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# More reliable responses for time integration analyses

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**Abstract.** One of the most versatile approaches for analyzing the dynamic behavior of structural systems is direct time integration of semi-discrete equations of motion. However responses computed by time integration are generally inexact and hence the corresponding errors would rather be studied in advance. In spite of the various error estimation formulations that exist in the literature, it is accepted practice to repeat the analyses with smaller time steps, followed by a comparison between the results. In this paper, after a review of this simple method and disregarding the round-off errors, a more efficient, reliable and yet simple method for estimating errors and enhancing the accuracy is proposed. The main objectives of this research are more realistic error estimation based on the concept of convergence, approximately controlling the reliability by comparing the actual rate of convergence with the integration method's order of accuracy, and enhancement of reliability by applying Richardson's extrapolation. Starting from the errors at specific time instants, the study is then generalized to cases in which the errors should be estimated and decreased at specific events e.g. peak responses. Numerical study illustrates the efficacy of the proposed method.

**Key words:** direct time integration; error estimation; reliable responses; rate of convergence; Richardson's extrapolation; round-off error; computational cost.

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## 1. Introduction

From the point of view of spatial discretization, the dynamic behavior of structural systems can be studied by analyzing the following mathematical model:

$$[M] \cdot \{\ddot{u}(t)\} + \{f_{\text{int}}\} = \{f_{ext}(t)\} \tag{1}$$

subjected to

$$\begin{aligned} \{u(t=0)\} &= \{u_0\} \\ \{\dot{u}(t=0)\} &= \{\dot{u}_0\} \end{aligned} \tag{2}$$

and for some special cases e.g. elastic-plastic dynamic systems, also subjected to

$$\{f_{\text{int}}(t=0)\} = \{f_0\} \tag{3}$$

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In Eqs. (1)-(3) $\{u\}, \{\dot{u}\}, \{\ddot{u}\}$ respectively denote the systems' displacement, velocity and acceleration, $[M]$ stands for the systems' mass, $\{f_{\text{int}}\}$ represents the systems' internal force, $\{f_{ext}\}$ is the external force, and finally $\{u_0\}, \{\dot{u}_0\}, \{f_0\}$ respectively denote the initial values of the systems' displacement, velocity and internal force. In the most general cases, in order to solve the initial value problems expressed by Eqs. (1)-(3), it is necessary to implement step-by-step methods (Chopra 1995, Mahin and Lin 1983). The Newmark time integration method (Newmark 1959) is one of the pioneering methods, which is still widely accepted in practice.

Since Newmark's suggestion in 1958, the versatility of direct time integration methods in handling linear and nonlinear dynamic problems with close natural frequencies, non-classical damping and sophisticated excitations, has led to their general acceptance in the engineering practice. Unfortunately, in order to have efficient analyses most of the time integration methods are based on simplifying assumptions (Chopra 1995, Gupta 1992), and hence generate approximate responses. Consequently, three problems namely the problems of numerical instability, insufficient accuracy and high computational cost might arise. In practical analysis, computational cost is the most important (Schueller and Pradlwarter 1999). Nevertheless, as a meaningful quantity, computational cost should be evaluated with respect to some level of accuracy. The purpose of this paper is to suggest a reliable and computationally efficient method to obtain responses with required accuracy. In other words, reliable accuracy with low computational cost for approximate time integration analyses is the topic of investigation here.

