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Some explicit solutions to plane equilibrium problem for no-tension bodies

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Abstract. A method is presented to integrate explicitly certain equilibrium problems for no-tension bodies, in absence of body forces and under the assumption that two of the principal stresses are everywhere null. The method is exemplified in the case of rectangular panels, clamped at their bottoms and loaded at their tops.

Key words: no-tension materials; pressureless gasses; explicit solutions.

1. Introduction

The constitutive description of the no-tension materials we consider can be summarized as follows (Di Pasquale 1984, Del Piero 1989): the infinitesimal strain is additively decomposed into an inelastic (or fracture), positive-semidefinite part and an elastic part depending on stress in a linear and isotropic manner; the stress, in turn, must be negative-semidefinite, and orthogonal to the inelastic part of the strain. These materials are often referred to as masonry-like, although the isotropy assumption e.g. is never exactly verified for masonry.

It is known (Giaquinta and Giusti 1985) that, whenever the general equilibrium problem for these materials has a solution, uniqueness in stress is as a rule accompanied by nonuniqueness in strain and displacement; moreover, such solutions, when they exist, are generally difficult to determine explicitly. However, the plane problems we here consider are fairly more tractable.

In plane-stress problems, at least one of the principal stresses must be null; as to the two remaining principal stresses, we expect the body to be divided into regions, in each of which one of the following three situations takes place: both stresses are negative, and the material behaves as if it were linearly elastic; one stress is negative, the other being null, and fracturing is in order; both stresses are null, and the material degenerates. It is the second situation which is both typical and interesting, and we here concentrate on cases when it prevails everywhere in the body (cf. Di Pasquale 1984).

The class of problems we later solve explicitly have a special feature: the stress state is determined independently of the states of strain and displacement. In fact, in the spirit of the semi-

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inverse method, we take two of the principal stresses to be everywhere null, and hence the unknown fields allowing for a representation of the stress state at each point of the body are two, just as many as the scalar equilibrium equations; moreover, the stress boundary conditions completely determine those two fields. Interestingly, the system of equilibrium equations we study, where the body forces are null, is formally identical to the nonlinear system of conservation laws ruling the dynamics of the one-dimensional, isentropic flow of a "pressureless" compressible gas (Courant and Friedrichs 1948, Brenier and Grenier 1998).

Given the stress field, the elastic part of the strain is immediately determined by the use of the relative constitutive equation. The orthogonality condition determines the inelastic part only to within an arbitrary, nonnegative scalar field. To proceed further, the compatibility condition for the total strain must be used; this condition yields a parabolic linear PDE for the scalar field, whose solution, for which we derive an explicit representation formula, can be fully determined by means of the displacement boundary conditions. At this point, it remains for us to check whether the scalar field we have obtained is indeed everywhere nonnegative; whenever this is the case, we are sure we have constructed a solution to the problem we posed; otherwise, we must conclude that, for the given loads, there is no solution of the form we have selected (in any event, to have an admissible equilibrium stress might be important for certain applications, e.g., to apply the theorems of Limit Analysis, as indicated by Del Piero, 1998).

As anticipated, we illustrate our method by applying it to a rectangular panel, clamped at the bottom and loaded at the top. The vertical load is taken uniform and constant, whereas the horizontal load is given either a bilinear distribution (Fig. 2) or a parabolic distribution (Fig. 4), graded by a scalar multiplier. In both cases we determine the equilibrium stress field explicitly, as well as the maximum value of the multiplier. Then, for the first load distribution only, we apply the representation formula for the components of the inelastic strain to compute the admissible total strain and the accompanying displacement field which complies with the given boundary conditions.

2. The equilibrium equations

We denote the stress tensor by $\boldsymbol{T}$ and assume $\boldsymbol{T}$ to be negative-semidefinite $(\boldsymbol{T} \leq \mathbf{0})$; and we denote by $\boldsymbol{E}$ the infinitesimal strain tensor. We also assume that $\boldsymbol{E}$ is the sum of an elastic part $\boldsymbol{E}^e$, on which $\boldsymbol{T}$ depends linearly and isotropically, and an inelastic part $\boldsymbol{E}^a$, positive-semidefinite $(\boldsymbol{E}^a \geq \mathbf{0})$ and orthogonal to $\boldsymbol{T}$. The constitutive law is therefore expressed by the relations

$$\boldsymbol{E} = \boldsymbol{E}^e + \boldsymbol{E}^a, \tag{1}$$

$$\boldsymbol{E}^e = \frac{1}{E}[(1+\nu)\boldsymbol{T} - \nu \operatorname{tr}(\boldsymbol{T})\boldsymbol{I}], \tag{2}$$

$$\boldsymbol{E}^a \geq \mathbf{0}, \tag{3a}$$

$$\boldsymbol{T} \leq \mathbf{0}, \tag{3b}$$

$$\boldsymbol{T} \cdot \boldsymbol{E}^a = 0, \tag{3c}$$

where E denotes Young's modulus, ν Poisson's ratio and $\boldsymbol{T} \cdot \boldsymbol{E}^a = \operatorname{tr}(\boldsymbol{T}\boldsymbol{E}^a)$ is the scalar product of $\boldsymbol{T}$ and $\boldsymbol{E}^a$. A well-known consequence of isotropy (Del Piero 1989) is that $\boldsymbol{T}$, $\boldsymbol{E}$ and $\boldsymbol{E}^a$ are coaxial and, moreover, that

$$\boldsymbol{T}\boldsymbol{E}^a = \boldsymbol{0}\,. \tag{4}$$

As mentioned, we assume that the stress state is plane and that at every point in the body two of the principal stresses are null. Therefore, in an orthogonal Cartesian reference system $(O;\, x, y)$, denoting the cotangent of the angle between the active isostatic line and the x axis by κ, we have

$$\boldsymbol{T} = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} = \sigma \begin{pmatrix} \kappa^2 & \kappa \\ \kappa & 1 \end{pmatrix}. \tag{5}$$

where we denote σ_y by σ. It can be seen that the principal non-zero stress equals $\sigma(1 + \kappa^2)$, and therefore in light of Eq. (3b), the admissibility condition for the stress is

$$\sigma \le 0\,. \tag{6}$$

In the absence of body forces the equilibrium equations,

$$(\sigma\kappa^2)_{,x} + (\sigma\kappa)_{,y} = 0, \tag{7a}$$

$$(\sigma\kappa)_{,x} + \sigma_{,y} = 0, \tag{7b}$$

make up a system of conservation laws. These coincide formally with the one-dimension Euler equations for the flow of an isentropic gas when the pressure gradient vanishes. More specifically, if $-\sigma$ and κ are interpreted as the density and velocity of the gas, respectively, and x, y as the spatial coordinate and time, Eq. (7a) expresses the conservation of linear momentum and Eq. (7b) is the continuity equation.

