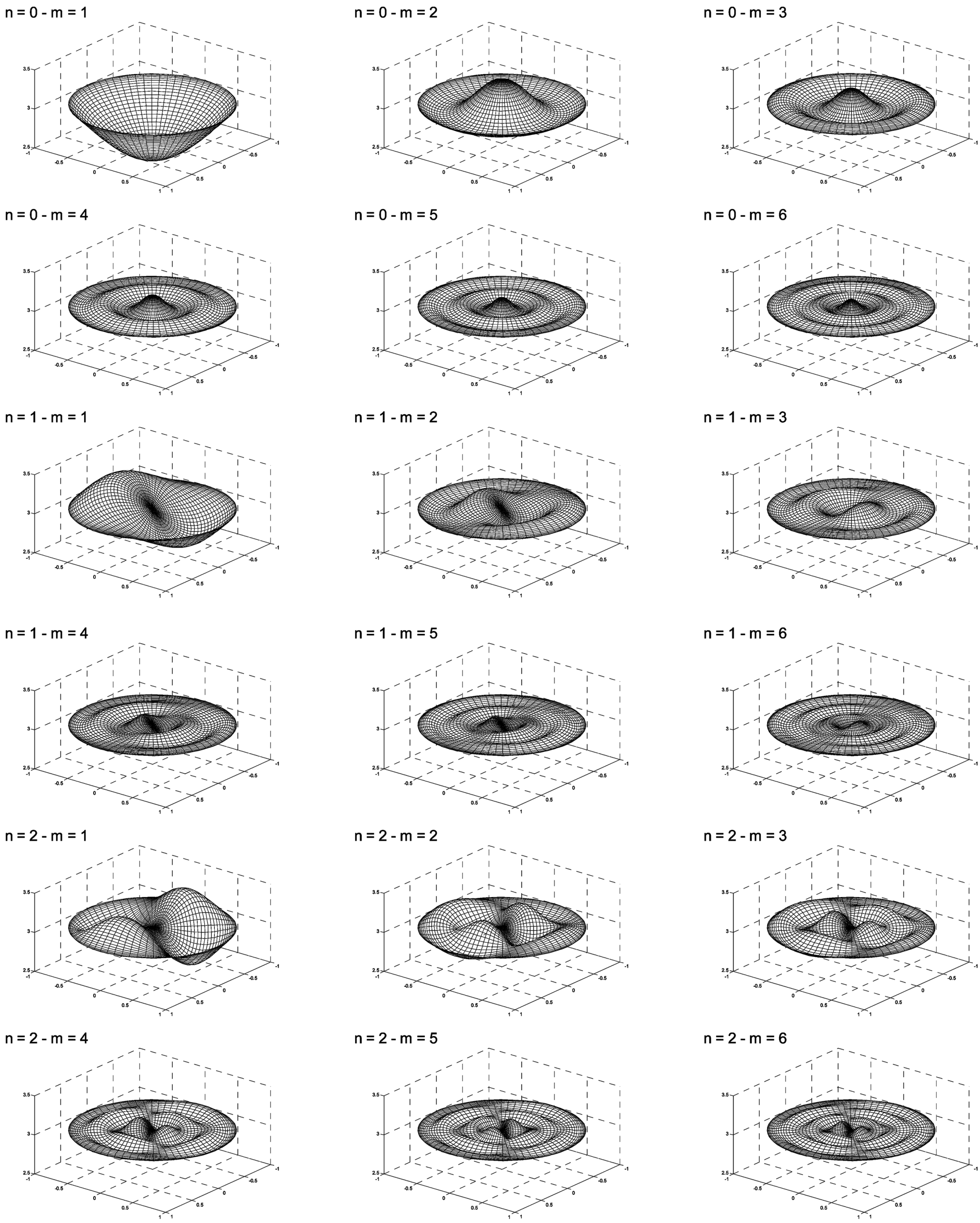

Fig. A.1 Mode shapes for the circular plate shown in Fig. 6