Despite the recent developments of time integration methods (Fung 1997, Golley 1998, Kim *et al*. 1999, …), and various adaptive time integration techniques (Zienkiewicz and Xie 1991, Zeng *et al*. 1992, Ruge 1999, …), a reliable method for estimating the responses' errors has not been presented yet (Choi and Chung 1996, Zienkiewicz *et al*. 1999). In fact, the new methods only provide means for more computationally efficient step-by-step analysis. Moreover the main objective of the adaptive time integration methods, is step-by-step analysis with less time steps without losing accuracy (Zienkiewicz and Xie 1991), and the errors are being computed locally and with just enough precision to attain this goal. Furthermore the results of adaptive time integration methods are generally highly sensitive to certain additional parameters that should be determined in advance (Zeng *et al*. 1992). Hence estimating responses' errors and their efficient reduction to the acceptable levels is a yet unresolved problem. This paper studies estimating and controlling errors of time integration in order to make an improvement in this field. Special attention is given to the conventional simple and practically accepted method (Clough and Penzien 1993). According to the conventional method, after a complete time integration analysis, the analysis is repeated with half time steps, and the error of the former is estimated by comparing it with the latter. If the difference of the two responses is small enough, the last response is assumed to be more accurate and is accepted as the exact response. In this method, not only the assumption of more accuracy for the second response is apparently vague, but also the comparison method of the two responses is not well defined. It is customary, not to check the validity of any assumption, and merely compare the responses according to the goals of the analyses e.g. maximum response. However to achieve higher reliability for analyses' responses and consequently for the corresponding structural designs, the following two questions should rather be studied further:

1. Having generated approximate responses for a special dynamic system by several step-by-step analyses with a common time integration method and different time step sizes, how should the errors be estimated reliably?
2. If the estimated errors were not acceptable, how should they be reduced efficiently?

This paper is an attempt at answering these two basic questions. In order to have a more comprehensive study the conventional method (Clough and Penzien 1993) is considered here in a more generalized form, in which the time step sizes may be scaled down by any positive integer.

Having this in mind and with attention to the definition of order of accuracy, i.e., rate of convergence (Belytschko and Hughes 1983), formulations for error estimation in linear systems are presented in section 2. In section 3 different complementary remarks are made about the efficacy and generalization of the derived formulation. An algorithm for reliably obtaining responses with desired accuracy is then presented in section 4. The discussion is once again generalized to nonlinear regimes in section 5, followed by a study of the computational cost in section 6. In section 7 the efficacy of the suggested algorithm is studied in view of some numerical examples, and finally the conclusions are set out in section 8.

## 2. Theory

### *2.1 Error estimation*

When the effect of round-off error is negligible, according to the convergence concept, the deviation of the responses generated by direct time integration from their exact values i.e., errors, can be expressed as

$$E = \sum_{j=0}^{\infty} C_j \cdot (\Delta t)^{q+j} \tag{4}$$

In Eq. (1) $\Delta t$ is the constant size of the time steps all along the time interval, $q$ is a positive integer called convergence rate (Wood 1990, Lambert 1983), and $C_0$, $C_1$, … are constants. For small enough time steps Eq. (4) can be simplified to

$$E \cong C_0 \cdot (\Delta t)^q \tag{5}$$

where $q$ depends on the integration method and $C_j$ s depend both on the integration method and on the response (Wood 1990, Hughes 1987). Also the forcing function on the RHS of equation of motion might affect $q$ and $C_j$ s (Wood 1990, Penry and Wood 1985). However this effect vanishes for most of the discretized engineering excitations.

Now consider an arbitrary system and a series of time integration analyses $i = 0, 1, 2, ...$ carried out for the system with a common time integration method and each obtained using the constant time step size $\Delta t/m^i$ all along the time interval, where $\Delta t$ is the time step size for the main analysis ($i = 0$) and $m \in \{2, 3, 4, \dots\}$. If the computed and exact responses for the $i^{\text{th}}$ analysis at an arbitrary time instant are respectively denoted with $u_i$ and $u$, the error of $u_i$ can be expressed as

$$E_i = u_i - u \tag{6}$$

Compared with Eq. (4) that is a means for studying the errors in approximate analyses (Belytschko and Hughes 1983, Wood 1990, Lambert 1983, Hughes 1987), Eq. (6) is a formal definition of error (Ralston and Rabinowitz 1978). Using Eq. (6), the deviation of $u_{i-1}$ from $u_i$,

$$\delta_{i-1} = u_{i-1} - u_i \tag{7}$$

obtained from two step-by-step analyses respectively with time steps equal to $\Delta t/m^{i-1}$ and $\Delta t/m^i$, might be rewritten as