From Eq. (7) we obtain

$$\kappa^2\sigma_{,x} + 2\sigma\kappa\kappa_{,x} + \kappa\sigma_{,y} + \sigma\kappa_{,y} = 0,$$

$$\kappa\sigma_{,x} + \sigma\kappa_{,x} + \sigma_{,y} = 0.$$

Multiplying the second equation by k, assumed to be non-zero, and substituting it into the first, we get

$$\sigma(\kappa\kappa_{,x} + \kappa_{,y}) = 0;$$

therefore, for $\sigma \ne 0$, system (7) becomes

$$\kappa\kappa_{,x} + \kappa_{,y} = 0, \tag{8a}$$

$$\kappa\sigma_{,x} + \sigma\kappa_{,x} + \sigma_{,y} = 0. \tag{8b}$$

The former equation coincides with the inviscid Burger equation and has already been derived in (Di Pasquale 1984); the second is a linear equation for σ.

In the applications we denote the vertical and horizontal loads distributed on the top of the panel by p and q, respectively (Fig. 1).

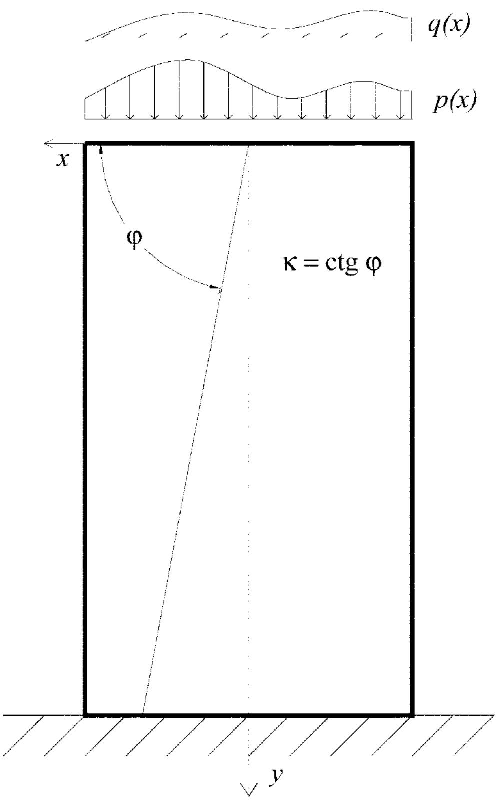

Fig. 1 Rectangular panel loaded at its top

Thus, for $f(x) = \dfrac{q(x)}{p(x)}$, in view of Eq. (5) we have

$$\sigma(x, 0) = -p(x)\,, \tag{9a}$$

$$\kappa(x, 0) = f(x)\,. \tag{9b}$$

Then, the solution to system (8) is implicitly defined by the relations

$$\kappa(x, y) = f(x - \kappa y)\,, \tag{10a}$$

$$\sigma(x, y) = -\frac{p(x - \kappa y)}{1 + y f'(x - \kappa y)}\,, \tag{10b}$$

where f' denotes the derivative of f (cf. Zauderer 1989).

If functions f and p are sufficiently smooth, then relations Eq. (10) define a regular solution to Eq. (8) for $y \le y_c = -1/f'(x_0)$, where x_0 is the value of x for which $-1/f'(x)$ reaches its smallest non-negative value.

In general, for $y > y_c$, systems (7) and (8) are not equivalent. It has recently been proved (Brenier and Grenier 1998), under very general conditions, that system (7) is equivalent to a single scalar conservation law that, in general, is not the inviscid Burger equation. However, we have not utilised this result in the applications presented herein, because we confine ourselves to consider the solution to system (8) in the interval $0 \le y \le y_c$.

As already shown in (Di Pasquale 1984), the active isostatic lines of our equilibrium problem are straight (Fig. 1); in effect, they coincide with the characteristics of system (8). Their equation is

$$x - yf(\lambda) - \lambda = 0, \tag{11}$$

where λ is the abscissa of the intersection point with the x axis. If f is not a constant function, the characteristics are not parallel and, in general, envelop a curve Γ, whose implicit equation is obtained by resolving the system

$$x - yf(\lambda) - \lambda = 0, \tag{12a}$$

$$yf'(\lambda) + 1 = 0. \tag{12b}$$

In particular, if f is linear, $f(\lambda) = c_1\lambda + c_2$, the characteristics meet at point $\left(-\frac{c_2}{c_1}, \frac{1}{c_1}\right)$. By comparing Eq. (10b) and Eq. (12b), we can deduce that, moving along any active isostatic, as we approach its point of tangency with the curve Γ, the principal non-zero stress $\sigma(1 + \kappa^2)$ tends to infinity.

When f is continuous but f' has a first-kind discontinuity at a point λ_0, κ is continuous, while σ is discontinuous along the isostatic of equation $x - yf(\lambda_0) - \lambda_0 = 0$. The fact that a discontinuity curve for σ, across which κ is continuous, must coincide with an active isostatic line can be easily deduced from the Rankine-Hugoniot jump conditions (Serre 1999) which for our system of conservation laws, (7), are

$$s[\sigma] = [\sigma\kappa], \tag{13a}$$

$$s[\sigma\kappa] = [\sigma\kappa^2], \tag{13b}$$

where $s = dx/dy$ is the slope of the discontinuity curve, and the square brackets denote the jump of the enclosed quantities across the discontinuity.

3. Examples of stress field determination

Let us now consider a rectangular panel of width b and height h, clamped at its bottom and subjected to horizontal and vertical loads, both distributed on the panel's top. We shall first consider the case in which the vertical load p is uniform, whereas the horizontal load q has a bilinear distribution (Fig. 2).