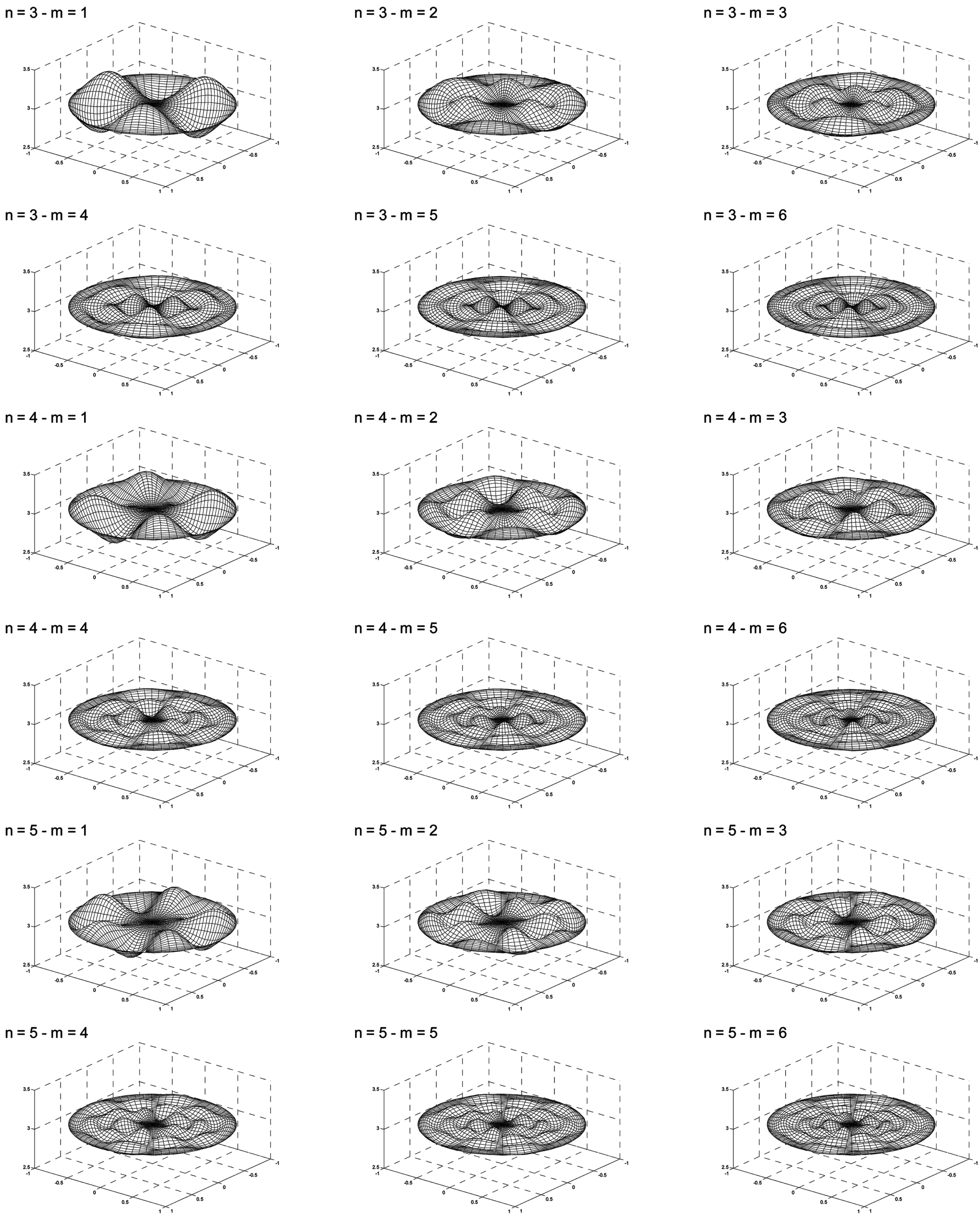

Fig. A.2 Mode shapes for the circular plate shown in Fig. 6

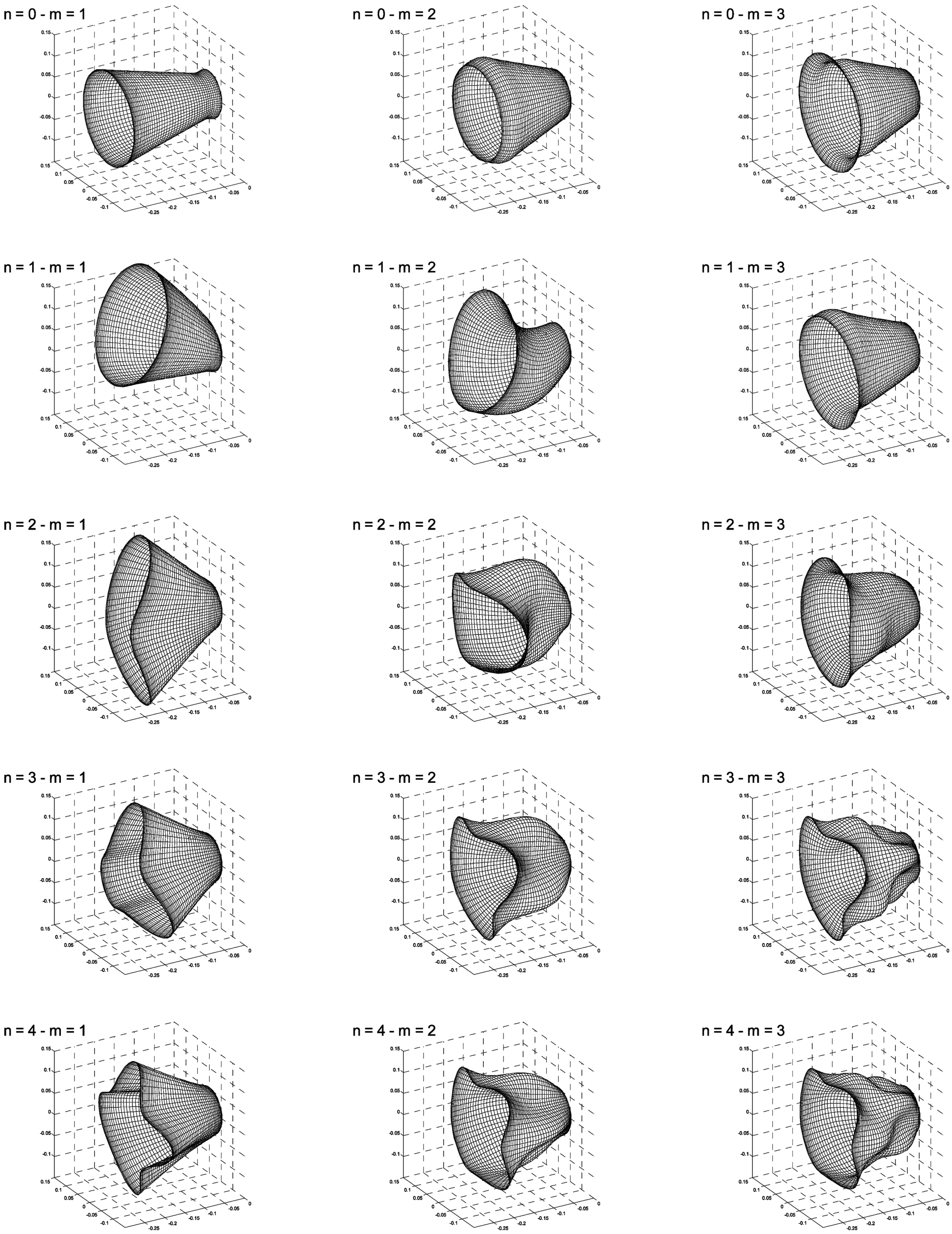