$$\delta_{i-1} = (u + E_{i-1}) - (u + E_i) = E_{i-1} - E_i \tag{8}$$

Assuming small enough time steps and constant $C_0$, and substituting for $E_i$ and $E_{i-1}$ from Eq. (5) results

$$\delta_{i-1} \cong C_0 \cdot \left(\frac{\Delta t}{m^{i-1}}\right)^q - C_0 \cdot \left(\frac{\Delta t}{m^i}\right)^q = \frac{C_0 \cdot (\Delta t)^q}{m^{i \cdot q}} \cdot (m^q - 1) \tag{9}$$

Though $C_0$ is unknown, $\delta_{i-1}$ at the LHS of Eq. (9) is available from Eq. (7). In other words, while without the exact response (in most of the practical cases), the errors $E_{i-1}$ and $E_i$ can not be computed, their difference, which is equal to the difference between the responses $i$ and $i-1$ i.e., the deviation of the response $i-1$ from the response $i$, is approximated by Eq. (9). Furthermore, assuming small enough time steps, Eq. (5) results in

$$E_i \cong C_0 \cdot \left(\frac{\Delta t}{m^i}\right)^q = \frac{C_0 \cdot (\Delta t)^q}{m^{i \cdot q}} \tag{10}$$

Consequently, comparing the RHSs of Eqs. (9) and (10) results in

$$E_i \cong \frac{\delta_{i-1}}{m^q - 1} \tag{11}$$

and thus the error of time integration analysis with time step size equal to $\Delta t/m^i$ has already been estimated by Eq. (11). Hence after two complete time integration analyses with time step sizes, $\Delta t/m^{i-1}$ and $\Delta t/m^i$, and then computing $\delta_{i-1}$ from Eq. (7), the actual error of the response $u_i$ can be estimated by Eq. (11). It is also interesting to note that even in the case that $C_0$ is much smaller than $C_1$, Eq. (11) is conservative.

In the conventional method (Clough and Penzien 1993), the difference of the current and previous analyses' responses i.e., $|\delta_{i-1}|$ for the $i^{\text{th}}$ and $(i-1)^{\text{th}}$ analyses, is directly compared with the maximum acceptable error i.e., $\overline{E}$. Therefore compared with the conventional method, Eq. (11) permits $|\delta_{i-1}|$ to be $(m^q - 1) \geq 1$ times more. This is a significant improvement specially when we consider the fact that $(m^q - 1)$ in Eq. (11) is constant and can be determined in advance. Nevertheless, due to the approximation implemented in Eqs. (9)-(11), the validity of the following assumptions should be maintained, for the sake of reliability:

1. Time steps should be small enough to provide good accuracy for Eq. (5).
2. Responses $u_i$ and $u_{i-1}$ should be close enough not to cause any significant changes in the value of $C_0$ during analyses $i-1$ and $i$.

### 2.2 Restricting conditions

The assumptions referred to in the end of subsection 2.1 might be met without sufficient accuracy.

In order to consider the effect of this shortcoming on Eq. (11), for each three analyses carried out respectively with time step sizes $\Delta t/m^{i-2}$, $\Delta t/m^{i-1}$ and $\Delta t/m^{i}$ all along the time interval, the parameters $C_0'$ and $q'$ are defined here, such that the following Eqs.:

