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Differential cubature method for buckling analysis of arbitrary quadrilateral thick plates

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Abstract. In this paper, a novel numerical solution technique, the differential cubature method is employed to study the buckling problems of thick plates with arbitrary quadrilateral planforms and non-uniform boundary constraints based on the first order shear deformation theory. By using this method, the governing differential equations at each discrete point are transformed into sets of linear homogeneous algebraic equations. Boundary conditions are implemented through discrete grid points by constraining displacements, bending moments and rotations of the plate. Detailed formulation and implementation of this method are presented. The buckling parameters are calculated through solving a standard eigenvalue problem by subspace iterative method. Convergence and comparison studies are carried out to verify the reliability and accuracy of the numerical solutions. The applicability, efficiency, and simplicity of the present method are demonstrated through solving several sample plate buckling problems with various mixed boundary constraints. It is shown that the differential cubature method yields comparable numerical solutions with 2.77-times less degrees of freedom than the differential quadrature element method and 2-times less degrees of freedom than the energy method. Due to the lack of published solutions for buckling of thick rectangular plates with mixed edge conditions, the present solutions may serve as benchmark values for further studies in the future.

Key words: differential cubature method; buckling analysis; critical load; thick quadrilateral plates; plates with mixed boundary conditions.

1. Introduction

Both thin and thick rectangular plates are extensively used in mechanical, civil, nuclear and aerospace structures. A good understanding of the buckling behaviors of these structural components is crucial to the design and performance evaluation of mechanical systems. A vast amount of literature for buckling analysis of plates is available. Srinivas and Rao (1969) presented an exact three dimensional analysis for the buckling of simply supported thick rectangular plates. Brunelle (1971) solved the buckling problems of thick rectangular plates with two opposite edges simply supported and the other edges arbitrary constrained. Brunelle and Robertson (1974) derived the governing equations of a transversely isotropic, initially stressed Mindlin plates, and presented the buckling solutions for a simply supported rectangular plate under combination of a uniform

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compressive stress and a uniform bending stress acting in the same direction. Hinton (1978) also studied the buckling of initially stressed Mindlin plates by the finite strip method, and obtained some results for plates with two opposite sides simply supported and various edge conditions on the other sides. Rao *et al.* (1975) analyzed the stability of moderately thick rectangular plates by using a high precision triangular finite element. Luo (1982) performed a hybrid finite element formulation for buckling of thin and thick plates. Dawe and Roufaeil (1982) solved the buckling problem of thick plates by using the finite strip method and Rayleigh-Ritz method.

In all the papers mentioned above, researchers have confined their studies to a simple rectangular domain, and only the uniform boundary conditions were considered. For the stability analysis of plates with arbitrary shape, Kennedy and Prabhakara (1979) used the classical Rayleigh-Ritz method to study the buckling of orthotropic skew plates. Based on the classical Rayleigh-Ritz method, Wang *et al.* (1992) and Kitipornchai *et al.* (1993) developed pb-2 Ritz functions for approximating the deflected shape of skew plates in order to study the buckling of skew plates with simply supported edges and internal supports. They presented design charts for skew plates with different edge conditions. Besides the energy method, other numerical methods are also usually used to study this problem. For examples, Wittrick (1956), Edwardes and Kabaila (1978) and Hegedus (1988) investigated the buckling problem by using the finite difference method, finite element method and finite strip method, respectively. Regarding the plates with mixed boundary conditions, Hamada *et al*. (1967) studied the buckling of thin plates with simply supported but partially clamped edges. Keer and Stahl (1972) presented an exact analytical solution of buckling problem for a simply supported thin plate having mixed boundary conditions. Using an approximate method, Sakiyama and Matsuda (1987) analyzed the buckling of rectangular Mindlin plates with mixed edge conditions. However, only the solutions for thin plates with mixed boundary conditions were presented. Up to now, solutions to the buckling of thick plates with mixed boundary conditions are extremely scarce in the open literature.

Recently, an efficient global solution technique, the differential quadrature(DQ) method was introduced by Bellman and his associates (Bellman and Casti 1971, 1972) for solving linear and nonlinear differential equations with a little computational cost. Since then, there have been numerous developments and applications of the method in structural mechanics (Bert and Malik 1996, Liew *et al*. 1996, Malik and Bert 1998). However, further application of the method has been greatly restricted by the disadvantage that it cannot be directly used to solve problems with discontinuities or with complex domains. Besides, although the differential quadrature method is also applicable to multidimensional problems, it is most suitable for solving one-dimensional problems, since it is based on a weighted linear sum of discrete function values in a single variable. To overcome these drawbacks, Civan (1994) developed a novel numerical technique, the differential cubature(DC) method as an accurate alternative to the differential quadrature method in dealing with multi-dimensional differential equations. The DC method is a direct discretization method, which approximates the partial derivatives of a function by means of polynomials, which expressed as a weighted linear sum of function values at the grid points in the overall physical domain. The practical importance of the DC method is that it needs to use only a few grid points which is able to obtain an acceptable accuracy in an arbitrary domain. And, the DC method is much simpler than the differential quadrature method when treating the multivariable problems. In his publication, Civan has shown that the differential cubature method is exceptionally efficient in solving the mathematical models of Buckley-Leverett problem for water flooding of naturally fractured oil bearing reservoirs and that the DC method is particularly advantageous over the differential quadrature method when dealing with mixed operations, such as $\partial^2/\partial x\partial y$. Furthermore, since the grid points are located in arbitrary position, the DC method can be easily used to solve the problems in various shaped domains and with various boundary constraints. Liew and Liu (1997, 1998) have used this method for static analysis of arbitrary shaped thin and thick plates. They concluded that the DC method can yield rapidly convergent numerical solutions and the results were in excellent agreement with the exact analytical solutions. Therefore it has been claimed to be a superior numerical method for solving the multi-dimensional problems in arbitrary domains. However, to the author's knowledge, the potential of this method for solution of a varied class of problems has not been explored and no any work has been reported on the application of the differential cubature method for plate buckling and vibration problems as yet. In view of the fact that few solutions to buckling of thick plates with non-uniform boundary conditions are reported and the potential capability of the differential cubature method, the authors attempt to exploit this new numerical method for buckling analysis of thick plates with arbitrary shape and mixed boundary conditions. In this paper, the suitability, efficiency, simplicity and convergence properties of this method were all demonstrated. The numerical accuracy is verified by the comparison of the present results with corresponding exact solutions or other numerical solutions in the open literature.