For $-\frac{1}{2}b < \lambda_0 \le \frac{1}{2}b$, let

$$f_1(x) = \begin{cases} \dfrac{\alpha(b + 2x)}{b + 2\lambda_0}, & \text{for} \quad -\frac{1}{2}b \le x \le \lambda_0, & \text{(14a)} \\[2ex] \dfrac{\alpha(b - 2x)}{b - 2\lambda_0}, & \text{for} \quad \lambda_0 \le x \le \frac{1}{2}b. & \text{(14b)} \end{cases}$$

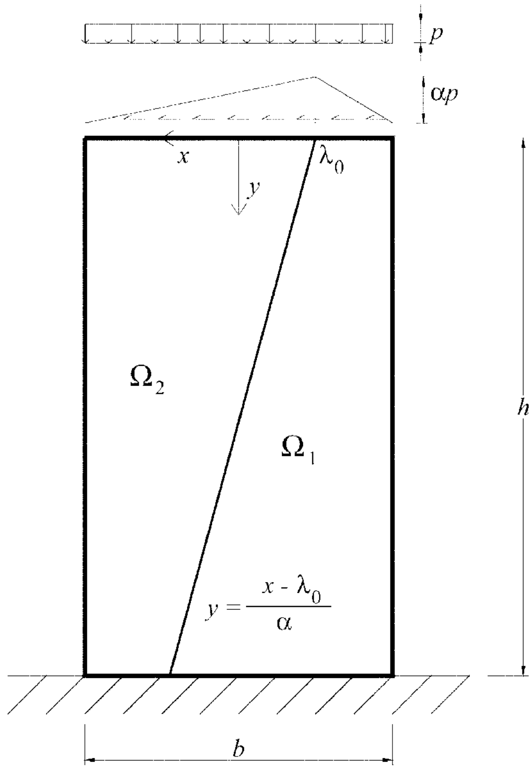

Fig. 2. Load-distribution laws on the top of the rectangular panel (example 1)

The solution to system (8) is regular in the two regions, Ω_1 and Ω_2, divided by the isostatic line, which, in view of Eq. (11) and Eq. (14), has the equation

$$y = \frac{x - \lambda_0}{\alpha} \tag{15}$$

(cf. the discussion at the end of Section 2). From Eq. (10) we get

$$\kappa(x, y) = \begin{cases} \dfrac{\alpha(2x + b)}{2\alpha y + b + 2\lambda_0}, & \text{for} \quad (x, y) \in \Omega_1, & \text{(16a)} \\[2ex] \dfrac{\alpha(2x - b)}{2\alpha y - b + 2\lambda_0}, & \text{for} \quad (x, y) \in \Omega_2 & \text{(16b)} \end{cases}$$

and

$$\sigma(x, y) = \begin{cases} \dfrac{-p(2\lambda_0 + b)}{2\alpha y + b + 2\lambda_0}, & \text{for} \quad (x, y) \in \Omega_1, & \text{(17a)} \\[2ex] \dfrac{-p(2\lambda_0 - b)}{2\alpha y - b + 2\lambda_0}, & \text{for} \quad (x, y) \in \Omega_2 & \text{(17b)} \end{cases}$$

and from Eq. (5)

$$\tau_{xy}(x, y) = \begin{cases} \dfrac{-p\alpha(2x+b)(2\lambda_0+b)}{(2\alpha y+b+2\lambda_0)^2}, & \text{for} \quad (x, y) \in \Omega_1, & (18a) \\ \dfrac{-p\alpha(2x-b)(2\lambda_0-b)}{(2\alpha y-b+2\lambda_0)^2}, & \text{for} \quad (x, y) \in \Omega_2, & (18b) \end{cases}$$

$$\sigma_x(x, y) = \begin{cases} \dfrac{-p\alpha^2(2x+b)^2(2\lambda_0+b)}{(2\alpha y+b+2\lambda_0)^3}, & \text{for} \quad (x, y) \in \Omega_1, & (19a) \\ \dfrac{-p\alpha^2(2x-b)^2(2\lambda_0-b)}{(2\alpha y-b+2\lambda_0)^3}, & \text{for} \quad (x, y) \in \Omega_2. & (19b) \end{cases}$$

This solution is well defined for $\alpha < \alpha_1^c$, with

$$\alpha_1^c = \frac{b}{2h} - \frac{\lambda_0}{h}. \tag{20}$$

The horizontal distributed load $q_1(x) = pf_1(x)$ has resultant $T = \frac{1}{2}\alpha pb$. Then, when p and λ_0 are fixed, the maximum value of T that can be reached is

$$T_1^c = \frac{pb(b-2\lambda_0)}{4h}. \tag{21}$$

For this load value, all the active isostatic lines of Eq. (11), with $\lambda_0 \le \lambda \le \frac{1}{2}b$, meet the panel's bottom at the corner $\left(\frac{1}{2}b, h\right)$; thus the region Ω_2 is free to rotate around this point (Fig. 3). If we interpret p as the permanent part of the load and q_1 as the part graded by the multiplier α, in the framework of Limit Analysis (Del Piero 1998), α_1^c represents the collapse multiplier.

As λ_0 varies, T_1^c reaches a maximum value of $T_m = \frac{pb^2}{2h}$, for $\lambda_0 = -\frac{1}{2}b$; T_m is precisely the load at which the whole panel overturns around the point with coordinates $\left(\frac{1}{2}b, h\right)$.

Finally, note that, in light of Eq. (14), the value of the slope of the horizontal load q_1 for $\lambda_0 \le x \le \frac{1}{2}b$, reached in correspondence to α_1^c, equals $-\frac{p}{h}$ and is therefore independent of λ_0.

Let us now consider the case in which the vertical load p is still uniform, whereas the horizontal load q_2 has a parabolic distribution (Fig. 4),

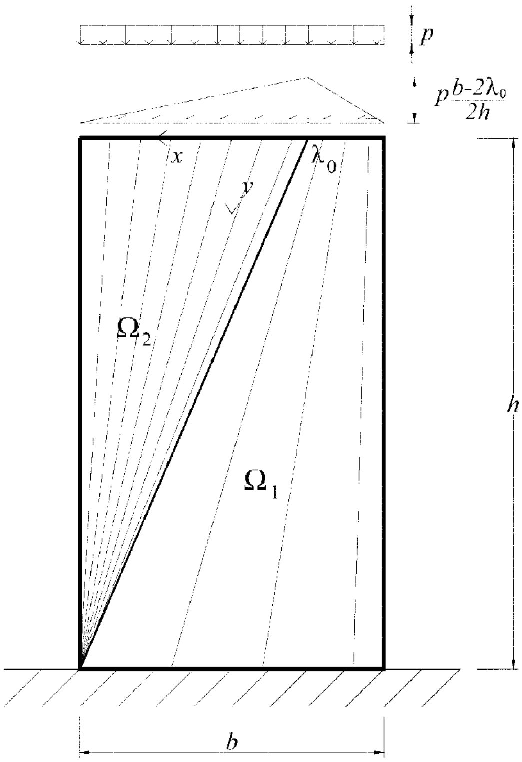

Fig. 3 Active isostatic lines for an arbitrary value of λ_0 and $\alpha = \alpha_1^c$

