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Harmonic differential quadrature (HDQ) for axisymmetric bending analysis of thin isotropic circular plates

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Abstract. Numerical solution to linear bending analysis of circular plates is obtained by the method of harmonic differential quadrature (HDQ). In the method of differential quadrature (DQ), partial space derivatives of a function appearing in a differential equation are approximated by means of a polynomial expressed as the weighted linear sum of the function values at a preselected grid of discrete points. The method of HDQ that was used in the paper proposes a very simple algebraic formula to determine the weighting coefficients required by differential quadrature approximation without restricting the choice of mesh grids. Applying this concept to the governing differential equation of circular plate gives a set of linear simultaneous equations. Bending moments, stresses values in radial and tangential directions and vertical deflections are found for two different types of load. In the present study, the axisymmetric bending behavior is considered. Both the clamped and the simply supported edges are considered as boundary conditions. The obtained results are compared with existing solutions available from analytical and other numerical results such as finite elements and finite differences methods. A comparison between the HDQ results and the finite difference solutions for one example plate problem is also made. The method presented gives accurate results and is computationally efficient.

Key words: harmonic differential quadrature; circular plates; deflection; bending moment; numerical methods.

1. Introduction

Real physical systems or engineering problems are often described by partial differential equations, either linear or nonlinear and in most cases, their closed form solutions are extremely difficult to establish. As a result, approximate numerical methods have been widely used to solve partial differential equations that arise in almost all engineering disciplines. With the modern computer technology, various numerical methods were well developed and widely used to solve various kinds of engineering and science problems, which are described by the differential equations. The most commonly used numerical methods for such applications are the finite element, finite difference and boundary element methods, and most engineering problems can be solved by these methods to satisfactory accuracy if a proper and sufficient number of grid points are used (Civalek 1998). Consequently, both CPU time and storage requirements are often considerable for

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the standard methods. In seeking a more efficient numerical method which requires fewer grid points yet achieves acceptable accuracy, the method of DQ, which is based on the assumptions that the partial derivatives of a function in one direction can be expressed as a linear combination of the function values at all mesh points along that direction, was introduced by Bellman *et al*. (1971). The method of DQ circumvents the above difficulties by computing a moderately accurate solution from only a few points. Since then, applications of DQ method to various engineering problems have been investigated and their successes have demonstrated the potential of the method as an attractive numerical analysis technique (Bert *et al*. 1993, Liew *et al*. 1997c, Liew and Teo 1999b, Liew *et al*. 2002, Bert and Malik 1996b, Farsa *et al*. 1993, Quan and Chang 1989).

Plates are initially flat structural elements, having thickness much smaller than the other dimensions. Many practical engineering applications fall into categories plates in bending. Circular plates have many applications in civil, aerospace, petroleum, nuclear and, mechanical engineering. They are used in these fields as the aircraft fuselage, rockets and turbo jets, reactor walls, ship and submarine parts, and holding tanks etc. There are many methods available in the literature to study the static and dynamic behavior of thin plates with different boundary and loading conditions. Recently, DQ and differential cubature (DC) methods are proposed for static and vibration analysis of circular and other type plates (Han and Liew 1997a, Liew and Liu 1997a, Liew and Teo 1999b). For details, one may refer to Timoshenko and Woinowsky-Krieger (1959) etc. Exact solutions for plate problems are rather difficult to obtain, except for a few simple cases. In many cases, one may have to resort to various approximate namely numerical methods. Each method has its own advantages and disadvantages. Of the various methods proposed in recent times, one can cite the Ritz, finite differences, finite and boundary element methods as the most efficient and universal methods for solving variant type plate problems. In this study, the static analysis of circular plates subjected to two different types of loads and support conditions are investigated by using harmonic differential quadrature. The accuracy, efficiency and convenience of HDQ are demonstrated throughout the numerical examples. Following, in section 2, the method of DQ approximation is briefly summarized. Weighting coefficients and main principles of the HDQ method are given in Section 3. The choice of sampling grid points is also given in this section. Numerical examples are given in Section 4 to illustrate the efficiency of the HDQ.