$$E_{i-2} \cong C_0' \cdot (\Delta t/m^{i-2})^{q'} \tag{12}$$

$$E_{i-1} \cong C_0' \cdot (\Delta t/m^{i-1})^{q'} \tag{13}$$

$$E_{i} \cong C_0' \cdot (\Delta t/m^{i})^{q'} \tag{14}$$

are satisfied with a least error criterion, defined in an appropriate manner. In spite of the similar typography of Eq. (5) and Eqs. (12)-(14), there exists a fundamental difference between them. That is, Eq. (5) is in fact a simplification of Eq. (4), and is acceptable for small enough time step sizes, whereas Eqs. (12)-(14) are just a means for defining $q'$ and $C_0'$, and are hence valid regardless of the time step size. Therefore, unlike $C_0$, which might differ for analyses with different time step sizes, $C_0'$ is constant for analyses with time step sizes $\Delta t/m^{i-2}$, $\Delta t/m^{i-1}$ and $\Delta t/m^{i}$. Also, $q'$ need not be an integer and even might differ for each set of three analyses. To clarify the matters further, Fig. 1 schematically demonstrates the errors $E_{i-2}$, $E_{i-1}$ and $E_i$ at an arbitrary time instant of a step-by-step time integration analysis, where the responses are computed by a common time integration method and with time step sizes respectively equal to $\Delta t_{i-2}$, $\Delta t_{i-1}$ and $\Delta t_i$

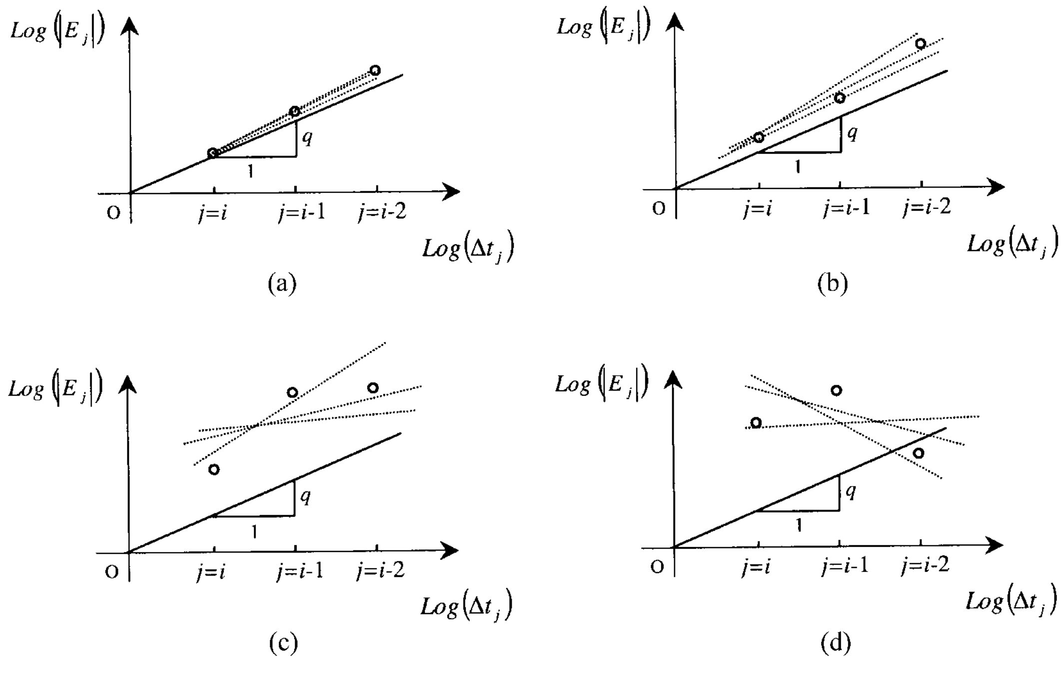


Fig. 1 Schematic illustration for the procedure resulting the definition of $q'$ and $C_0'$ (a) When the time steps are very small, (b) When the time steps are small enough, (c) When the time steps are moderate, (d) When the time steps are large

$$\Delta t_j = \frac{\Delta t}{m^j} \qquad j = i-2, i-1, i \tag{15}$$

Each of the dashed lines in Fig. 1 is generated by a specific least error criterion and the slopes of these lines are indeed $q'$. As illustrated in Fig. 1, since Eqs. (12)-(14) consider no restriction for time steps' sizes, the definition of $q'$ and $C_0'$ by Eqs. (12)-(14) implies the following facts:

1. $q'$ need not be an integer and even yet positive,
2. The values of $q'$ and $C_0'$ are not unique and depend on the least error criterion,
3. Based on the selected least error criterion, $q'$ and $C_0'$ are being determined specifically for each set of three analyses.