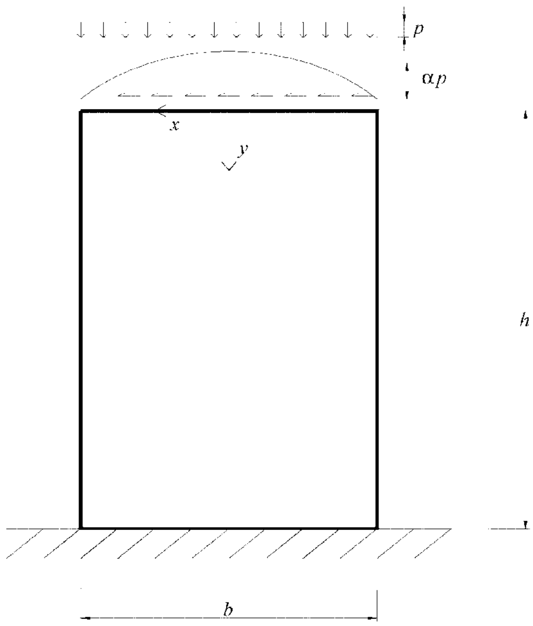

Fig. 4 Load-distribution laws on the top of the rectangular panel (example 2)

$$f_2(x) = \alpha\left(1 - \frac{4x^2}{b^2}\right). \tag{22}$$

The solution to system (8)

$$\kappa(x, y) = \frac{b\sqrt{-16\alpha xy + 16\alpha^2 y^2 + b^2} + 8\alpha xy - b^2}{8\alpha y^2} \tag{23}$$

and

$$\sigma(x, y) = \frac{-bp}{\sqrt{-16\alpha xy + 16\alpha^2 y^2 + b^2}} \tag{24}$$

are regular throughout the panel for

$$\alpha < \alpha_2^c = \frac{b}{4h}. \tag{25}$$

Moreover, in view of Eq. (5), we deduce

$$\tau_{xy}(x, y) = \frac{-bp(b\sqrt{-16\alpha xy + 16\alpha^2 y^2 + b^2} + 8\alpha xy - b^2)}{8\alpha y^2\sqrt{-16\alpha xy + 16\alpha^2 y^2 + b^2}} \tag{26}$$

and

$$\sigma_x(x, y) = \frac{-bp(b\sqrt{-16\alpha xy + 16\alpha^2 y^2 + b^2} + 8\alpha xy - b^2)^2}{64\alpha^2 y^4\sqrt{-16\alpha xy + 16\alpha^2 y^2 + b^2}}. \tag{27}$$

Now, in light of Eq. (11), the active isostatic lines have equation

$$y = \frac{b^2(x - \lambda)}{\alpha(b^2 - 4\lambda^2)}$$

and, by Eq. (12), for $0 \le \lambda \le \frac{1}{2}b$, envelop the curve

$$y = \frac{2x + \sqrt{(4x^2 - b^2)}}{4\alpha}, \quad \text{with} \quad x \ge \frac{1}{2}b \quad \text{and} \quad y \ge h$$

whose asymptote is the isostatic $y = \dfrac{x}{\alpha}$ (Fig. 5).

For $\alpha = \alpha_2^c$, the isostatic of equation $x = \frac{1}{2}b$ intersects the envelope at the panel's corner with coordinates $\left(\frac{1}{2}b, h\right)$, at which point the value of the principal non-zero stress is unbounded.

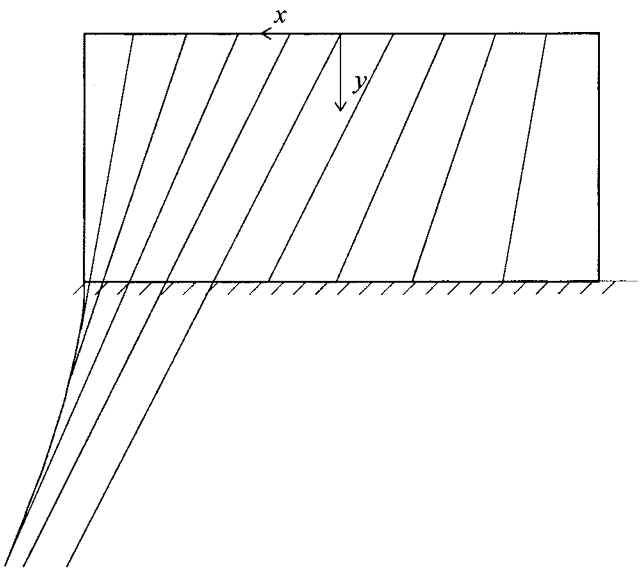

Fig. 5 Envelope of the active isostatic lines, for $\alpha = \alpha_2^c$

The horizontal distributed load $q_2(x) = pf_2(x)$ has resultant $T = \frac{2}{3}\alpha pb$; thus, the value of T corresponding to α_2^c is

$$T_2^c = \frac{pb^2}{6h}. \tag{28}$$

Figs. 6 and 7 show the behaviour of $\sigma(x, y)$ and $\tau_{xy}(x, y)$ for $\alpha = \alpha_2^c$, fixed parameter values (b = 5 m, h = 10 m, p = 1 MPa) and different values of y.

In the analogous gas-dynamic problem, where y, $-\sigma$ and $-\tau_{xy}$ play the role of the time, density and linear momentum per unit volume, respectively, we have the progressive formation of a shock wave taking place at "time" h, as shown in Figs. 6 and 7.

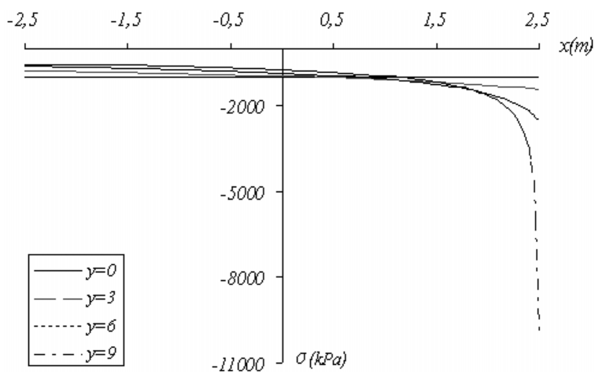

Fig. 6 σ vs x for different values of y and $\alpha = \alpha_2^c$

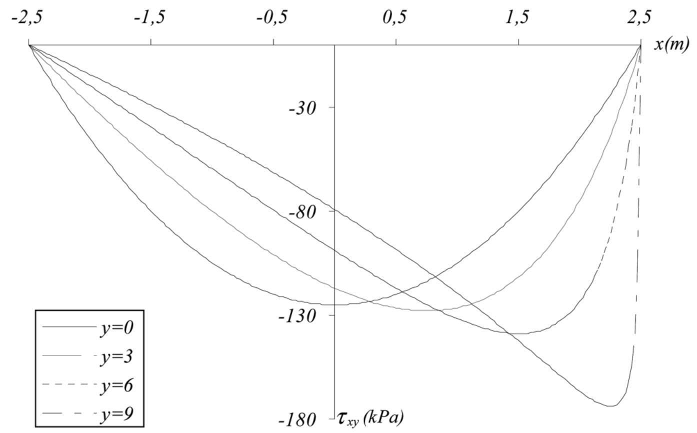

Fig. 7 τ_{xy} vs x for different values of y and $\alpha = \alpha_2^c$

These examples show that, with fixed vertical load, the maximum value of the resultant of the horizontal loads that can be assigned to the panel depends strictly on the law according to which these loads are distributed.