2. Differential Quadrature Method (DQM)

As with other numerical analysis techniques, such as finite element or finite difference methods, the DQ method also transforms the given differential equation into a set of analogous algebraic equations in terms of the unknown function values at the preselected sampling points in the field domain. In many practical applications the numerical solutions of the governing differential equations are required at only a few points in the physical domain. Frequently, for reasonable accuracy, conventional finite difference and finite element methods require the use of a large number of grid points. Therefore, even though solutions at only a few specified points may be desired, numerical solutions must be produced at all grid points. The problem areas in which the applications of differential quadrature method may be found in the available literature include fluid mechanics, static and dynamic structural mechanics (Bert and Malik 1996a, Liew *et al*. 1997a, Striz *et al*. 1994, Civalek 2001).

During recent years, the DQ method has been largely promoted by Bert and associates who were the first to introduce the method as a tool for structural analysis (Bert *et al.* 1987). After this, various problems in structural mechanics have been solved successfully by this method (Bert *et al.* 1994, Du *et al.* 1994). Liew and his co-workers also made most important study on DQ and differential cubature (DC), (Liew *et al.* 1997a, Liu and Liew 1998, Liew and Han 1997b). In fact, Liew and his associates had made most effective supplement to the theory and application on DQ and DC methods. Recent works of Liew and associates have been mainly on the three dimensional vibration, bending and stability analysis of plates (Liew *et al.* 1999a, Liew and Teo 1999b, Liew *et al.* 2001). Han and Liew applied differential quadrature method to the solutions of thick circular (Han and Liew 1997a) and annular Reissner/Mindlin plates (Han and Liew 1998). Han and Liew proposed an eight-node differential quadrature formulation (Han and Liew 1997b). Recently, Liew and his co-workers (Liew *et al.* 2002) also proposed a new kind DQ method. This new proposed method has been called the moving least squares differential quadrature (MLSDQ). This new approach (MLSDQ) exploits the merits of both the DQ and meshless method. Authors applied the DQ method to the stability, vibration and bending analysis of elastic bars (Civalek 2001), vibration and buckling of beams, columns and plates (Civalek 2002). Striz *et al.* (1988) have investigated nonlinear bending analysis of circular plates employing the simplified version of DQ.

It has been claimed that the DQ method has the capability of producing highly accurate solutions with minimal computational effort. The method has seemingly a high potential as an alternative to the above mentioned conventional numerical solution techniques such as the finite element and finite difference methods (Jang *et al.* 1989, Bert and Malik 1996c, Liew and Teo 1999b, Shu and Chew 1998, Du *et al.* 1995). Therefore research on extension and application of the method becomes an important endeavor. In the differential quadrature method, a partial derivative of a function with respect to a space variable at a discrete point is approximated as a weighted linear sum of the function values at all discrete points in the region of that variable. For simplicity, we consider a one-dimensional function $\Psi(x)$ in the $[-1, 1]$ domain, and N discrete points. Then the first derivatives at point i, at $x = x_i$ is given by

$$\Psi_x(x_i) = \left.\frac{\partial\Psi}{\partial x}\right|_{x = x_i} = \sum_{j=1}^{N} A_{ij}\Psi(x_j); \quad i = 1, 2, \ldots, N, \tag{1}$$

where x_j are the discrete points in the variable domain, $\Psi(x_j)$ are the function values at these points and A_{ij} are the weighting coefficients for the first order derivative attached to these function values. Bellman *et al.* (1972) suggested two methods to determine the weighting coefficients. The first one is to let Eq. (1) be exact for the test functions

$$\Psi_k(x) = x^{k-1}, \qquad k = 1, 2, \ldots, N, \tag{2}$$

which leads to a set of linear algebraic equations

$$(k-1)x_i^{k-2} = \sum_{j=1}^{N} A_{ij}x_j^{k-1}; \quad \text{for} \quad i = 1, 2, \ldots, N \quad \text{and} \quad k = 1, 2, \ldots, N. \tag{3}$$

which represents N sets of N linear algebraic equations. This equation system has a unique solution because its matrix is of Vandermonde form. This equation may be solved for the weighting coefficients analytically using the Hamming's method (Hamming 1973) or numerical method using the certain special algorithms for Vandermonde equations, such as the method of Bjorck and Pereyra (1970).