The second point mentioned above implicitly indicates the reason for using approximation signs in Eqs. (12)-(14). However due to the similar typography of Eqs. (12)-(14) and Eqs. (4) and (5), the error induced in defining $q'$ and $C_0'$ decreases for smaller time steps. In other words, after determining $q'$ and $C_0'$ by a least error criterion, substituting them in the RHS of Eqs. (12)-(14) would generate LHSs for these equations. Although they are different from the corresponding RHSs i.e., $E_{i-2}, E_{i-1}$ and $E_i$, gradually converge for smaller time step sizes (Fig. 1). Since only the typography of Eqs. (12)-(14) and not the values of $q'$ and $C_0'$ are considered in this discussion, the range of acceptable time step sizes are here much wider compared to the range for which Eq. (4) may be reliably simplified to Eq. (5). Nevertheless, a practical and simple method for controlling whether the time steps' sizes are small enough for Eqs. (12)-(14) to be acceptable is attainable. In this regard, equating Eq. (4) with

$$E = C_0' \cdot (\Delta t)^{q'} \tag{16}$$

which is implicit when we use Eqs. (12)-(14), results in

$$\begin{aligned} C_0' &\neq C' \\ q' &\neq q \end{aligned} \tag{17}$$

for the general case. Further, from Eqs. (4) and (16),

$$\begin{aligned} \lim_{\Delta t \to 0} q' &= q \\ \lim_{\Delta t \to 0} C_0' &= C_0 \end{aligned} \tag{18}$$

and hence for small time steps

$$q' \cong q \tag{19}$$

(This is also apparent in Fig. 1). Therefore considering Eqs. (18) and (19), we see that the size of the time steps can be controlled by $q'$. To attain a restricting relation for $q'$, it should be noted that from Eqs. (7), (8), (12)-(14) relations similar to Eqs. (9) and (10) and consequently the following equation:

$$E_i \cong \frac{\delta_i}{m^{q'} - 1} \tag{20}$$

can be deduced instead of Eq. (11) only if,

$$\lim_{i \to \infty} E_i = 0 \tag{21}$$

In view of Eqs. (12)-(14), Eq. (21) is equivalent to

$$q' > 0 \tag{22}$$

From the other side of view, regardless of the selected least error criterion,

$$\frac{\delta_{i-1}}{\delta_{i-2}} = \frac{E_{i-1} - E_i}{E_{i-2} - E_{i-1}} = \frac{C_0' \cdot \left(\frac{\Delta t}{m^{i-1}}\right)^{q'} - C_0' \cdot \left(\frac{\Delta t}{m^i}\right)^{q'}}{C_0' \cdot \left(\frac{\Delta t}{m^{i-2}}\right)^{q'} - C_0' \cdot \left(\frac{\Delta t}{m^{i-1}}\right)^{q'}} = \frac{1}{m^{q'}} \tag{23}$$

$$q' = \frac{1}{\log(m)} \cdot \log\left(\frac{\delta_{i-2}}{\delta_{i-1}}\right) \tag{24}$$

Hence Eq. (11) is acceptable if $q'$ exists or according to Eq. (24)

$$\frac{\delta_{i-2}}{\delta_{i-1}} > 0, \tag{25}$$

and further Eq. (22) is satisfied. A question that might arise in this stage is whether Eq. (11) is replaceable with Eq. (20). In fact $q'$ represents the rate of convergence between $i-2^{\text{th}}$ and $i^{\text{th}}$ analyses, whereas what is needed is the rate of convergence between $i-1^{\text{th}}$ and $j^{\text{th}}$ analyses when $j$ approaches infinity. Although this rate is different from both $q$ and $q'$, since the time step size is to be kept small enough; Eq. (11) is preferred here in view of the role of $q$ and $q'$ in Eqs. (5) and (12)-(14). It is also instructive to note that for the time steps' sizes that are acceptable for the presented theory, Eq. (25) is valid for all values of $i$. Thus from mathematical induction