2. The differential cubature method

Basically, the differential cubature method is a numerical procedure expressing a linear operation such as a continuous function or any orders of partial derivatives of multivariable function or combinations of them as a weighted linear sum of discrete function values chosen within the overall domain of a problem. For a two dimensional problem, supposing that there are n arbitrary located grid points indexed in one dimensional, the cubature approximation at the ith discrete point can be expressed as

$$\Re\{f(x,y)\}_i = \sum_{j=1}^{n} c_{ij} f(x_j, y_j) \tag{1}$$

where $\Re$ denotes a linear differential operator, c_{ij} is the cubature weighting coefficients. According to Civan (1994) and Liew (1997, 1998), the weighting coefficients c_{ij} can be determined by the following expression

$$\Re\{x^{\alpha-\beta}y^{\beta}\}_i = \sum_{j=1}^{n} c_{ij}(x_j^{\alpha-\beta}y_j^{\beta}); \quad \beta = 0, 1, 2, \ldots, \alpha; \quad \alpha = 0, 1, 2, \ldots, k-1 \tag{2}$$

The n-monomials, $x^{\alpha-\beta}y^{\beta}$, are used to obtain a unique solution of Eq. (2). Once the grid points (x_i, y_i) are given, the cubature weighting coefficients can be obtained by solving an $n \times n$ orders linear algebraic equations.

3. Governing equations and boundary conditions

3.1 Basic governing equations

Consider a flat, isotropic, thick plate of uniform thickness h, length a and oblique width b. On the basis of Mindlin's concept, the buckling equation of the plate subjected to in-plane compressive forces N_x, N_y is given by

$$\frac{D}{2}\left[(1-v)\nabla^2\psi_x + (1+v)\frac{\partial\varphi}{\partial x}\right] - kGh\left(\psi_x + \frac{\partial w}{\partial x}\right) = 0 \tag{3a}$$

$$\frac{D}{2}\left[(1-v)\nabla^2\psi_y + (1+v)\frac{\partial\varphi}{\partial y}\right] - kGh\left(\psi_y + \frac{\partial w}{\partial y}\right) = 0 \tag{3b}$$

$$kGh(\nabla^2 w + \varphi) - \left(N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{3c}$$

where

$$\varphi = \frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y} \tag{4}$$

$$D = \frac{Eh^3}{12(1-v^2)} \tag{5}$$

and w is the transverse deflection; ψ_x and ψ_y are the rotations of the normal about the x-axis and y-axis respectively; D is the plate flexural rigidity; E, G and v are the Young's modulus, the shear modulus and the Poisson's ratio; k is the shear correction factor taken to be 5/6; ∇^2 is Laplace's two-dimensional operator;

In light of the relationship between force resultants and deformation variables, the bending moments, the twisting moments and the shear force can be expressed in terms of the plate deflection and the rotations as follows

$$M_x = D\left(\frac{\partial\psi_x}{\partial x} + v\frac{\partial\psi_y}{\partial y}\right); \quad M_y = D\left(v\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}\right); \quad M_{xy} = \frac{1-v}{2}D\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right) \tag{6}$$

$$Q_x = kGh\left(\psi_x + \frac{\partial w}{\partial x}\right); \quad Q_y = kGh\left(\psi_y + \frac{\partial w}{\partial y}\right) \tag{7}$$

To normalize the above equations, the following non-dimensional parameters are introduced

$$\xi = \frac{x}{a};\ \eta = \frac{y}{b};\ W = \frac{w}{h};\ \alpha = \frac{a}{b};\ \beta = \frac{h}{a};\ p = \frac{N_x}{kGh};\ \delta = \frac{h}{b};\ \theta = \frac{6k(1-v)}{\beta^2};\ t = \frac{N_y}{N_x} \tag{8}$$

Substituting Eq. (8) into Eq. (3), normalizing and rearranging them, we can obtain

$$\frac{\partial^2\psi_x}{\partial\xi^2} + \frac{1-v}{2}\alpha^2\frac{\partial^2\psi_x}{\partial\eta^2} - \theta\psi_x + \frac{1+v}{2}\alpha\frac{\partial^2\psi_y}{\partial\xi\partial\eta} - \theta\beta\frac{\partial W}{\partial\xi} = 0 \tag{9a}$$

$$\frac{1+v}{2}\alpha\frac{\partial^2\psi_x}{\partial\xi\partial\eta} + \alpha^2\frac{\partial^2\psi_y}{\partial\eta^2} + \frac{1-v}{2}\frac{\partial^2\psi_y}{\partial\xi^2} - \theta\psi_y - \theta\delta\frac{\partial W}{\partial\eta} = 0 \tag{9b}$$

$$\frac{\partial\psi_x}{\partial\xi} + \alpha\frac{\partial\psi_y}{\partial\eta} + \beta\frac{\partial^2 W}{\partial\xi^2} + \delta\alpha\frac{\partial^2 W}{\partial\eta^2} - p\left(\beta\frac{\partial^2 W}{\partial\xi^2} + t\delta\alpha\frac{\partial^2 W}{\partial\eta^2}\right) = 0 \tag{9c}$$

Substituting Eq. (8) into Eqs. (6), (7), the bending moments and the shear resultants are normalized as follows

$$\bar{M}_x = \frac{\partial \psi_x}{\partial \xi} + \nu\alpha\frac{\partial \psi_y}{\partial \eta}; \quad \bar{M}_y = \nu\frac{\partial \psi_x}{\partial \xi} + \alpha\frac{\partial \psi_y}{\partial \eta}; \quad \bar{M}_{xy} = \frac{1-\nu}{2}\left(\alpha\frac{\partial \psi_x}{\partial \eta} + \frac{\partial \psi_y}{\partial \xi}\right) \tag{10}$$

$$\bar{Q}_x = \psi_x + \beta\frac{\partial W}{\partial \xi}; \quad \bar{Q}_y = \psi_y + \delta\frac{\partial W}{\partial \eta} \tag{11}$$

where, $\bar{M}_x = \frac{M_x a}{D}$; $\bar{M}_y = \frac{M_y a}{D}$; $\bar{M}_{xy} = \frac{M_{xy} a}{D}$; $\bar{Q}_x = \frac{Q_x}{kGh}$; $\bar{Q}_y = \frac{Q_y}{kGh}$

3.2 Boundary conditions

The boundary conditions for an arbitrary edge of Mindlin plates are (Kitipornchai *et al.* 1993)
(1) Generalized soft simply supported edge (S')

$$\begin{aligned} W &= 0 \\ \bar{M}_n &= \bar{M}_x n_x^2 + \bar{M}_y n_y^2 + 2\bar{M}_{xy} n_x n_y = 0 \\ \bar{M}_{nt} &= (\bar{M}_y - \bar{M}_x) n_x n_y + \bar{M}_{xy}(n_x^2 - n_y^2) = 0 \end{aligned} \tag{12}$$