In order to reduce the complexity of the derivative approximation formulae and thereby conserve on computational effort, it is advantageous to use quadrature approximation formulae for also the second, third and higher order derivatives. Thus, the weighting coefficients for each formula will be different from those for the first-order derivative. As similar to the first order, the second order derivative can be written as

$$\Psi_{xx}(x_i) = \left.\frac{\partial^2\Psi}{\partial x^2}\right|_{x = x_i} = \sum_{j=1}^{N} B_{ij}\Psi(x_j); \quad i = 1, 2, \ldots, N, \tag{4}$$

where the B_{ij} are the weighting coefficients for the second order derivative. Eq. (4) can be written also as

$$\Psi_{xx}(x_i) = \left.\frac{\partial^2\Psi}{\partial x^2}\right|_{x = x_i} = \sum_{j=1}^{N} A_{ij}\sum_{k=1}^{N} A_{jk}\Psi(x_k); \quad i = 1, 2, \ldots, N, \tag{5}$$

The function given by Eq. (2) is used again so that the second order derivative is

$$(k-1)(k-2)x_i^{k-3} = \sum_{j=1}^{N} B_{ij}x_j^{k-1} \tag{6}$$

which can be solved in the same manner as indicated for Eq. (3) above. Weighting coefficients of the second, third and fourth order derivatives B_{ij}, C_{ij}, D_{ij}, can be obtained by following formulations;

$$B_{ij} = \sum_{k=1}^{N} A_{ik}A_{kj}, \quad C_{ij} = \sum_{k=1}^{N} A_{ik}B_{kj}, \quad D_{ij} = \sum_{k=1}^{N} A_{ik}C_{kj}. \tag{7,8,9}$$

The second method proposed also by Bellman *et al.* (1972) to obtain the weighting coefficients is similar to the first one with the exception that a different set of trial or test functions are chosen for satisfying Eq. (1) exactly;

$$\Psi_k(x) = \frac{L_N(x)}{(x-x_k)L_N^{(1)}(x_k)}, \qquad k = 1, 2, \ldots, N. \tag{10}$$

where $L_N(x)$ is the Nth order Legendre polynomial and $L_N^{(1)}(x)$ the first order derivative of $L_N(x)$. N is the number of grid points as with the first one. However, it requires that x_k ($k = 1, 2, ..., N$) have to be chosen to be roots of the shifted Legendre polynomial. This means that once the number of grid points N is specified the roots of the shifted Legendre polynomial are given, thus the distribution of the grid points are fixed regardless of the physical problems being considered. By choosing x_k to be the roots of the shifted Legendre polynomial and substituting Eq. (10) into Eq. (1), we obtain a direct simple algebraic expression for the weighting coefficients A_{ij}

$$A_{ij} = \frac{L'_N(x_i)}{(x_i-x_j)L'_N(x_j)} \quad \text{for} \quad i \neq j; \quad \text{and} \quad i, j = 1, 2, \ldots, N, \tag{11}$$

$$A_{ii} = \frac{1-2x_i}{2x_i(x_i-1)} \quad \text{for} \quad i = j; \quad \text{and} \quad i, j = 1, 2, \ldots, N. \tag{12}$$

In this second approach, the weighting coefficients that were defined at Eqs. (11) and (12) are easy to obtain without solving algebraic equations or having a singularity problem as with the first one.

3. Harmonic Differential Quadrature (HDQ)

Despite the increasing application of the DQ method in structural analysis, a draw back regarding its ill conditioning of the weighting coefficients with increasing number of grid points used as well as the increasing order of derivatives was pointed out by Bellman (1971, 1972). A recent variation of the original differential quadrature approximation called the harmonic differential quadrature (HDQ) has been proposed by Striz *et al.* (1995) and Liew *et al.* (1999a). Unlike the DQ method that uses the polynomial functions, such as Lagrange interpolates and Legendre polynomials as the test functions, HDQ uses harmonic or trigonometric functions as the test functions. As the name of the test function suggested, this method is called the HDQ method. The harmonic test function $h_k(x)$ used in the HDQ method is defined as (Shu and Xue 1997);

$$h_k(x) = \frac{\sin\frac{(x-x_0)\pi}{2}\ldots\sin\frac{(x-x_{k-1})\pi}{2}\sin\frac{(x-x_{k+1})\pi}{2}\ldots\sin\frac{(x-x_N)\pi}{2}}{\sin\frac{(x_k-x_0)\pi}{2}\ldots\sin\frac{(x_k-x_{k-1})\pi}{2}\sin\frac{(x_k-x_{k+1})\pi}{2}\ldots\sin\frac{(x_k-x_N)\pi}{2}}$$

$$\text{for } k = 0, 1, 2, \ldots, N \tag{13}$$

According to the HDQ, the weighting coefficients of the first-order derivatives A_{ij} for $i \neq j$ can be obtained by using the following formula:

$$A_{ij} = \frac{(\pi/2)P(x_i)}{P(x_j)\sin[(x_i-x_j)/2]\pi}, \quad i, j = 1, 2, 3, \ldots, N, \tag{14}$$

where

$$P(x_i) = \prod_{j=1, j\neq i}^{N} \sin\left(\frac{x_i-x_j}{2}\pi\right), \quad \text{for} \quad j = 1, 2, 3, \ldots, N. \tag{15}$$