$$\exists\, i > 2: \quad \delta_{i-1} > 0, \delta_{i-2} > 0 \quad \Rightarrow \quad \forall j > 0 \quad \delta_j > 0$$

$$\exists\, i > 2: \quad \delta_{i-1} < 0, \delta_{i-2} < 0 \quad \Rightarrow \quad \forall j > 0 \quad \delta_j < 0$$

$$\nexists\, i > 2: \quad \delta_{i-1} \cdot \delta_{i-2} < 0 \tag{26}$$

Consequently, since $m \ge 2$ from Eqs. (23), (22) and (25),

$$\forall i > 2 \qquad 0 < \frac{\delta_{i-1}}{\delta_{i-2}} < 1 \tag{27}$$

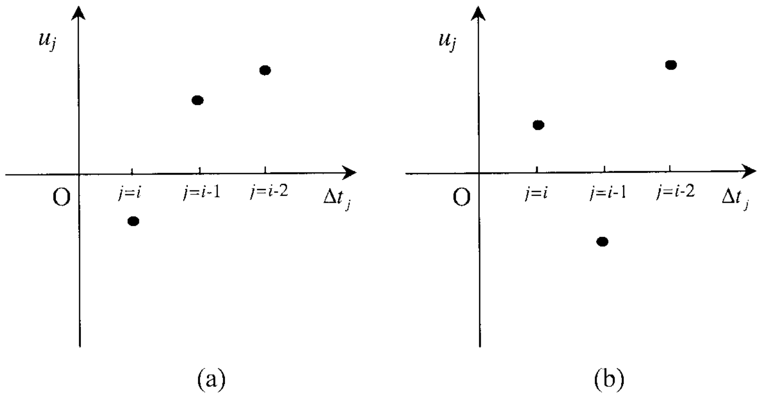


Fig. 2 Cases in which in spite of convergence the presented theory does not work (a) Eq. (22) is not satisfied ($\delta_{i-2} \le \delta_{i-1}$), (b) Eq. (25) is not satisfied ($\delta_{i-2} \cdot \delta_{i-1} \le 0$)

In other words, the theory explained above is valid only for responses that converge, and converge only from one side i.e., even for convergent responses, the schematic case illustrated in Fig. 2 is omitted from the discussion.

Eventually, compared to the conventional method where

$$E_i \cong \delta_{i-1} \tag{28}$$

making use of Eq. (11) subjected to Eqs. (25) and (22) not only is a more reasonable response to the first question stated in the introduction, but also might lead to considerable reduction in computational cost as a response to the second question.

## 3. Complementary remarks

### *3.1 More reliability*

Eq. (11) subject to Eqs. (25) and (22) forms a more rational basis for error estimation compared to the conventional Eq. (28). However since Eqs. (22) and (24) are based on Eqs. (12)-(14), and these relations can only be satisfied approximately, the reliability of Eq. (11) - even when constrained to Eqs. (25) and (22) - needs more discussion before it can be adopted for practical analysis purposes. To control the reliability, one way is to show that when subjected to Eqs. (25) and (22), the RHS of Eq. (11) is never less than the actual error, i.e.,

$$|E_i| \le \left|\frac{\delta_{i-1}}{m^q - 1}\right| \tag{29}$$

Another approach, which is simple and practical but not as reasonable as the first, is to enhance the accuracy of the last response and considerably reduce the actual error from what is estimated through Eq. (11). In this regard Richardson's extrapolation (Bismarck-Nasr and De Oliveira 1991) is an appropriate tool and is applied here. According to this technique, which is computationally very inexpensive, if the responses $u_i$ and $u_{i-1}$ obtained from two time integration analyses respectively carried out with time steps $\Delta t_i$ and $\Delta t_{i-1}$ all along the time interval, converge with the rate $q$, a much more accurate response would be derived from:

$$u_R = \frac{\Delta t_{i-1}^q \cdot u_i - \Delta t_i^q \cdot u_{i-1}}{\Delta t_{i-1}^q - \Delta t_i^q} \tag{30}$$