4. A representation formula for the strain

In this section, given the stress field, a method for deducing the strain is presented. From Eq. (2) and Eq. (5), we obtain

$$\boldsymbol{E}^e = \frac{\sigma}{E}\begin{pmatrix} \kappa^2 - \nu & \kappa(1+\nu) \\ \kappa(1+\nu) & 1 - \nu\kappa^2 \end{pmatrix}. \tag{29}$$

Moreover, from Eq. (4) and Eq. (5) we deduce

$$\boldsymbol{E}^a = a\begin{pmatrix} 1 & -\kappa \\ -\kappa & \kappa^2 \end{pmatrix}, \tag{30}$$

where, in light of Eq. (3a), a must be a scalar non-negative field,

$$a(x, y) \geq 0. \tag{31}$$

Requiring that $\boldsymbol{E} = \boldsymbol{E}^e + \boldsymbol{E}^a$ satisfy the compatibility equation, with the help of (30) and Eq. (29), we obtain

$$\begin{gathered}(\sigma\kappa^2 - \nu\sigma + Ea)_{,yy} + (\sigma - \nu\sigma\kappa^2 + Ea\kappa^2)_{,xx} + \\ -2((1+\nu)\sigma\kappa - Ea\kappa)_{,xy} = 0,\end{gathered} \tag{32}$$

from which, accounting for the fact that by system (7) we have

$$(\sigma\kappa^2)_{,xx} + 2(\sigma\kappa)_{,xy} + \sigma_{,yy} = 0, \tag{33}$$

we obtain for a the equation

$$(a\kappa^2)_{,xx} + 2(a\kappa)_{,xy} + a_{,yy} = \theta, \tag{34}$$

where

$$E\theta = [\sigma(1+\kappa^2)]_{,xx} + [\sigma(1+\kappa^2)]_{,yy} \tag{35}$$

is the Laplacian of the trace of the stress.

Therefore, for no-tension materials, stress compatibility does not require that the stress trace be harmonic, as in the linear elastic case, but calls for the existence of a non-negative function a that satisfies Eq. (34). Moreover, note that a is independent of the value of ν.

Eq. (34) is parabolic and its characteristic has the equation

$$\frac{dx}{dy} = \kappa. \tag{36}$$

In view of Eq. (8a), this suggests the change of variables

$$\xi = \kappa(x, y), \tag{37a}$$

$$\eta = y. \tag{37b}$$

Since $\eta_{,x} = \eta_{,xx} = \eta_{,xy} = \eta_{,yy} = 0$ and $\eta_{,y} = 1$, we have the relations

$$a_{,x} = a_{,\xi}\kappa_{,x} \tag{38a}$$

$$a_{,y} = a_{,\xi}\kappa_{,y} + a_{,\eta} \tag{38b}$$

$$a_{,xx} = a_{,\xi\xi}\kappa_{,x}^2 + a_{,\xi}\kappa_{,xx} \tag{38c}$$

$$a_{,xy} = a_{,\xi\xi}\kappa_{,x}\kappa_{,y} + a_{,\xi\eta}\kappa_{,x} + a_{,\xi}\kappa_{,xy} \tag{38d}$$

$$a_{,yy} = a_{,\xi\xi}\kappa_{,y}^2 + 2a_{,\xi\eta}\kappa_{,y} + a_{,\eta\eta} + a_{,\xi}\kappa_{,yy} \tag{38e}$$

and from Eq. (34) we can, through a number of steps, deduce

$$\begin{aligned}&(\kappa\kappa_{,x} + \kappa_{,y})^2 a_{,\xi\xi} + 2(\kappa\kappa_{,x} + \kappa_{,y})a_{,\xi\eta} + a_{,\eta\eta} + \\ &(\kappa^2\kappa_{,xx} + 2\kappa\kappa_{,xy} + \kappa_{,yy})a_{,\xi} + 2\kappa_{,x}a_{,\eta} = \theta.\end{aligned} \tag{39}$$

From Eq. (8a) it is a simple matter to show that

$$\kappa^2\kappa_{,xx} + 2\kappa\kappa_{,xy} + \kappa_{,yy} = 0 \tag{40}$$

and thereby from Eq. (39) obtain

$$a_{,\eta\eta} + 2\kappa_{,x}a_{\eta} = \theta. \tag{41}$$

Now, setting

$$a' = a_{,\eta}, \tag{42}$$

we can write

$$a'_{,\eta} + 2\kappa_{,x}a' = \theta, \tag{43}$$

which, once $\kappa_{,x}$ and θ are expressed as functions of ξ and η, can be treated as an ordinary linear first-order differential equation. To this aim, under the assumption that relations (37) can be inverted, we consider the inverse transformation

$$x = x(\xi, \eta), \tag{44a}$$

$$y = \eta, \tag{44b}$$

and obtain

$$\kappa_{,x} = \frac{1}{x_{,\xi}} \tag{45a}$$

$$\kappa_{,y} = -\frac{x_{,\eta}}{x_{,\xi}}. \tag{45b}$$

Moreover, in view of Eq. (37a), Eq. (45a) and Eq. (45b), Eq. (8a) implies

$$x_{,\eta} = \xi \tag{46}$$

from which we arrive at

$$x = \xi\eta + \psi(\xi), \tag{47}$$

where, in light of Eq. (10a) and Eq. (37a), $\psi = f^{-1}$. In effect, as f is generally a non-injective function, this relation is valid only locally. Moreover, ψ is not differentiable along any isostatic originating in a point of x axis where f' vanishes. From Eq. (46) and Eq. (47) we obtain

$$x_{,\xi} = \eta + \psi'(\xi), \tag{48a}$$

$$x_{,\xi\eta} = 1, \tag{48b}$$

$$x_{,\xi\xi} = \psi''(\xi), \tag{48c}$$

$$x_{,\eta\eta} = 0. \tag{48d}$$

Eqs. (45a), (46) and (48) allow us to express $\kappa_{,xx}$ and $\kappa_{,yy}$ as a function of ξ and η. In fact, recalling that $\eta_{,x} = 0$ and $\eta_{,y} = 1$, we have

$$\kappa_{,xx} = (\kappa_{,x})_{,x} = (\kappa_{,x})_{,\xi}\xi_{,x} = -\frac{x_{,\xi\xi}}{(x_{,\xi})^3} = -\frac{\psi''(\xi)}{(\eta + \psi'(\xi))^3}; \tag{49}$$

$$\kappa_{,yy} = (\kappa_{,y})_{,y} = (\kappa_{,y})_{,\xi}\xi_{,y} + (\kappa_{,y})_{,\eta}\eta_{,y} = \frac{2x_{,\xi}x_{,\eta} - x_{,\xi\xi}(x_{,\eta})^2}{(x_{,\xi})^3} =$$

$$\frac{2\xi(\eta + \psi'(\xi)) - \xi^2\psi''(\xi)}{(\eta + \psi'(\xi))^3}. \tag{50}$$