Fig. A.3 Mode shapes for the truncated conical shell shown in Fig. 7

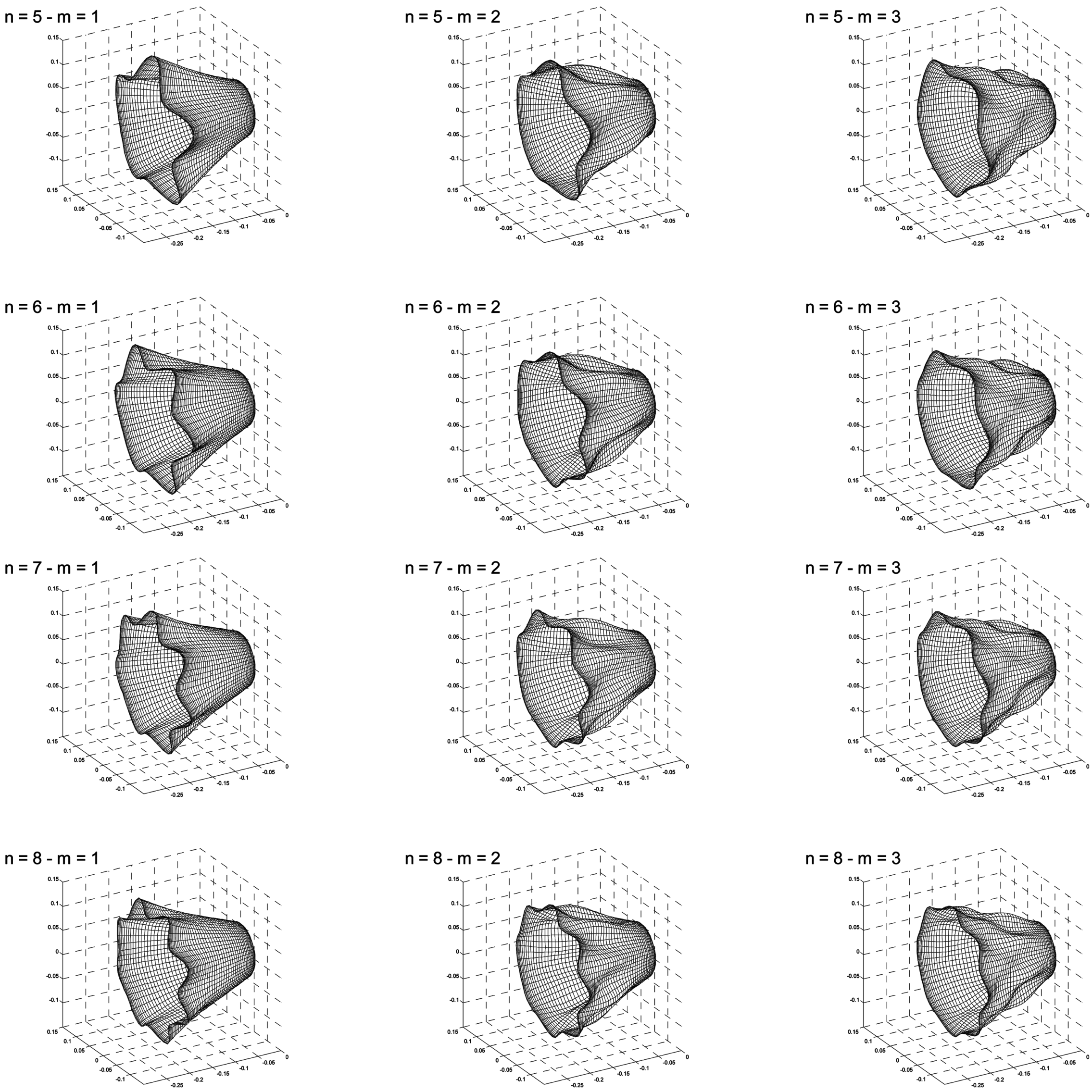