Further assuming that the time steps are scaled down with a constant factor i.e., $m$, Eq. (30) would be simplified to

$$u_R = \frac{m^q \cdot u_i - u_{i-1}}{m^q - 1} \tag{31}$$

To demonstrate the accuracy of $u_R$, substituting for $u_i$ and $u_{i-1}$ from Eq. (6) and then substituting for $E_i$ and $E_{i-1}$ from Eq. (5), results in

$$u_R \cong u \tag{32}$$

Therefore, even though, when Eqs. (25) and (22) are satisfied and the outcome of Eq. (11) is acceptable, $u_i$ is assumed to be accurate enough, applying Richardson's extrapolation would considerably increase the reliability.

### *3.2 Unequal time step sizes*

From the beginning stages of this paper it is implicitly assumed that in each time integration analysis the time step sizes are all equal. However, according to the formal definition of convergence (Henrici 1962, Lambert 1983),

$$\forall t_k: \quad \lim_{i \to \infty} \begin{Bmatrix} u \\ \dot{u} \end{Bmatrix}_k = \begin{Bmatrix} u(t_k) \\ \dot{u}(t_k) \end{Bmatrix} \tag{33}$$

where $\begin{Bmatrix} u \\ \dot{u} \end{Bmatrix}_k$ and $\begin{Bmatrix} u(t_k) \\ \dot{u}(t_k) \end{Bmatrix}$ respectively denote the computed and exact responses at time instant $t_k$ and at an arbitrary degree of freedom and $i$ indicates the number of sequential time integration analysis generating $\begin{Bmatrix} u \\ \dot{u} \end{Bmatrix}_k$, the concept of convergence is independent of time steps' sizes. Hence, the time steps configuration are considered here such that the time steps' sizes satisfy

$$\Delta t_i^j = r_j \cdot \Delta t_i \tag{34}$$

In Eq. (34) $\Delta t_i$ is a representative of time step sizes in the $i^{\text{th}}$ successive time integration analysis (initialized with $i = 0$) and is subjected to Eq. (15) for all $i$ $s$, $\Delta t_i^j$ denotes the size of the $j^{\text{th}}$ time step in the $i^{\text{th}}$ successive time integration analysis, and $r_j$ s are positive constants independent of $i$.

With this selection, changes of time steps' sizes can be studied merely in view of $\Delta t_i$. Accordingly in view of Eqs. (15) and (34), similar formulation can be derived for the cases where the time step sizes are not equal. Therefore, for any arbitrarily selected configuration of time steps in the main analysis i.e., $i = 0$, in order to preserve the validity of the aforementioned conclusions, it is sufficient to select time steps' configuration for the other analyses i.e., $i > 0$ such that Eqs. (15) and (34) are satisfied. These would be automatically met when for each new analysis ($i > 0$) all the time steps are divided by a constant integer referred by $m(m \geq 2)$.

### 3.3 Responses' errors at specific events

Up to this stage, the presented theory only estimates the responses' errors at arbitrarily prescribed time instants. However this is in contradiction with the objective of many practical analyses. Estimating the error of the maximum response in Fig. 3 is an example. To extend the derived formulation to such cases, a point worthy of attention is that in these cases after determining the time instant at which a special event has occurred, Eqs. (11), (25) and (22) are still valid, but $\delta_{i-1}$ and $\delta_{i-2}$ are not directly computable by Eq. (7). This is because a special event might occur at different time instants in analyses with different time step sizes, e.g. in Fig. 3: $t_1 \neq t_2 \neq t_3$. In addition, due to the discretized nature of time integration analyses, $u_{i-1}$ and $u_{i-2}$ i.e., the responses analogous to $u_i$ but for the $i-1^{\text{th}}$ and $i-2^{\text{th}}$ analyses, are not necessarily determined during the $i-1^{\text{th}}$ and $i-2^{\text{th}}$ step-by-step analysis. To overcome these problems it is sufficient first to save the time history of the $i-1^{\text{th}}$ and $i-2^{\text{th}}$ responses and then use linear interpolation where necessary for computing $u_{i-1}$ and $u_{i-2}$. It is evident that some additional error would be generated by linear interpolation. However considering the discretized nature of most of the engineering excitations, this kind of error is practically negligible (Gupta 1992).