Moreover, from the relations

$$\kappa(\xi, \eta) = \xi, \tag{51a}$$

$$\sigma(\xi, \eta) = -\frac{p(\psi(\xi))\psi'(\xi)}{\eta + \psi'(\xi)}, \tag{51b}$$

as a consequence of Eq. (10), Eq. (44), Eq. (47) and the relation $\psi = f^{-1}$, we can calculate all derivatives of σ and $\sigma\kappa^2$ with respect to variables ξ and η, and therefore, with the help of Eq. (35), Eq. (38) and Eq. (45), determine $\theta = \theta(\xi, \eta)$. Proceeding in this way we obtain

$$\theta = \frac{\varphi\zeta^2\mu\psi''' - 3\zeta^2\mu(\psi'')^2 + 3\varphi\zeta\psi''(\zeta\mu' + 4\xi\mu)}{E\varphi^5} +$$

$$-\frac{(\zeta^2\mu'' + 4(2\xi\zeta\mu' + \mu(3\xi^2 + 1)))}{E\varphi^3}, \tag{52}$$

where, for the sake of brevity, we have set

$$\mu(\xi) = -\psi'(\xi)p(\psi(\xi)), \tag{53a}$$

$$\zeta = 1 + \xi^2, \tag{53b}$$

$$\varphi(\xi, \eta) = \eta + \psi'(\xi). \tag{53c}$$

Hence, by accounting for Eq. (45a) and Eq. (48a), from Eq. (43) and Eq. (53c), we obtain

$$a'_{,\eta} + \frac{2}{\varphi}a' = \theta, \tag{54}$$

whose solution can be expressed in the well-known form

$$a'(\xi, \eta) = \frac{c(\xi) + \int\varphi(\xi, \eta)^2\theta(\xi, \eta)d\eta}{\varphi(\xi, \eta)^2}, \tag{55}$$

where c is an arbitrary function. The integral at the right side of Eq. (55) can be calculated explicitly. In fact, with the help of Eq. (52) and Eq. (53), we obtain

$$a' = \frac{-\ln\varphi(4\mu + 12\mu\xi^2 + 8\xi\zeta\mu' + \zeta^2\mu'')}{E\varphi^2} - \frac{3\psi''(\zeta^2\mu' + 4\xi\zeta\mu) + \zeta^2\mu\psi'''}{E\varphi^3} +$$

$$\frac{3\zeta^2\mu(\psi'')^2}{2E\varphi^4} + \frac{c}{\varphi^2}, \tag{56}$$

from which, integrating with respect to η, we get

$$a = \frac{\ln\varphi(\zeta^2\mu'' + 4(2\xi\zeta\mu' + 3\mu\xi^2 + \mu))}{E\varphi} - \frac{\zeta^2\mu(\psi'')^2}{2E\varphi^3} +$$

$$\frac{\zeta^2\mu\psi''' + 3\zeta^2\mu'\psi'' + 12\xi\zeta\mu\psi''}{2E\varphi^2} +$$

$$\frac{\zeta^2\mu'' + 8\xi\zeta\mu' + 4\mu(1 + 3\xi^2)}{E\varphi} - \frac{c}{\varphi} + d, \tag{57}$$

where d is an arbitrary function of ξ. Once a is known as a function of ξ and η, with the help of Eq. (37), a can be determined as a function of x and y.

In particular, for $\theta = 0$, from Eqs. (55), (53c), (47), (45) and (37a), we obtain

$$a = -c(k)k_{,x} + d(k). \tag{58}$$

5. Examples of strain field determination

We now propose to verify whether the stress state determined by the Eqs. (16)-(19) solves the boundary-value problem for the case shown in Fig. 2. In other words, we wish to see whether there exists a displacement field satisfying the boundary conditions whose corresponding strain satisfies the constitutive equation for some $a \geq 0$. For the sake of simplicity, we shall limit ourselves to considering the case in which $\lambda_0 = 0$ and $\nu = 0$.

We begin by looking at the region Ω_1. From Eq. (47), Eq. (16) and Eq. (17) it is a simple matter to obtain

$$\psi(\xi) = \frac{b(\xi - \alpha)}{2\alpha} \tag{59}$$

and therefore from Eq. (53a) we deduce

$$\mu = -\frac{pb}{2\alpha}. \tag{60}$$

Thus, from Eq. (57) we obtain

$$a(\xi, \eta) = \frac{-2pb(1 + 3\xi^2)\left(1 + \ln\left(\eta + \frac{b}{2\alpha}\right)\right)}{E\alpha\left(\eta + \frac{b}{2\alpha}\right)} - \frac{c(\xi)}{\eta + \frac{b}{2\alpha}} + d(\xi). \tag{61}$$

In order to determine the displacement in region Ω_1 it is convenient to make reference to a polar

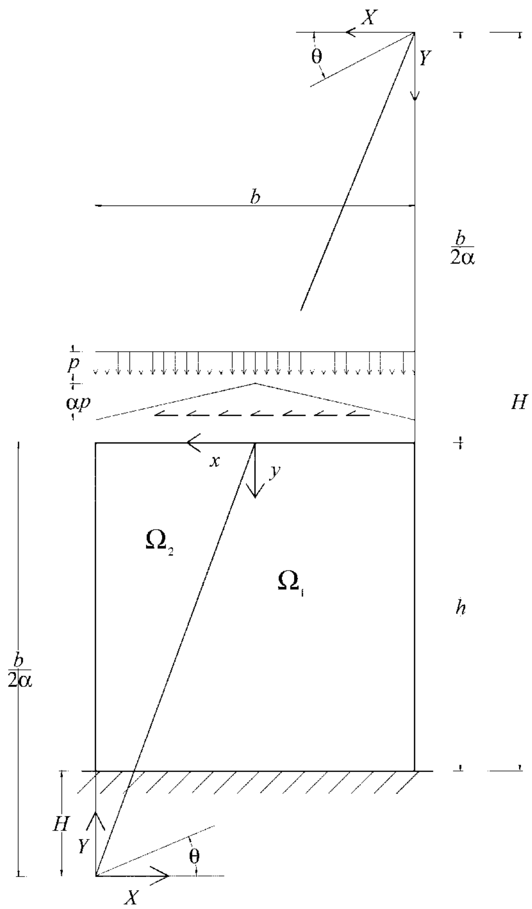

Fig. 8 Polar coordinates used for displacement determination

coordinate system r, ϑ, with origins at point $\left(-\frac{1}{2}b, -\frac{b}{2\alpha}\right)$, the intersection of the straight line $x = -\frac{1}{2}b$ with the isostatic of equation $x = \alpha y$ (Fig. 8).