The weighting coefficients of the second-order derivatives B_{ij} for $i \neq j$ can be obtained using following formula:

$$B_{ij} = A_{ij}\left[2A_{ii}^{(1)} - \pi ctg\left(\frac{x_i-x_j}{2}\right)\pi\right], \quad i, j = 1, 2, 3, \ldots, N, \tag{16}$$

The weighting coefficients of the first-order and second-order derivatives $A_{ij}^{(p)}$ for $i = j$ are given as

$$A_{ii}^{(p)} = -\sum_{j=1, j\neq i}^{N} A_{ij}^{(p)}; \quad p = 1 \quad \text{or} \quad 2\,; \quad \text{and} \quad \text{for} \;\; i = 1, 2, ..., N. \tag{17}$$

The weighting coefficients of the third and fourth order derivatives can be computed easily from A_{ij} and B_{ij} as with the Eqs. (8) and (9). It should be mentioned that in the DQ solutions, the sampling points in the various coordinate directions might be different in number as well as in their type. A natural, and often convenient, choice for sampling points is that of equally spaced points. This type of sampling point spacing (Type-I) is given as

$$x_i = \frac{i-1}{N-1}; \quad i = 1, 2, \ldots, N, \tag{18}$$

in the related directions. Some times, the differential quadrature solutions deliver more accurate results with unequally spaced sampling points. A better choice for the positions of the grid points between the first and the last points at the opposite edges is that corresponding to the zeros of orthogonal polynomials such as; the zeros of Chebyshev polynomials. Furthermore, another choice that is found to be even better than the Chebyshev and Legendre polynomials is the set of points proposed by Shu and Richards (1992). These points are given as

$$x_i = \frac{1}{2}\left[1 - \cos\left(\frac{2i-1}{N-1}\right)\pi\right]; \qquad i = 1, 2, \ldots, N. \tag{19}$$

in the related directions. We called this kind of grid distribution as Type-II in this study. In addition, the use of zeros of shifted Legendre polynomials has been shown to produce very accurate results (Chen *et al*. 2000, Liew and Teo 1999b, Striz *et al*. 1994, Bert and Malik 1996c). DQ and HDQ methods use same algorithm except for computation of the weighting coefficients. The main advantage of HDQ over the DQ is its ease of the computation of the weighting coefficients without any restriction on the choice of grid points.

4. Numerical applications and results

To verify the analytical formulation presented in the previous section; circular plates with two different types of boundary and load conditions are considered. The axisymmetric bending behavior is considered. Following, the governing differential equation for deflection and bending of circular plate is presented. The present formulation is based on classical small deflection theory. Then, the HDQ and DQ methods have been applied to this differential equation.

4.1 Deflection and bending analysis of circular plates

Consider a thin circular plate of uniform thickness (Fig. 1). The governing differential equation for small deflection is given as

$$\frac{d^4u}{dr^4} + \frac{2}{r}\left(\frac{d^3u}{dr^3}\right) - \frac{1}{r^2}\left(\frac{d^2u}{dr^2}\right) + \frac{1}{r^3}\left(\frac{du}{dr}\right) = \frac{q(r)}{D}, \tag{20}$$

where D is the flexural rigidity, q is the normal pressure or uniformly distributed load on the plate, r

is the radial position and u the normal deflection of the circular plate. By normalizing of Eq. (20) for uniformly distributed load, we obtain

$$\frac{d^4U}{dR^4} + \frac{2}{R}\left(\frac{d^3U}{dR^3}\right) - \frac{1}{R^2}\left(\frac{d^2U}{dR^2}\right) + \frac{1}{R^3}\left(\frac{dU}{dR}\right) = 1. \tag{21}$$