Fig. A.4 Mode shapes for the truncated conical shell shown in Fig. 7

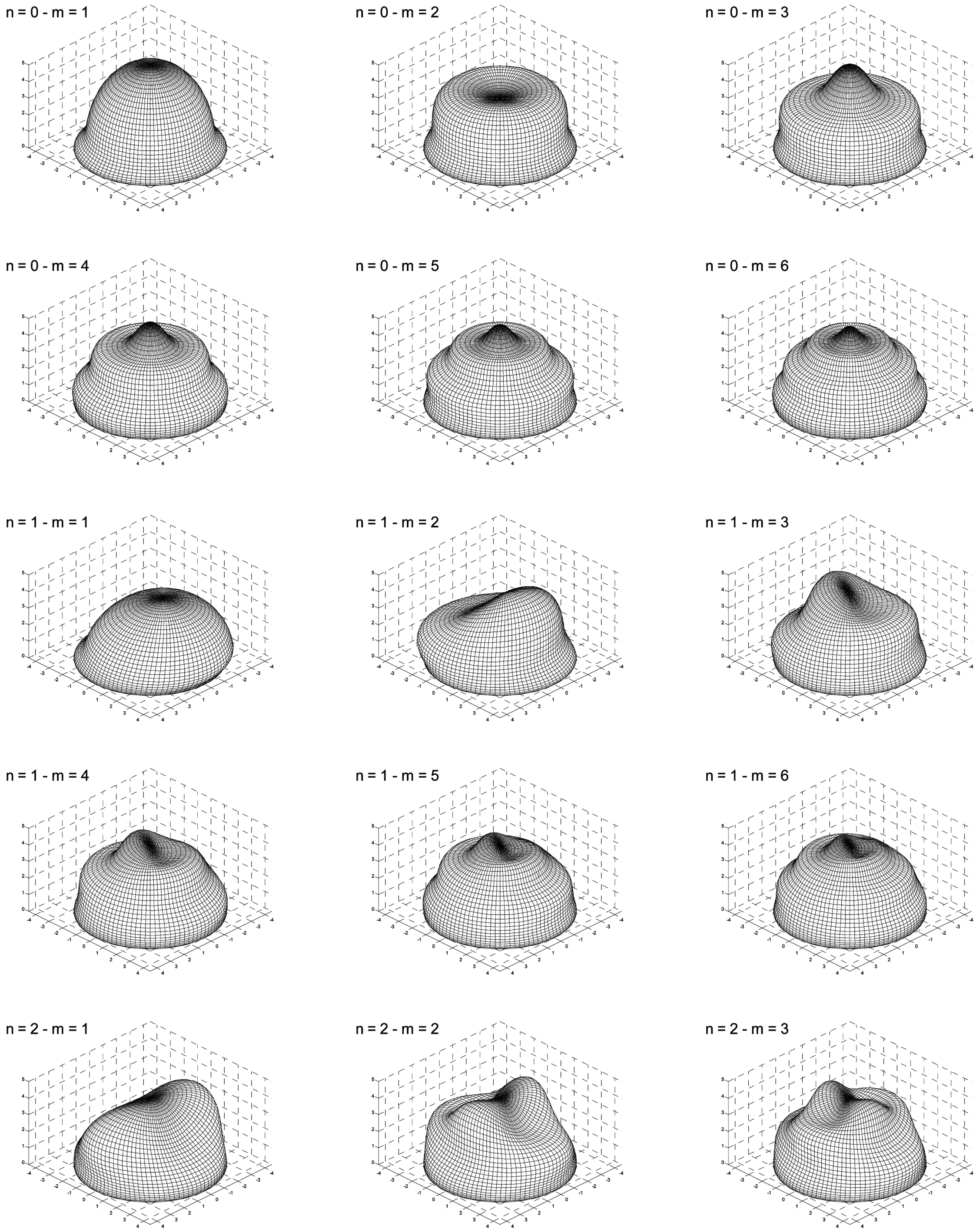