By virtue of Eq. (5), Eq. (30) and Eq. (29), for $\nu = 0$, we have

$$\sigma_r = \sigma(1 + \kappa^2), \tag{62a}$$

$$\tau_{r\vartheta} = \sigma_\vartheta = 0; \tag{62b}$$

$$\varepsilon^a_\vartheta = a(1 + \kappa^2), \tag{63a}$$

$$\varepsilon^a_r = \varepsilon^a_{r\vartheta} = 0; \tag{63b}$$

$$\varepsilon^e_r = \frac{\sigma(1 + \kappa^2)}{E}, \tag{64a}$$

$$\varepsilon^e_\vartheta = \varepsilon^e_{r\vartheta} = 0. \tag{64b}$$

As it holds that

$$\eta = r\sin\vartheta - \frac{b}{2\alpha} \tag{65}$$

and that, in light of Eq. (16), Eq. (17) and Eq. (37a), we have

$$\sigma = -\frac{pb}{2\alpha r\sin\vartheta}, \tag{66a}$$

$$\xi = \kappa = \operatorname{ctg}\vartheta, \tag{66b}$$

from Eq. (63) and Eq. (64), we get

$$\varepsilon_r^e = -\frac{pb}{2\alpha E r\sin^3\vartheta}, \tag{67a}$$

$$\varepsilon_\vartheta^a = \frac{a}{\sin^2\vartheta}, \tag{67b}$$

where, by virtue of Eq. (61), Eq. (65) and Eq. (66b),

$$a(r, \vartheta) = -\frac{2bp}{E\alpha r\sin\vartheta}(1 + 3\operatorname{ctg}^2\vartheta)(1 + \ln(r\sin\vartheta)) - \frac{C(\vartheta)}{r\sin\vartheta} + D(\vartheta), \tag{68}$$

with C and D are arbitrary functions of ϑ.

Denoting the radial displacement by u_r, from the relation

$$\varepsilon_r = \frac{\partial u}{\partial r}, \tag{69}$$

with the help of Eq. (67a) we obtain

$$u_r = -\frac{bp\ln r}{2E\alpha\sin^3\vartheta} + F(\vartheta), \tag{70}$$

where F is an arbitrary function. As the panel's bottom is clamped, we have $u_r\left(\frac{H}{\sin\vartheta}, \vartheta\right) = 0$, for

$$H = h + \frac{b}{2\alpha} \tag{71}$$

and can therefore write

$$u_r = \frac{bp}{2E\alpha\sin^3\vartheta}\ln\left(\frac{H}{r\sin\vartheta}\right). \tag{72}$$

Analogously, denoting the circumferential displacement by u_ϑ, from the relation

$$\varepsilon_\vartheta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\vartheta}{\partial\vartheta}, \tag{73}$$

with the help of Eq. (67b), Eq. (68) and Eq. (72), we obtain

$$u_\vartheta = \frac{bp((\ln(1-\cos\vartheta)(1-8\ln H)) + 3\ln(1+\cos\vartheta))}{32E\alpha} +$$

$$\frac{bp\ln(\sin\vartheta)}{2E\alpha}\left(\frac{\ln H}{2} + \frac{3\cos\vartheta}{\sin^4\vartheta} - \frac{1}{4}\right) +$$

$$\frac{bp\cos\vartheta}{2E\alpha\sin^2\vartheta}\left(\frac{5}{4} + \frac{\ln H}{2} + \frac{3\ln r}{\sin^2\vartheta} + \frac{9}{8}\mathrm{ctg}^2\vartheta + \frac{21}{8\sin^2\vartheta}\right) +$$

$$r\hat{D}(\vartheta) - \hat{C}(\vartheta) + G(r)\,, \tag{74}$$

where G is an arbitrary function, $\hat{C}(\vartheta)$ is a primitive of $C(\vartheta)/\sin^3\vartheta$ and $\hat{D}(\vartheta)$ is a primitive of $D(\vartheta)/\sin^2\vartheta$. Functions C, D and G can be determined by requiring that the displacement vanish at the panel's bottom and using the well-known relation

$$2\varepsilon_{r\vartheta} = \frac{1}{r}\frac{\partial u_r}{\partial\vartheta} + \frac{\partial u_\vartheta}{\partial r} - \frac{u_\vartheta}{r}. \tag{75}$$

From this, in light of Eq. (72) and Eq. (74), we obtain

$$C(\vartheta) = \frac{bp}{8E\alpha\sin^4\vartheta}[16\ln H(2\cos^4\vartheta - \cos^2\vartheta - 1) + \cos^3\vartheta(\sin^4\vartheta + \sin^2\vartheta + 1) +$$

$$\cos^2\vartheta(8 - 20\sin^2\vartheta) + \cos\vartheta(\sin^6\vartheta - 1) - 8] \tag{76}$$

which, by substituting into Eq. (74), furnishes a new expression for u_ϑ, with the help of which we can, from Eq. (75), obtain

$$G'(r) - \frac{G(r)}{r} = 0$$

which in turn implies that

$$G(r) = \hat{c}r, \tag{77}$$

where $\hat{c}$ is an arbitrary constant. Recalculating u_ϑ with the help of Eq. (76) and Eq. (77) and imposing the boundary condition

$$u_\vartheta\left(\frac{H}{\sin\vartheta}, \vartheta\right) = 0,$$

we obtain

$$D(\vartheta) = \frac{bp}{E\alpha H}(1 + 3\mathrm{ctg}^2\vartheta) \tag{78}$$

and

$$\hat{c} = 0, \tag{79}$$

which finally allows us to write

$$a(r,\vartheta) = \frac{bp}{E\alpha}\Bigg[-2\left(\frac{1+2\cos^2\vartheta}{r\sin^3\vartheta}\right)\ln\left(\frac{r\sin\vartheta}{H}\right) + 3\operatorname{ctg}^2\vartheta\left(\frac{1}{H}-\frac{1}{2r\sin\vartheta}\right)-\frac{1}{r\sin^3\vartheta}+\frac{1}{H}\Bigg], \tag{80}$$

$$u_\vartheta = \frac{bp\cos\vartheta}{E\alpha\sin^3\vartheta}\left[\frac{3}{2\sin\vartheta}\ln\left(\frac{r\sin\vartheta}{H}\right)+\frac{1}{\sin\vartheta}-\frac{r}{H}\right]. \tag{81}$$