(2) Generalized hard simply supported edge (S)

$$\begin{aligned} W &= 0 \\ \bar{M}_n &= \bar{M}_x n_x^2 + \bar{M}_y n_y^2 + 2\bar{M}_{xy} n_x n_y = 0 \\ \psi_t &= -\psi_x n_y + \psi_y n_x = 0 \end{aligned} \tag{13}$$

(3) Clamped edge (C)

$$W = 0; \quad \psi_n = \psi_x n_x + \psi_y n_y = 0; \quad \psi_t = -\psi_x n_y + \psi_y n_x = 0 \tag{14}$$

(4) Generalized free edge (F)

$$\begin{aligned} \bar{Q}_n &= \bar{Q}_x n_x + Q_y n_y = 0 \\ \bar{M}_n &= \bar{M}_x n_x^2 + \bar{M}_y n_y^2 + 2\bar{M}_{xy} n_x n_y = 0 \\ \bar{M}_{nt} &= (\bar{M}_y - \bar{M}_x) n_x n_y + \bar{M}_{xy}(n_x^2 - n_y^2) = 0 \end{aligned} \tag{15}$$

where $\bar{Q}_n$, $\bar{M}_n$ and $\bar{M}_{nt}$ denotes the shear force, the bending and twisting moments on the edge of the plate respectively; ψ_n and ψ_t is the rotations of the midplane in the normal plane and in the tangent plane, to the plate boundary, respectively; and the subscripts n and t represents, respectively, the normal and the tangent directions of the edge. Substituting Eqs. (10), (11) into Eqs. (12)-(15), one can obtain the dimensionless boundary conditions in terms of the transverse displacement and rotations

(1) Generalized soft simply supported edge

$$W = 0$$

$$(n_x^2 + \nu n_y^2)\frac{\partial \psi_x}{\partial \xi} + \alpha(\nu n_x^2 + n_y^2)\frac{\partial \psi_y}{\partial \eta} + (1-\nu)\alpha n_x n_y \frac{\partial \psi_x}{\partial \eta} + (1-\nu) n_x n_y \frac{\partial \psi_y}{\partial \xi} = 0$$

$$(\nu-1) n_x n_y \frac{\partial \psi_x}{\partial \xi} + \alpha(1-\nu) n_x n_y \frac{\partial \psi_y}{\partial \eta} + \frac{1-\nu}{2}\alpha(n_x^2 - n_y^2)\frac{\partial \psi_x}{\partial \eta} + \frac{1-\nu}{2}(n_x^2 - n_y^2)\frac{\partial \psi_y}{\partial \xi} = 0 \tag{16}$$

(2) Generalized hard simply supported edge

$$W = 0$$

$$(n_x^2 + \nu n_y^2)\frac{\partial \psi_x}{\partial \xi} + \alpha(\nu n_x^2 + n_y^2)\frac{\partial \psi_y}{\partial \eta} + (1-\nu)\alpha n_x n_y \frac{\partial \psi_x}{\partial \eta} + (1-\nu) n_x n_y \frac{\partial \psi_y}{\partial \xi} = 0$$

$$-\psi_x n_y + \psi_y n_x = 0 \tag{17}$$

(3) Clamped edge

$$W = 0; \qquad \psi_x n_x + \psi_y n_y = 0; \qquad -\psi_x n_y + \psi_y n_x = 0 \tag{18}$$

(4) Generalized free edge

$$(n_x^2 + \nu n_y^2)\frac{\partial \psi_x}{\partial \xi} + \alpha(\nu n_x^2 + n_y^2)\frac{\partial \psi_y}{\partial \eta} + (1-\nu)\alpha n_x n_y \frac{\partial \psi_x}{\partial \eta} + (1-\nu) n_x n_y \frac{\partial \psi_y}{\partial \xi} = 0$$

$$(\nu-1) n_x n_y \frac{\partial \psi_x}{\partial \xi} + \alpha(1-\nu) n_x n_y \frac{\partial \psi_y}{\partial \eta} + \frac{1-\nu}{2}\alpha(n_x^2 - n_y^2)\frac{\partial \psi_x}{\partial \eta} + \frac{1-\nu}{2}(n_x^2 - n_y^2)\frac{\partial \psi_y}{\partial \xi} = 0$$

$$n_x \psi_x + n_y \psi_y + \beta n_x \frac{\partial W}{\partial \xi} + \delta n_y \frac{\partial W}{\partial \eta} = 0 \tag{19}$$

4. Discretization of governing equations and boundary conditions

First, we define the following linear operators, which will be required in the discretization of the governing equations and boundary conditions

$$\Re_1 = \frac{\partial^2}{\partial \xi^2} + \frac{1-\nu}{2}\alpha^2 \frac{\partial^2}{\partial \eta^2} - \theta; \qquad \Re_2 = \frac{\partial^2}{\partial \xi \partial \eta}; \qquad \Re_3 = \frac{\partial}{\partial \xi}$$

$$\Re_4 = \alpha^2 \frac{\partial^2}{\partial \eta^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial \xi^2} - \theta; \qquad \Re_5 = \frac{\partial}{\partial \eta}; \qquad \Re_6 = \beta \frac{\partial^2}{\partial \xi^2} + \delta \alpha \frac{\partial^2}{\partial \eta^2};$$

$$\Re_7 = \beta \frac{\partial^2}{\partial \xi^2} + t\delta\alpha \frac{\partial^2}{\partial \eta^2}; \tag{20}$$

Using the differential cubature procedures, the normalized governing Eqs. (9a-c) can be discretized as follows

$$\sum_{j=1}^{n} C_{ij}^{(1)}\psi_{xj} + \frac{1+\nu}{2}a\sum_{j=1}^{n} C_{ij}^{(2)}\psi_{yj} - \theta\beta\sum_{j=1}^{n} C_{ij}^{(3)}W_j = 0 \tag{21a}$$

$$\frac{1+\nu}{2}a\sum_{j=1}^{n} C_{ij}^{(2)}\psi_{xj} + \sum_{j=1}^{n} C_{ij}^{(4)}\psi_{yj} - \theta\delta\sum_{j=1}^{n} C_{ij}^{(5)}W_j = 0 \tag{21b}$$

$$\sum_{j=1}^{n} C_{ij}^{(3)}\psi_{xj} + a\sum_{j=1}^{n} C_{ij}^{(5)}\psi_{yj} + \sum_{j=1}^{n} C_{ij}^{(6)}W_j + p\sum_{j=1}^{n} C_{ij}^{(7)}W_j = 0 \tag{21c}$$

where n is the total number of the discrete points, and i is the index number, $i = 1, 2, ..., n$; $C_{ij}^{(m)}$ is the cubature weighting coefficients corresponding to the linear operator $\Re_m$, $m = 1, 2, \ldots, 7$.