Fig. A.5 Mode shapes for the spherical cap shown in Fig. 9

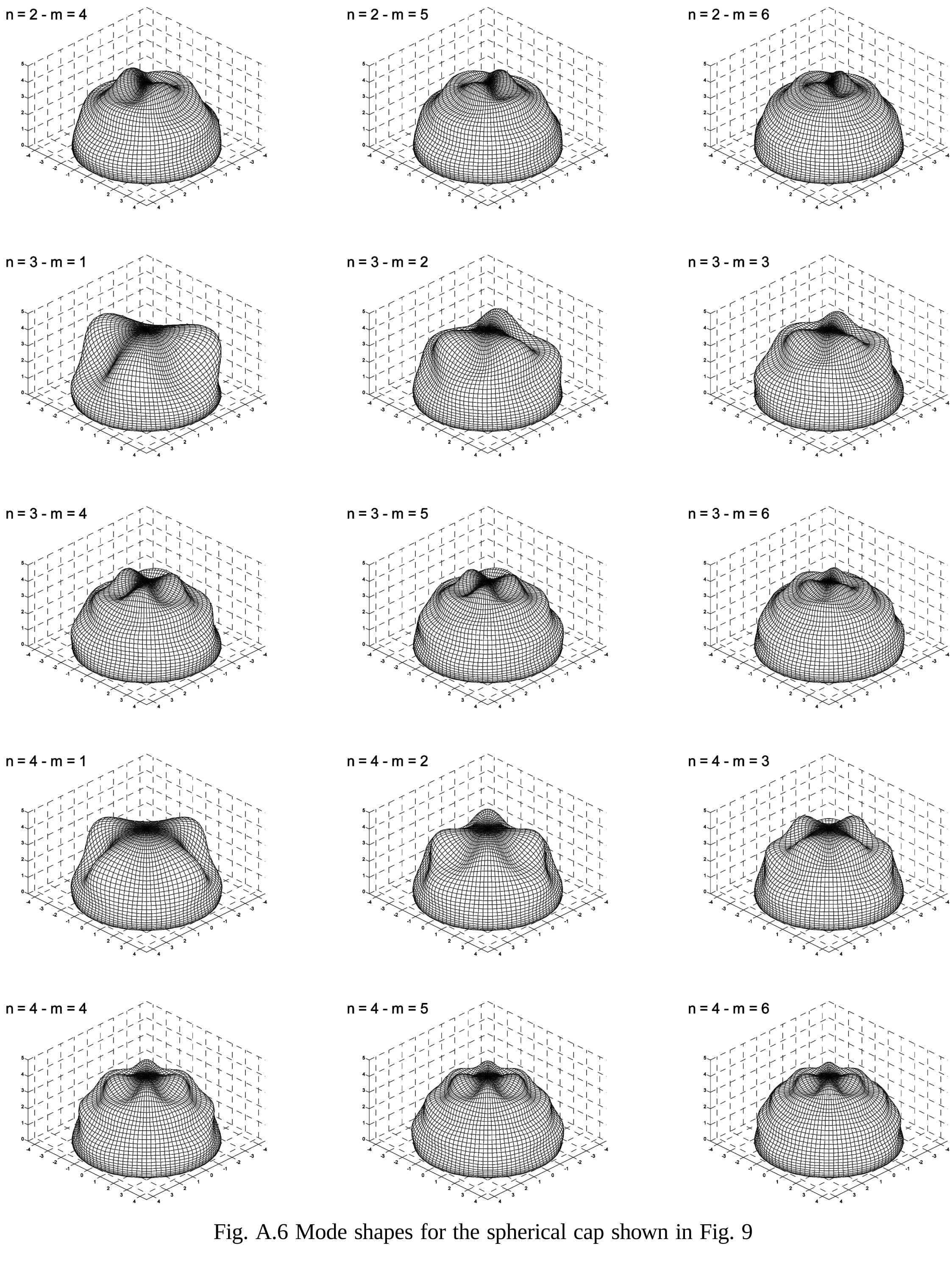

Fig. A.6 Mode shapes for the spherical