In order that the strain and displacement fields obtained solve our boundary-value problem for Ω_1, at this point it is sufficient to check that $a(r,\vartheta)\geq 0$ in Ω_1, that is to say for

$$\operatorname{arcctg}(\alpha)\leq\vartheta\leq\frac{1}{2}\pi \tag{82a}$$

and

$$\frac{b}{2\alpha\sin\vartheta}\leq r\leq\frac{H}{\sin\vartheta}. \tag{82b}$$

To this end, we observe that, in light of Eq. (82b), we have

$$\frac{\partial a}{\partial r} = \frac{bp}{E\alpha r^2\sin^3\vartheta}\left[2(2\cos^2\vartheta+1)\ln\left(\frac{r\sin\vartheta}{H}\right)-\frac{5\cos^2\vartheta}{2}-1\right]<0 \tag{83}$$

and therefore, for every fixed ϑ, a assumes its minimum value for $r = H/\sin\vartheta$, so that

$$a(r,\vartheta)\geq a\left(\frac{H}{\sin\vartheta},\vartheta\right) = \frac{bp}{2E\alpha H}\left(\frac{1}{\sin^2\vartheta}-1\right)\geq 0. \tag{84}$$

Regarding region Ω_2, from Eq. (47), Eq. (16) and Eq. (17), we deduce, in place of Eq. (59) and Eq. (60), that

$$\psi(\xi) = \frac{b(\alpha-\xi)}{2\alpha}, \tag{85a}$$

$$\mu = \frac{pb}{2\alpha}, \tag{85b}$$

and then, in light of Eq. (57) we can write

$$a(\xi,\eta) = \frac{-2pb(1+3\xi^2)\left(1+\ln\left(\frac{b}{2\alpha}-\eta\right)\right)}{E\alpha\left(\frac{b}{2\alpha}-\eta\right)}+\frac{c(\xi)}{\left(\frac{b}{2\alpha}-\eta\right)}+d(\xi). \tag{86}$$

For calculation of the displacement it is now convenient to make reference to a polar coordinate system with origin at point $(b/2, b/2\alpha)$, as shown in Fig. 8. Therefore, in Ω_2 it holds that

$$\operatorname{arcctg}(\alpha) \leq \vartheta \leq \frac{1}{2}\pi, \tag{87a}$$

$$\frac{b - 2\alpha h}{2\alpha \sin\vartheta} \leq r \leq \frac{b}{2\alpha \sin\vartheta}. \tag{87b}$$

In this case, in place of Eq. (65), we have $\eta = \dfrac{b}{2\alpha} - r\sin\vartheta$ and, in light of Eq. (16), Eq. (17) and Eq. (37a), Eqs. (62), (63), (64), (66) and (67) maintain their validity, with

$$a(r,\vartheta) = -\frac{2bp}{E\alpha r \sin\vartheta}(1 + 3\operatorname{ctg}^2\vartheta)(1 + \ln(r\sin\vartheta)) + \frac{C(\vartheta)}{r\sin\vartheta} + D(\vartheta), \tag{88}$$

which differs from Eq. (68) in the sign of the term $\dfrac{C(\vartheta)}{r\sin\vartheta}$.

Proceeding in a manner analogous to the foregoing case, we arrive at expressions for a, u_r and u_ϑ that are formally identical to those obtained for Ω_1, provided that in Eqs. (80), (72) and (81) we replace Eq. (71) with

$$H = \frac{b}{2\alpha} - h. \tag{89}$$

In this case, however, a is not non-negative throughout. To verify this fact, it is enough to consider the restriction $\hat{a}$ of a to the segment

$$\vartheta = \frac{1}{2}\pi, \tag{90a}$$

$$H \leq r \leq \frac{b}{2\alpha}. \tag{90b}$$

Then, from Eq. (87) we in fact obtain

$$\hat{a}(r) = a\left(r, \frac{1}{2}\pi\right) = \frac{bp}{E\alpha}\left[-\frac{2}{r}\ln\left(\frac{r}{H}\right) - \frac{1}{r} + \frac{1}{H}\right] \tag{91}$$

and therefore have $\hat{a}(H) = 0$ and, moreover, $\dfrac{d\hat{a}(r)}{dr} < 0$ in a right-sided neighborhood of H.

Therefore, the stress state defined by Eqs. (16)-(19) does not allow determining a solution to our boundary-value problem.

Nevertheless, if we consider a body whose shape is that of the region Ω_1, constrained and loaded as shown in Fig. 9, the Eqs. (16a)-(19a), (80), (72) and (81) are the solution of the boundary-value problem. An interesting outcome is that the solution is unique with respect to both the strain and the displacement.

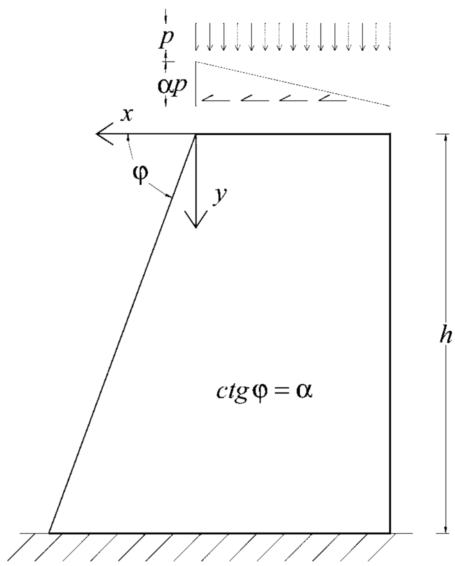

Fig. 9 The boundary value problem that has been solved

5. Conclusions

For a broad class of boundary value problems for plane bodies made of no-tension materials, the method here proposed provides a way to determine a stress state that is both admissible and equilibrated with the assigned loads. In some cases, once the stress state is known, it is also possible to determine an admissible strain state and the associated displacement field satisfying the assigned boundary conditions, thereby providing a complete solution to the given boundary value problem.

As highlighted in the examples, by examining load processes depending on a specific multiplier, the method can also be used to determine the corresponding collapse load.

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