Similarly the normalized boundary conditions for an arbitrary edge are discretized as

(1) Generalized soft simply supported edge

$$W_i = 0$$

$$(n_x^2 + \nu n_y^2)\sum_{j=1}^{n} C_{ij}^{(3)}\psi_{xj} + \alpha(\nu n_x^2 + n_y^2)\sum_{j=1}^{n} C_{ij}^{(5)}\psi_{yj}$$

$$+ (1-\nu)\alpha n_x n_y\sum_{j=1}^{n} C_{ij}^{(5)}\psi_{xj} + (1-\nu)n_x n_y\sum_{j=1}^{n} C_{ij}^{(3)}\psi_{yj} = 0$$

$$(\nu-1)n_x n_y\sum_{j=1}^{n} C_{ij}^{(3)}\psi_{xj} + \alpha(1-\nu)n_x n_y\sum_{j=1}^{n} C_{ij}^{(5)}\psi_{yj} +$$

$$\frac{1-\nu}{2}\alpha(n_x^2 - n_y^2)\sum_{j=1}^{n} C_{ij}^{(5)}\psi_{xj} + \frac{1-\nu}{2}(n_x^2 - n_y^2)\sum_{j=1}^{n} C_{ij}^{(3)}\psi_{yj} = 0 \tag{22}$$

(2) Generalized hard simply supported edge

$$W_i = 0$$

$$(n_x^2 + \nu n_y^2)\sum_{j=1}^{n} C_{ij}^{(3)}\psi_{xj} + \alpha(\nu n_x^2 + n_y^2)\sum_{j=1}^{n} C_{ij}^{(5)}\psi_{yj}$$

$$+ (1-\nu)\alpha n_x n_y\sum_{j=1}^{n} C_{ij}^{(5)}\psi_{xj} + (1-\nu)n_x n_y\sum_{j=1}^{n} C_{ij}^{(3)}\psi_{yj}\frac{\partial\psi_y}{\partial\xi} = 0$$

$$-n_y\psi_{xi} + n_x\psi_{yi} = 0 \tag{23}$$

(3) Clamped edge

$$W_i = 0; \qquad n_x\psi_{ix} + n_y\psi_{yi} = 0; \qquad -n_y\psi_{xi} + n_x\psi_{yi} = 0 \tag{24}$$

(4) Generalized free edge

$$(n_x^2 + \nu n_y^2)\sum_{j=1}^{n} C_{ij}^{(3)}\psi_{xj} + \alpha(\nu n_x^2 + n_y^2)\sum_{j=1}^{n} C_{ij}^{(5)}\psi_{yj}$$

$$+ (1-\nu)\alpha n_x n_y \sum_{j=1}^{n} C_{ij}^{(5)}\psi_{xj} + (1-\nu) n_x n_y \sum_{j=1}^{n} C_{ij}^{(3)}\psi_{yj} = 0$$

$$(\nu-1) n_x n_y \sum_{j=1}^{n} C_{ij}^{(3)}\psi_{xj} + \alpha(1-\nu) n_x n_y \sum_{j=1}^{n} C_{ij}^{(5)}\psi_{yj}$$

$$+ \frac{1-\nu}{2}\alpha(n_x^2 - n_y^2)\sum_{j=1}^{n} C_{ij}^{(5)}\psi_{xj} + \frac{1-\nu}{2}(n_x^2 - n_y^2)\sum_{j=1}^{n} C_{ij}^{(3)}\psi_{yj} = 0$$

$$n_x\psi_{xi} + n_y\psi_{yi} + \beta n_x \sum_{j=1}^{n} C_{ij}^{(3)} W_j + \delta n_y \sum_{j=1}^{n} C_{ij}^{(5)} W_j = 0 \tag{25}$$

where i is the index number of the points on boundary.

5. Eigenvalue problem for buckling factors

Combining the discretized governing Eq. (21) at each discrete point in the physical domain and the boundary condition Eqs. (22)-(25) at each boundary point, and rewriting them in terms of matrix, we obtain

$$([K] - p[M])\{q\} = 0 \tag{26}$$

where $[K]$ and $[M]$ are the bending stiffness matrix and the geometric stiffness matrix respectively,

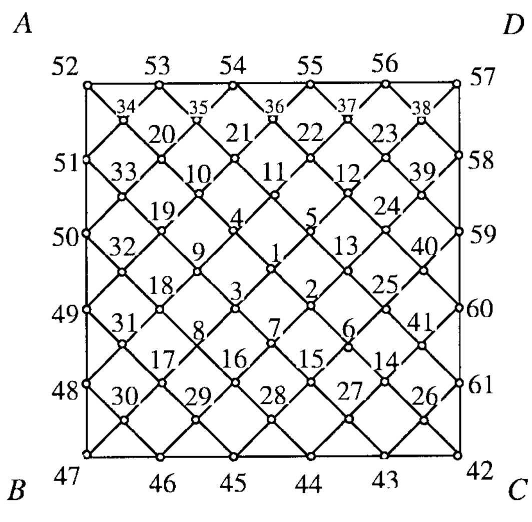

Fig. 1 Grid point pattern for square plate problem

the elements of which are determined by Eqs. (21a-c) or Eqs. (22)-(25); $\{q\}$ is the displacement vector, which is expressed as

$$\{q\} = \{W_1, \psi_{x1}, \psi_{y1}, W_2, \psi_{x2}, \psi_{y2}, \ldots, W_n, \psi_{xn}, \psi_{yn}\}^T \tag{27}$$

Eq. (26) is a standard eigenproblem, of which the eigenvalues and eigenvectors can be calculated by using an ordinary eigenvalue equation system solver.