cap shown in Fig. 9

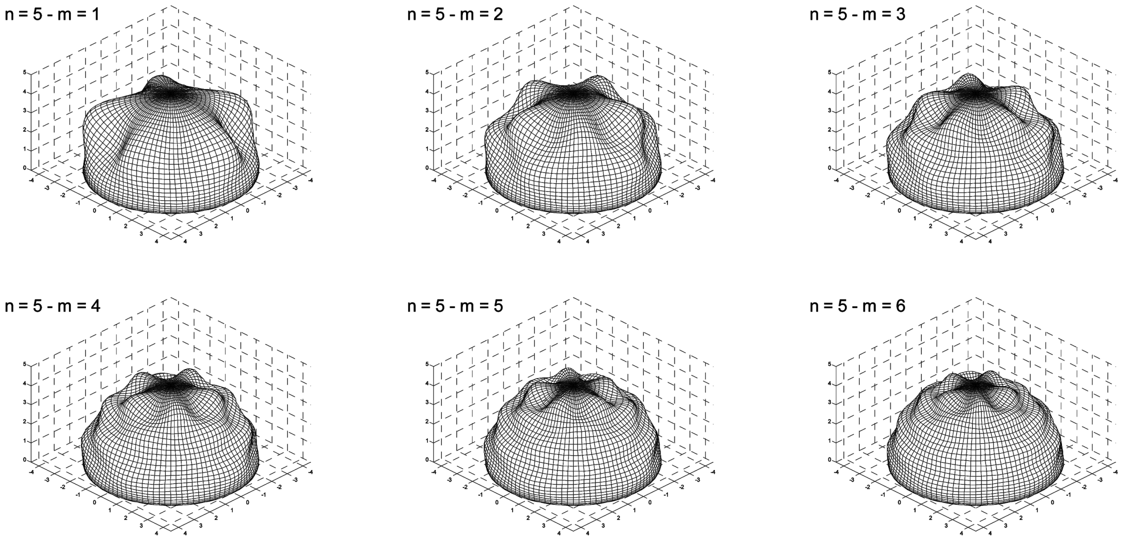

Fig. A.7 Mode shapes for the spherical cap shown in Fig. 9