where $R = r/a$, $U = u/\xi$, and $\xi = qa^4/D$ and a is known as the outside radius of the plate. The bending moments and stress resultants in the radial and tangential directions are given (Ugural 1999, Timoshenko and Woinowsky-Krieger 1959) as;

$$M_r = -D\left(\frac{d^2u}{dr^2} + \frac{v}{r}\frac{du}{dr}\right), \quad M\varphi = -D\left(\frac{1}{r}\frac{du}{dr} + v\frac{d^2u}{dr^2}\right), \tag{22a, 22b}$$

$$\sigma_r = -D\frac{12}{h^3}z\left[\frac{d^2u}{dr^2} + v\left(\frac{1}{r}\frac{du}{dr}\right)\right], \quad \sigma_\varphi = -D\frac{12}{h^3}z\left[\frac{1}{r}\frac{du}{dr} + v\frac{d^2u}{dr^2}\right], \tag{23a, 23b}$$

where v is the Poisson's ratio, h the thickness of the plate. In case of the simply supported outside, the boundary conditions are

$$U = 0 \quad \text{and} \quad D\left(\frac{d^2U}{dR^2} + \frac{v}{R}\frac{dU}{dR}\right) = 0 \quad \text{at} \quad R = 1. \tag{24,25}$$

In addition to above boundary conditions the regularity condition must be given for solid circular plates. This condition is necessary to assure that the plate slope is zero at the origin to avoid a singularity at this location. The regularity condition at the center of the plate is given by

$$\frac{dU}{dR} = 0 \quad \text{at} \quad R = 0, \tag{26}$$

Applying the differential quadrature approximation to the normalized plate deflection equation, boundary and regularity conditions given by Eqs. (21), (24), (25), and (26) one obtains

$$\sum_{j=1}^{N} D_{ij}U_j + \frac{2}{R_i}\sum_{j=1}^{N} C_{ij}U_j - \frac{1}{R_i^2}\sum_{j=1}^{N} B_{ij}U_j + \frac{1}{R_i^3}\sum_{j=1}^{N} A_{ij}U_j = 1, \tag{27}$$

for $i = 2, 3, ..., (N-2)$

$$U_N = 0, \quad \sum_{j=1}^{N} B_{Nj}U_j + \frac{\upsilon}{R}\sum_{j=1}^{N} A_{Nj}U_j = 0, \quad \text{and} \quad \sum_{j=1}^{N} A_{1j}U_j = 0. \tag{28,29,30}$$

Notice that, we only keep the discretized equations for $i = 2$ to $(N-2)$ in Eq. (27) because there is one regularity condition at $R = 0$ and there are two boundary conditions at $R = 1$ point.
The boundary conditions for a clamped outside edge are

$$U = 0 \quad \text{at} \quad R = 1, \tag{31}$$

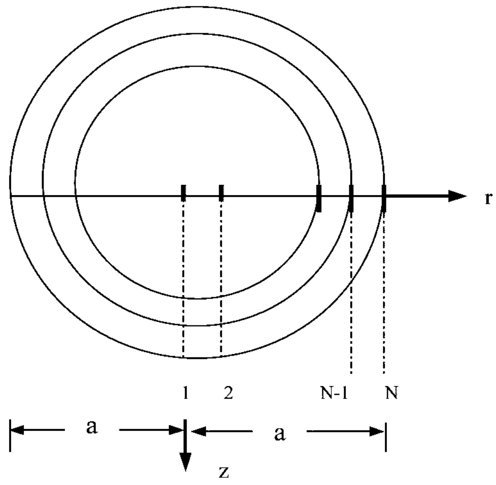

Fig. 1 Typical circular plate and grid points

and

$$dU/dR = 1 \quad \text{at} \quad R = 1, \tag{32}$$

Applying the differential quadrature approximation to these boundary conditions at each discrete point on the grid yields