6. Numerical results and discussion

To demonstrate the applicability of the DC method for buckling analysis of moderately thick plates, numerical calculations have been performed for plates with different mixed boundary

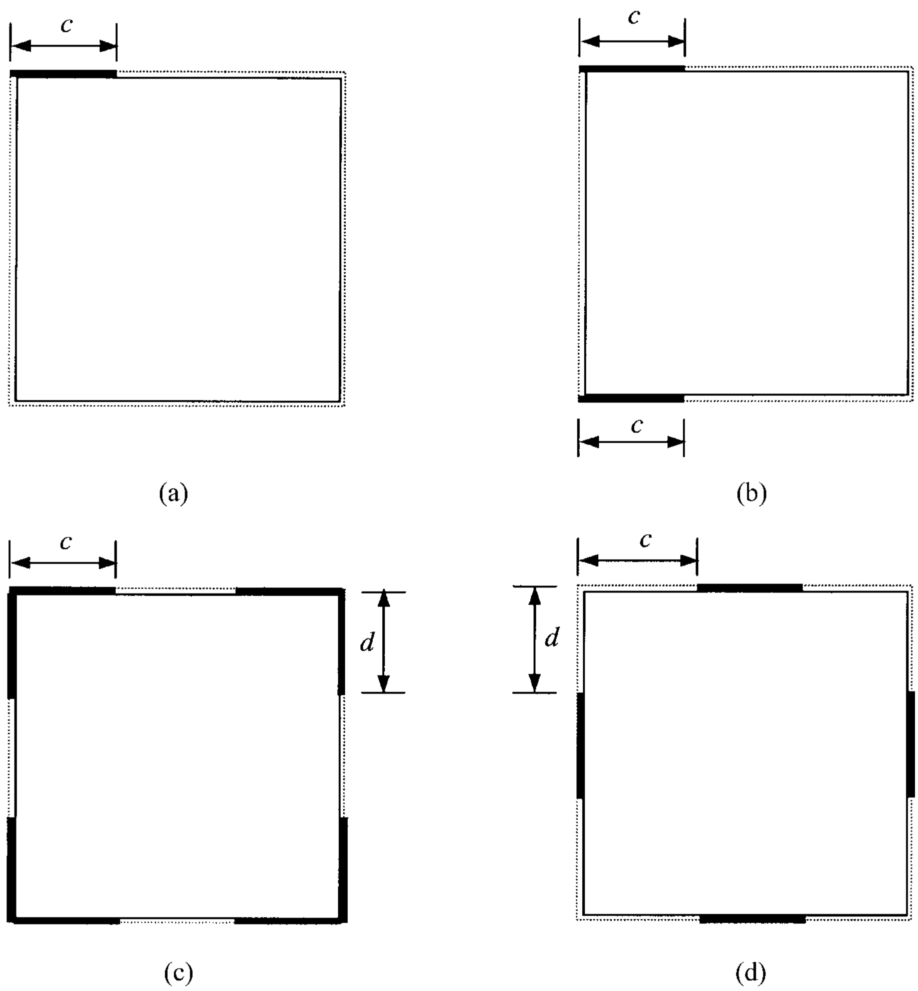

Fig. 2 Configurations of thick plates with various mixed boundary conditions

conditions, different thickness to span ratios, and different aspect ratios. The numerical procedure based on the differential cubature method proposed in this paper has been implemented in FORTRAN computer code and applied to rectangular plates and skew plates with different boundary conditions. Several examples are selected in this section to show the performance of the differential cubature method. For all cases considered herein, the Poisson's ratio ν is taken to be 0.3. For convenience of presentation and comparison of the numerical results, a dimensionless buckling parameter $\bar{p}$ has been defined as

$$\bar{p} = N_x b^2 / (\pi^2 D) \tag{28}$$

where N_x, b, D are of the same meanings as in Eq. (3).

6.1 Rectangular plates

First, convergence studies are carried out to establish the minimum grid points required for obtaining accurate solutions for a square plate with various boundary conditions under uniaxial pressing load. The grid pattern for this problem is shown in Fig. 1, and the number of grid points is changing from 13 to 181. In order to denote the support edge conditions of the plate, a four-letter symbol consisting of a combination of letters C, S and F has been used. The first, second, third and fourth letters represent respectively, the support conditions along the edges AB, BC, CD and DA(see Fig. 1). The symbol, CSCF, for example, represents clamped, simply supported, clamped and free support conditions along the edges AB, BC, CD, DA, respectively.

The convergence characteristic of buckling parameters $\bar{p}$ for this problem is studied by gradually increasing the number of grid points for the selected plate with different boundary conditions and different thickness to span ratios. The numerical results are presented in Table 1. From Table 1, it is found that the buckling parameters converge to stable values for thin and thick plates as the number of grid points increases, and the convergence of the present solutions with grid refinement demonstrates fluctuant characteristic for all boundary conditions considered in this paper. It is evident that at least 41 grid points are needed to acquire solutions with acceptable precision. When the mesh size becomes 61, a converged solution for the buckling parameter to at least three significant figures could always be achieved except for thin plate. The relative error between the numerical results obtained using 61 grid points and the converged results is within 6.88%, regardless of the plate thickness. For thin plates with SSSS boundary conditions (h/a = 0.01), 61 grid points provide acceptable results with a maximum discrepancy of 6.88%. For moderately thick plates with arbitrary boundary conditions, however, the discrepancies are all within 0.1%. In order to ensure high accuracy of the present solutions, 61 grid points are therefore employed to general all the numerical results in the following studies. Furthermore, it is observed from the comparison studies that a better convergece characteristic of the DC method is achieved for the CCCC boundary conditions than the SSSS boundary conditions, especially for thin plates. Table 1 also shows that the thicker a plate, the faster the convergence rate will be.

To validate the numerical accuracy and efficiency of the solution method, the present results are also compared with other existing exact and numerical solutions. It is evident in Table 1 that all the present results agree very well with the other solutions for plates with both SSSS and CCCC boundary conditions. One can easily find that the present results seem to be somewhat higher than the 3-d exact solutions. Compared to the 3-d exact solutions, the present results using 61 grid points

Table 1 Convergence study of buckling parameters $\overline{p}$ for a square plate under uniaxial compressing load

Boundary conditions	n	h/a				
		0.01	0.05	0.1	0.2	0.3
SSSS	13	1520.8	61.758	16.132	4.6216	2.3742
	25	3.9263	3.6997	3.7712	3.6778	2.3270
	41	3.9736	3.9535	3.7826	3.2578	2.6520
	61	4.2766	3.9549	3.7860	3.2642	2.6529
	85	3.9958	3.9511	3.7858	3.2640	2.6523
	113	4.0154	3.9514	3.7859	3.2638	2.6523
	145	4.0011	3.9512	3.7859	3.2638	2.6523
	181	4.0012	3.9512	3.7859	3.2637	2.6523
	DQE solution[a]	3.99775	3.94439	3.78645	3.26373	---
	Finite strip solution[b]	---	3.944	3.786	3.264	---
	Rayleigh-Ritz method[c]	---	3.929	3.731	---	---
	Exact solution[d]	---	3.911	3.741	3.15	---
	Thin plate solution[e]	4.000	---	---	---	---
	p-Ritz solution[f]	---	3.944	3.786	3.264	---
CCCC	13	713.42	32.377	10.807	4.6067	2.7713
	25	8.6753	8.4240	7.7093	5.0474	2.9732
	41	10.296	9.5527	8.2532	5.3400	3.2522
	61	10.490	9.5592	8.2900	5.3146	3.1967
	85	10.043	9.5517	8.2898	5.3147	3.1973
	113	10.078	9.5516	8.2899	5.3147	3.1973
	145	10.072	9.5516	8.2899	5.3147	3.1972
	181	10.074	9.5517	8.2899	5.3147	3.1973
	DQE solution[a]	10.052	9.5586	8.2916	5.3156	----
	p-Ritz solution[f]	----	9.5588	8.2917	5.3156	----

a; Liu (2001), *b*; Hinton (1978) *c*; Dawe and Roufaeil (1982), *d*; Srinivas and Rao (1969), *e*; Alexander (1982), *f*; Xiang (1993)

have a maximum discrepancy of 1.1%. Nevertheless, the present results are in close agreement with other numerical solutions, especially with the differential quadrature element solutions obtained by Liu. The present values of the buckling parameters are almost identical to those of DQE solutions. The main reason for this is that the present results and the DQE results given by Liu are all generated by using polynomials as trial functions and based on the same plate theory, i.e., the first order shear deformation theory. Indeed, the differential cubature method is an extension of differential quadrature method. However, it should be noted that Liu's DQE results were generated by using 13×13 grid points, while the present results were generated only by using 61 grid points. This means that the total degrees of freedom using by DQE method would be as 2.77 times large as those using by the differential cubature method. So, we can conclude that the present differential cubature method is much more efficient than the differential quadrature element method.

Based on the previous convergence and comparison studies, the present numerical method has now been employed to determine the critical load of moderately thick plates. In this paper, the

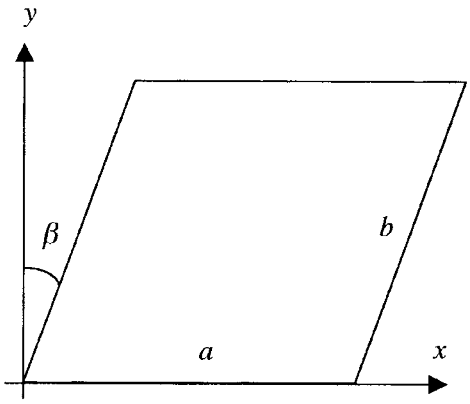

Fig. 3 Configuration of a skew plate

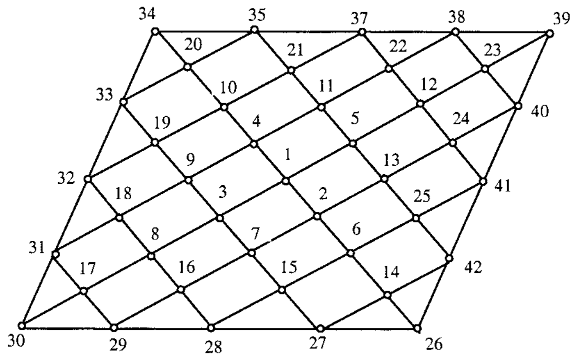

Fig. 4 Grid point pattern for skew plate problem

authors calculated the buckling parameters for a square plate having various mixed boundary constraints subjected to uniaxial and biaxial compress loads (Fig. 2(a-d)). The numerical results are tabulated in Tables 2 to 5 respectively. It is observed from these tables that the buckling parameters increase as the clamped portions ratio increases for all the relative thickness varying from 0.1 to 0.3. Also, the authors studied the influence of the aspect ratio to buckling factors of a rectangular plate with different relative thickness and mixed boundary conditions (Fig. 2(c, d)) under uniaxial loading condition, and the numerical results are presented in Tables 6-7. From these two tables, it is observed that regardless of the relative thickness, the buckling parameter decreases as the aspect ratio a/b increases. However, the variety pattern does not demonstrate the monotonic properties. It should be noted that when the aspect ratio a/b varies, the clamped ratio or the simply supported ratio $c/a = d/b$ is taken to be a constant 0.2; i.e., the buckling parameter is not only the function of

Table 2 Buckling parameters $\overline{p}$ for a simply supported square plate partially clamped along one edge under uniaxial compressing load (Fig. 3a)

h/a	c/a					
	0.0	0.2	0.4	0.6	0.8	1.0
0.1	3.786045	4.083044	4.450221	5.121418	5.216457	5.215809
0.2	3.264236	3.410466	3.675255	4.024441	4.152924	4.151449
0.3	2.652978	2.711804	2.748032	2.799802	2.817187	2.815337

Table 3 Buckling parameters $\overline{p}$ for a simply supported square plate partially clamped along two opposite edges subjected to uniaxial compressing load (Fig. 3b)

h/a	c/a					
	0.0	0.2	0.4	0.6	0.8	1.0
0.1	3.785150	4.297997	5.043557	5.830410	6.376959	6.400690
0.2	3.263690	3.515746	3.880054	4.203066	4.342392	4.342935
0.3	2.653240	2.724059	2.757091	2.850446	2.883739	2.888282

Table 4 Buckling parameters $\overline{p}$ for a simply supported square plate partially clamped along four edges symmetrically from the corners (Fig. 3c)

Loading conditions	c/a	h/a		
		0.1	0.2	0.3
Uniaxial(N_x)	0.2	6.079890	4.480995	3.069418
	0.4	8.317241	5.355851	3.240878
Biaxial($N_x = N_y$)	0.2	3.335272	2.415733	1.746879
	0.4	4.684618	3.287795	2.229513

Table 5 Buckling parameters $\overline{p}$ for a simply supported square plate but partially clamped along central portions of four edges subjected to uniaxial compressing load (Fig. 3d)

Loading conditions	c/a	h/a		
		0.1	0.2	0.3
Uniaxial(N_x)	0.2	8.229237	5.255400	3.011846
	0.4	6.798482	4.621438	3.011416
Biaxial($N_x = N_y$)	0.2	4.485126	3.207488	2.186402
	0.4	3.767321	2.699908	1.891833

Table 6 Buckling parameters $\overline{p}$ for a simply supported rectangular plate but partially clamped along two opposite edges subjected to uniaxial compressing load (Fig. 3c)

a/b	h/b			a/b	h/b		
	0.1	0.2	0.3		0.1	0.2	0.3
0.5	12.47308	6.223735	3.404814	1.0	5.043557	3.880054	2.757091
0.6	8.326212	5.056317	3.083836	1.1	4.630253	3.627263	2.660839
0.7	6.308306	4.253829	2.836055	1.2	4.261912	3.461529	2.625069
0.8	5.448079	3.880453	2.740824	1.3	4.046765	3.386294	2.642043
0.9	5.168001	3.878681	2.814005	1.4	4.052723	3.420290	2.335669