$$U_N = 0 \quad \text{and} \quad \sum_{j=1}^{N} A_{1j} U_j = 0. \tag{33,34}$$

where the repeated index j means summation from 1 to N. In the numerical applications, two different types of loads and support conditions are considered. Results are obtained for each case using various numbers of grid points. It is observed that the method has very good convergence. Reasonably accurate results can be achieved by using 11 grid points. The numerical results for various example circular plate problems are tabulated (Table 1, Table 2), and plotted (Figs. 2-5) and the comparison of the present results with the exact or other numerical values available in the literature, when possible, are made. Solving the set of combined algebraic Eqs. (27), (28), (29), and (30), the non-dimensional deflections U at various grid points can be found for circular plate in the case of the simply supported boundary conditions. Central concentrated load is taken into consideration in this example. For clamped edge Eqs. (31) and (34) will be used as boundary condition equations. Results are obtained for N = 11 grid points at ν = 0.3. Table 1 and Table 2 give the results together with the exact analytical solutions and FEM solutions (Civalek 1998) for comparisons. In Table 1 and Table 2, the percentage errors of HDQ solution for Type-II grid points from its exact value are given. Reasonably accurate results can be achieved by using 11 grid points in HDQ for Type-II grid sampling (cosine distributed grid) in the related directions. For the deflections, N = 11 grid points provide acceptable results with a maximum discrepancy of 0.87% for the clamped support condition and a maximum discrepancy of 1.01% for simple support condition. It is found that the HDQ method possesses both the advantages of DQ and the flexibility of the FEM. Figs. 2-5 demonstrate the influence of non-dimensional radial coordinate *r/a* on the bending moment in radial and tangential direction (M_r and M_φ) for two different types of support conditions. In this application, uniformly distributed load is taken into consideration. The method presented is to give good results with a small number of discrete points. It can be observed from these figures that all the HDQ results agree with the exact results to within 2.1%. The exact solution can be found in the literature (Ugural 1999, Berktay 1992, Timoshenko and Woinowsky-Krieger 1959).

Table 1 Non-dimensionalized deflections for simply supported circular plate under central concentrated load (N = 11; ν = 0.3)

r/a	U^* Exact (Ugural 1999)	*U* FEM (Civalek 1998)	*U* HDQ Type-II Grid Points	*U* HDQ Type-I Grid Points	% |Error|[a]
0.0	0.1586	0.1792	0.1586	0.1557	0.00
1/10	0.1542	0.1706	0.1536	0.1495	0.39
2/10	0.1443	0.1663	0.1442	0.1484	0.07
3/10	0.1308	0.1443	0.1308	0.1366	0.00
4/10	0.1149	0.1374	0.1151	0.1100	0.17
5/10	0.0973	0.1196	0.0969	0.0967	0.41
6/10	0.0786	0.0967	0.0788	0.0781	0.25
7/10	0.0591	0.0681	0.0597	0.0594	1.01
8/10	0.0393	0.0405	0.0391	0.0388	0.50
9/10	0.0195	0.0228	0.0194	0.0190	0.51
1.0	0.0000	0.0000	0.0000	0.0000	0.00

$^*U = u\pi D/P$, [a]% | Error | = $|(U_{exact} - U_{HDQ}$ for Type-II grid points$) / U_{exact}| * 100$

Table 2 Non-dimensionalized deflections for clamped circular plate under central concentrated load (N = 11; ν = 0.3)

r/a	U^* Exact (Berktay 1992)	*U* FEM (Civalek 1998)	*U* HDQ Type-II Grid Points	*U* HDQ Type-I Grid Points	% |Error|[a]
0.0	0.0625	0.06011	0.0625	0.0629	0.00
1/10	0.0589	0.0596	0.0586	0.0581	0.52
2/10	0.0520	0.0574	0.0520	0.0527	0.00
3/10	0.0433	0.0456	0.0432	0.0428	0.23
4/10	0.0342	0.0357	0.0345	0.0345	0.87
5/10	0.0252	0.0283	0.0251	0.0250	0.40
6/10	0.0170	0.0184	0.0170	0.0169	0.00
7/10	0.0100	0.0115	0.0099	0.0106	1.00
8/10	0.0047	0.0054	0.0047	0.0051	0.00
9/10	0.0012	0.0016	0.0012	0.0009	0.00
1.0	0.0000	0.0000	0.0000	0.0000	0.00

$^*U = u\pi D/P$, [a]% | Error | = $|(U_{exact} - U_{HDQ}$ for Type-II grid points$) / U_{exacty}| * 100$

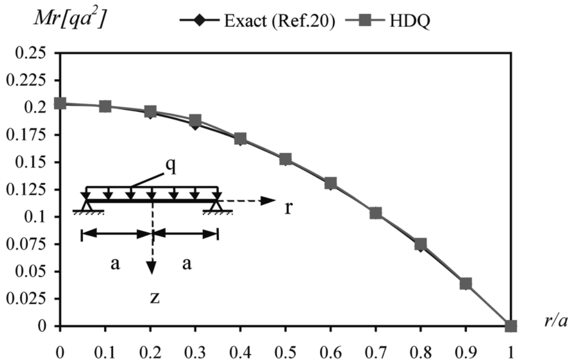

Fig. 2 Bending moment in radial direction for simply supported circular plate (ν = 0.25)