Table 7 Buckling parameters $\overline{p}$ for a simply supported rectangular plate but partially clamped along central portions of four edges subjected to uniaxial compressing load (Fig. 3d)

a/b	h/b			a/b	h/b		
	0.1	0.2	0.3		0.1	0.2	0.3
0.5	12.77294	6.560153	3.560303	1.0	8.229237	5.255400	3.011846
0.6	10.81427	6.128338	3.472331	1.1	7.810606	4.958963	2.842505
0.7	9.574515	5.810514	3.389223	1.2	7.292115	4.754931	2.673244
0.8	8.840962	5.593931	3.267632	1.3	6.983388	4.493064	2.523094
0.9	8.436534	5.420046	3.148860	1.4	6.716741	4.409866	2.348395

aspect ratio, but also the function of clamped side length. These numerical values can be served as a benchmark for future studies.

Table 8 Comparison studies of buckling parameters $\overline{p}$ for a skew thin plate with *a/b* = 1 under uniaxial compressing load (*h/a* = 0.001)

Researchers	SSSS				CCCC			
	$\beta = 0^o$	$\beta = 15^o$	$\beta = 30^o$	$\beta = 45^o$	$\beta = 0^o$	$\beta = 15^o$	$\beta = 30^o$	$\beta = 45^o$
Mizusawa (1980)	4.000	4.3514	5.6177	8.6410	----	----	----	----
Wang (1992)	4.000	4.4341	6.1867	10.600	10.080	10.889	13.749	20.680
Kitipornchai (1993)	4.000	4.3939	5.9217	10.104	10.074	10.834	13.538	20.112
Saadatpour (1998)	4.000	4.3928	5.8684	9.7560	----	----	----	----
Navin (1995)	3.999	4.4020	5.8970	10.103	10.073	10.835	13.548	20.616
Present method	4.2765	4.4338	6.0110	10.746	10.490	10.778	13.744	20.485

Table 9 Convergence of buckling parameters $\overline{p}$ for a skew plate with *a/b* = 1 under uniaxial compressing load

Boundary conditions	n	*h/a* = 0.001				*h/a* = 0.2			
		$\beta = 0^o$	$\beta = 15^o$	$\beta = 30^o$	$\beta = 45^o$	$\beta = 0^o$	$\beta = 15^o$	$\beta = 30^o$	$\beta = 45^o$
SSSS	13	1520.8	2237.4	3812.3	3773.6	4.6216	4.7432	8.3201	12.235
	25	3.9263	4.6786	6.7854	12.343	3.6778	4.0456	4.5880	7.8345
	41	3.9735	4.4012	8.3233	14.779	3.2577	3.5157	4.4483	6.1347
	61	4.2765	4.4338	6.0110	10.746	3.2642	3.5325	4.4567	6.1587
	85	3.9958	4.3955	5.8874	10.354	3.2640	3.5330	4.4515	6.1573
	113	4.0154	4.3976	5.9118	10.421	3.2638	3.5329	4.4510	6.1564
	145	4.0011	4.3938	5.8989	10.807	3.2638	3.5328	4.4509	6.1574
	181	4.0012	4.3939	5.8970	10.503	3.2637	3.5329	4.4509	6.1542
CCCC	13	713.42	874.82	1078.4	1843.6	4.6067	4.7784	5.0483	5.3881
	25	8.6753	8.8932	11.478	16.669	5.0474	5.3140	5.9192	6.5843
	41	10.296	10.997	14.630	21.038	5.3400	5.5539	6.1478	6.9937
	61	10.490	10.778	13.744	20.485	5.3146	5.4747	6.0574	6.9719
	85	10.043	10.865	13.661	20.632	5.3147	5.4932	6.0637	6.9730
	113	10.078	10.854	13.544	20.232	5.3147	5.4932	6.0553	6.9715
	145	10.072	10.851	13.545	20.535	5.3147	5.4932	6.0552	6.9725
	181	10.074	10.852	13.545	20.135	5.3147	5.4932	6.0553	6.9716
CFCF	13	1883.5	2272.8	3684.5	3378.1	4.6782	4.9065	5.5485	6.2734
	25	4.7328	4.8994	6.8341	10.345	2.7432	3.4044	3.8843	4.5503
	41	3.8906	4.3445	5.4515	8.4995	2.6258	2.7883	3.4329	4.0438
	61	4.1472	4.5457	5.8334	8.6432	2.6976	2.8516	3.4402	4.2312
	85	4.0171	4.3061	5.7421	8.2114	2.7061	2.8743	3.4578	4.2585
	113	3.8992	4.2809	5.6485	8.4062	2.6965	2.8621	3.4447	4.1773
	145	3.9189	4.2822	5.6072	8.5963	2.6962	2.8622	3.4203	4.2459
	181	3.9193	4.2824	5.6159	8.0948	2.6961	2.8622	3.4071	4.0458

6.2 Skew plates

In this subsection, a thick skew plate is considered to demonstrate the versatility and the simplicity of the DC method to plate problems with a complex domain. The geometry of the plate and the grid point pattern employed for this problem is shown in Fig. 3 and Fig. 4, respectively. First we carried out a comparison study of the present results for the buckling factors with other exact or numerical solutions for a clamped and a simply supported thin plate with different sharp angle. The relative thickness of the plate is taken to be 0.001. The numerical results are tabulated in Table 8. Again, the present results are in good agreement with all other exact solutions and numerical solutions, especially with Navin's Rayleigh-Ritz solutions and Kitipornchai's pb-2 Ritz solutions. The maximum relative error between the present results and Navin's results or Kitipornchai's is 6.93%. It is interesting to note that the results from Navin's analyses were carried out using a 66-term series(330 degrees of freedom); the Kitipornchai's results were generated using 120-term series(360 degrees of freedom); while the present results are calculated by using only 61 grid points(183 degrees of freedom). That is to say that the total number of degrees of freedom using in Navin's method or Kitipornchai's method is as almost 2 times large as that using in the present method. It should also be noted that the present method need not geometric map, while the Ritz energy method used by Navin and Kitipornchai need to transform the physical domain to a computational domain, and that is not very convenient for some operators especially for higher order operators.