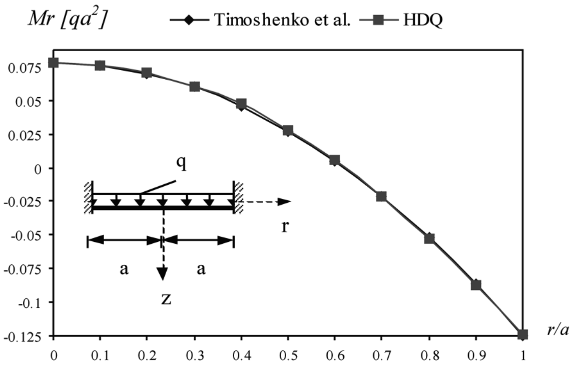

Fig. 3 Bending moment in radial direction for clamped circular plate (ν = 0.25)

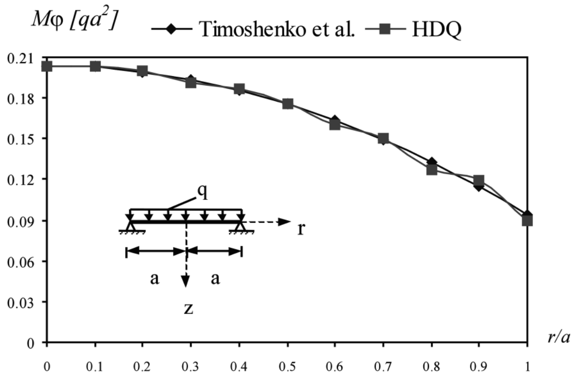

Fig. 4 Bending moment in tangential direction for simply circular plate (ν = 0.25)

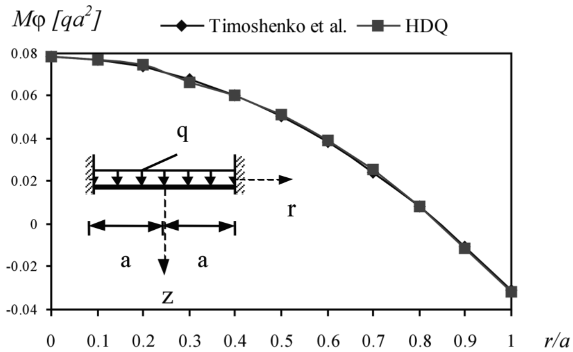

Fig. 5 Bending moment in tangential direction for clamped circular plate (ν = 0.25)

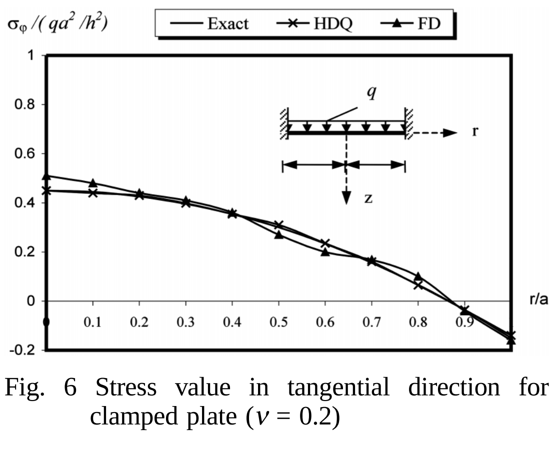

Fig. 6 Stress value in tangential direction for clamped plate (ν = 0.2)

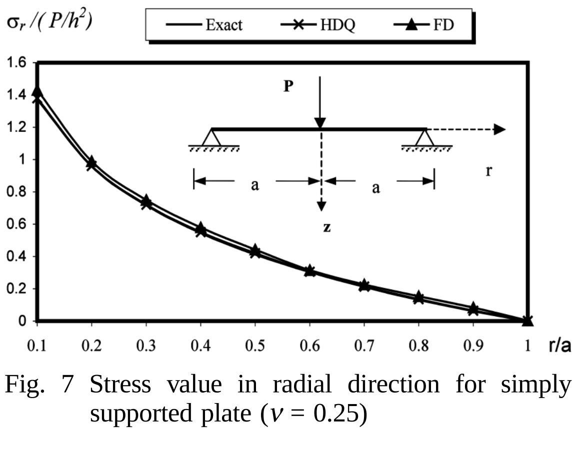

Fig. 7 Stress value in radial direction for simply supported plate (ν = 0.25)