The convergence characteristics of the buckling parameters are presented in Table 9 for plates having various sharp angles with various relative thickness and different boundary conditions. Similar properties to rectangular plates have been found. The buckling parameters converge to stable values very quickly for most cases. However, when the skew angle is 45^{o}, the convergence of the present DC results is not quite satisfactory because of the singularities of stresses at the obtuse corners. It is also observed that the convergence rate for plate with SSSS or CCCC boundary conditions is faster than that for plate with CFCF boundary conditions. Generally speaking, the convergence rate will be slow when the plate having free edges.

7. Conclusions

In this paper, the differential cubature method has been applied to solve the buckling problems of moderately thick plates with arbitrary shape and discontinuous mixed edge conditions. This is the first endeavor to exploit the DC method for buckling analysis of thick plates. Several examples have been selected to demonstrate the convergency, accuracy and applicability of the DC procedures. The numerical results have been compared with the opening literature. It has been shown that the DC method yields quickly convergent and accurate solutions for thin and thick plate buckling problems with a rectangular domain, and the DC results are in excellent agreement with the exact analytical solutions and other numerical solutions. For comparable accuracy, the differential cubature method required 2.77-times less degrees of freedom than the differential quadrature element method and 2-times less degrees of freedom than Rayleigh-Ritz method. However, for skew plate problems with a small sharp angle, the convergence rate is not quite satisfactory because of the stresses singularities at obtuse corners. Therefore, we can conclude that this method can be used as an alternative to other existing numerical methods for the solutions of thick rectangular plate buckling problems. For skew plate, however, much more grid points should be employed to acquire more accurate solutions.

References

Alexander Chajes, (1974), *Principles of Structural Stability Theory*, Prentice Hall, Inc.
Bellman, R.E. and Casti, J. (1971), "Differential quadrature and long term integration", *Journal of Mathematical*

Analysis and Applications, **34**, 235-238.
Bellman, R.E. and Casti, J. (1972), "Differential quadrature: A technique for the rapid solution of nolinear partial differential equations", *Journal of Computational Physics*, **10**, 40-52.
Bert, C.W. and Malik, M. (1996), "Differential quadrature method in computational mechanics: A review", *Applied Mechanics Review*, **48**, 1-28.
Brunelle, E.J. (1971), "Buckling of transversely isotropic Mindlin plates", *AIAA J*., **9**, 1018-1022.
Brunelle, E.J. and Robertson, S.R. (1974), "Initially stressed Mindlin plates", *AIAA J*., **12**, 1036-1045.
Civan, F. (1994), "Solving multivariable mathematical models by the quadrature and cubature methods", *Numerical Method for Partial Differential Equations*, **10**, 545-567.
Dawe, D.J. and Roufaeil, O.L. (1982), "Buckling of rectangular Mindlin plates", *Comput. Struct*., **15**, 461-471.
Edwardes, R.J. and Kabaila, A.P. (1978), "Buckling of simply supported skew plates", *Int. J. Numer. Methods Engrg*., **12**, 779-785.
Hamada, M., Inoue, Y. and Hashimoto, H. (1967), "Buckling of simply supported but partially clamped rectangular plates uniformly compressed in one direction", *Bull*. JSME, **10**, 35-40.
Hegedus, T. (1988), "Finite strip buckling analysis of skew plates under combined loading", in: Evayi M, *Stability of Steel Structures*, Akademi Kiado, Budapest, 633-641.
Hinton, E. (1978), "Buckling of initially stressed Mindlin plates using a finite strip method", *Comput. Struct*., **8**, 99-105.
Keer, L.M. and Stahl, B. (1972), "Eigenvalue problems of rectangular plates with mixed edge conditions", *Trans.* ASME *J. Appl. Mech*., **39**, 513-520.
Kennedy, J.B. and Prabhakara, M.K. (1979), "Combined-load local buckling of orthotropic skew plates", *J. Engrg. Mech. Div*., ASCE, **105**, 71-79.
Kitipornchai, S., Xiang, L., Wang, C.M. and Liew, K.M. (1993), "Buckling of thick skew plates", *Int. J. Numer. Methods Engrg*, **36**, 1299-1310.
Liew, K.M., Han, J.B. and Xiao, Z.M. (1996), "Differential quadrature method for Mindlin plates on Winkler foundations", *Int. J. Mech. Sci*., **38**, 405-421.
Liew, K.M. and Liu, F.L. (1997), "Differential cubature method: A solution technique for Kirchhoff plates of arbitrary shape", *Comput. Meth. Appl. Mech. Eng.*, **13**, 73-81.
Liu, F.L. and Liew, K.M. (1998), "Differential cubature method for static solutions of arbitrary shaped thick plates", *Int. J. Solids Struct.*, **35**(28-29), 3655-3674.
Liu, F.L. (2001), "Differential quadrature element method for buckling analysis of rectangular Mindlin plates having discontinuities", *Int. J. Solids Struct*., **38**, 2305-2321.
Luo, J.W. (1982), "A hybrid/mixed model finite element analysis for buckling of moderately thick plates", *Comput. Struct*., **15**, 359-364.
Malik, M. and Bert, C.W. (1998), "Three dimensional elasticity solutions for free vibrations of rectangular plates by the differential quadrature method", *Int. J. Solids Struct*., **35**, 299-381.
Mizusawa, T., Kajita, T. and Naruko, M. (1980), "Analysis of skew plate problems with various constraints", *J. Sound Vib*., **73**(4), 575-584.
Navin, J., Knight, N.F. and Ambur, D.R. (1995), "Buckling of arbitrary quadrilateral anisotropic plates", *AIAA J*., **33**, 938-944.
Rao, V.G., Venkataramana, J. and Raju, K.K. (1975), "Stability of moderately thick rectangular plates using a high precision triangular finite element", *Comput. Struct*., **5**, 257-259.
Saadatpour, M.M., Azhari, M. and Bradford, M.A. (1998), "Buckling of arbitrary quadrilateral plates with intermediate supports using the Galerkin method", *Comput. Methods Appl. Mech. Engrg*., **164**, 297-306.
Sakiyama, T. and Matsuda, H. (1987), "Elastic buckling of rectangular Mindlin plates with mixed boundary conditions", *Comput. Struct.*, **25**, 801-808.
Srinivas, S. and Rao, A.K. (1969), "Buckling of rectangular thick plates", *AIAA J*., **7**, 1645-1746.
Wang, C.M., Liew, K.M. and Alwis, W.A. (1992), "Buckling of skew plates and corner condition for simply supported edges", *J. Engrg. Mech*., ASCE, **118**, 651-662.
Wittrick, W.H. (1956), "On the buckling of oblique plates in shear", *Aircraft Engrg*., **28**, 25-27.
Xiang, Y. (1993), "Numerical developments in solving the buckling and vibration of Mindlin plates", Ph.D thesis, The University of Queensland, Australia.