Figs. 6, 7 demonstrate the influence of non-dimensional radial coordinate r/a on the stresses, σ_r, σ_φ for two different type support conditions. In this application, uniformly distributed and central concentrated load are taken into consideration. The method presented is shown to give excellent results with a small number of discrete points. It can be observed from these figures all the HDQ results agree with the exact results. Results obtained from polar finite differences method are indicated by FD. For the FD solution 15 × 15 grid size is used. As can be seen, the HDQ results compare very well with the analytical solutions for only 11 × 11 grids point. Analytical solutions are found in related literatures (Ugural 1999, Timoshenko and Woinowsky-Krieger 1959).

As can be seen from the obtained results, the deflection values are more accurate than the bending moments and the stress values. This is an expected case just like in other numerical analysis methods. Because, the statement of the bending moment involve the first-and the second-order derivative of deflection. The variation of the error with number of grid points was shown in Fig. 8 for HDQ, DQ, finite elements (FE), and FD method. The percentage error had been reduced as parallel to the increase the grid points. In this figure clamped support case is taken as boundary condition for uniformly distributed load. The FE results were obtained for uniformly distributed load and clamped edges by first author (Civalek 1998). The best solution is obtained for 13 × 13 grids sizes by using HDQ method. But, a reasonably converged solution may be achieved for 19 × 19 grids by FD. In addition to this, a reasonably converged solution may be obtained for 17 × 17 grid points using FEM (Civalek 1998). From the figure, the convergence of the HDQ and DQ method is seen to be very good. It is shown that in this table, HDQ method produces better convergent solutions than the FEM and FD when a similar number of grid points are used. It is also concluded that the HDQ method displays an oscillatory convergence, however, the DQ method results in a monotonic convergence. This consequence has also been stated by Liew *et al.* (2001).

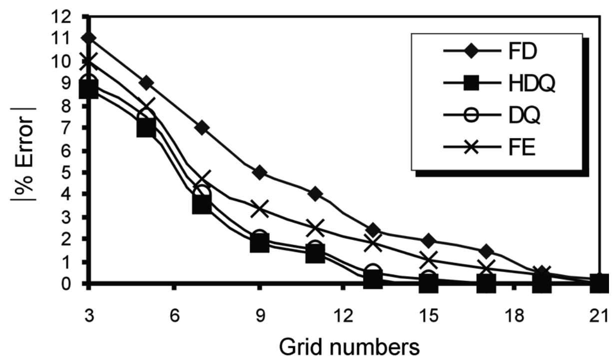

Fig. 8 Percentage error with grid numbers for uniformly loaded simply supported plates (ν = 0.2)

It was found that the HDQ method requires less than three seconds of CPU time for almost all cases on a standard personal computer (Pentium-II processor having 64 MB RAM). To the authors' knowledge, it is the first time that the harmonic differential quadrature and differential quadrature method has been successfully applied to thin, isotropic circular plate problems for the analysis of deflection and especially for bending moments and stresses in radial and tangential directions.

5. Conclusions

A harmonic type of DQ method was introduced to study the static analysis of thin, isotropic plates with various support and load conditions. The conventional small deflection theory is used in the study with the governing differential equations transformed into a set of linear algebraic equations by the harmonic differential quadrature formulation. The method of HDQ that was used in the paper proposes a very simple algebraic formula to determine the weighting coefficients required by differential quadrature approximation without restricting the choice of mesh grids. The known boundary conditions are easily incorporated in the HDQ as well as the other type of DQ. A good comparative accuracy of DQ and HDQ methods for static and vibration analysis of plates is presented by Liew *et al*. (1999a, 2001). More detailed information can be found in these references. The discretizing and programming procedures are straightforward and easy. An attractive advantage of the HDQ method is that it can produce the acceptable accuracy of numerical results with very few grid points in the solution domain and therefore can be very useful for rapid evaluation in engineering design. This has verified the accuracy and applicability of the HDQ method to the class of problem considered in this study.

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