### 3.4 Guaranteed superiority

A shortcoming of Eqs. (25) and (22) is that they are only applicable after three time integration

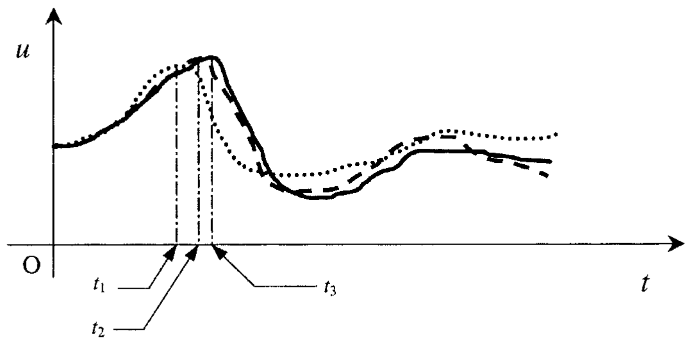


Fig. 3 A schematic illustration for a response, where an event is the purpose of the analysis (Peak Response) and $t_1 \neq t_2 \neq t_3$

...... denotes the responses generated by time integration with $\Delta t$ as the time step size

– – – denotes the response generated by time integration with $\Delta t/2$ as the time step size

—— denotes the responses generated by time integration with $\Delta t/4$ as the time step size analyses (when $\delta_0$ and $\delta_1$ are determined). Hence, for the sake of efficiency and in order to bound the maximum computational cost to the computational cost of the conventional method, the proposed method would rather be applied only when possible. In this regard, after the second iterative analysis i.e., $i = 1$, or when Eqs. (25) and (22) are not satisfied, the proposed method would be converted to the conventional method. Richardson's extrapolation would not be applicable, and thus the reliability would be reduced to that of the conventional method. Considering this technique besides the previous discussions in an algorithm presented in section 4, not only the reliability of the new method would be more than the conventional method, but also it would never be computationally more expensive. This is numerically illustrated in section 7.

## 4. Algorithm

Based on the concept presented in the previous sections, when the effect of round-off error is negligible, and the time integration method fulfills convergence requirements, responses with desired accuracy i.e., $\bar{E}$, either for arbitrarily selected time instants i.e., $t_0$, or at instants in correspondence with specific events e.g. maximum response, are attainable by the following procedure:

1. $i = 0$,
2. Select an integration method, an integer from the set {2, 3, 4, …} as $m$ (2 is recommended), and also a configuration for the time steps all along the time interval i.e., $T$, ($T = \sum_j \Delta t_i^j$) bearing in mind the linear stability requirements, accuracy recommendations, the system's least vibrational period, and the temporal discretization of excitation,
3. Carry out time integration analysis with time step sizes $\Delta t_i^j$, and work out the objective of the analysis, $u_0$,
4. $i = i + 1$,
5. $\forall j: \quad \Delta t_i^j = \Delta t_{i-1}^j / m$,
6. Carry out time integration analysis with time step sizes equal to $\Delta t_i^j$, and work out the objective of the analysis, $u_i$ and determine $t_0$ if it is not prescribed,
7. If the time instant at which the error should be estimated is prescribed compute $\delta_{i-1}$ directly from Eq. (7). Otherwise, considering the time histories of the previous analyses and linear interpolation where necessary, determine $u_{i-1}$ and $u_{i-2}$ at the time instants at which $u_i$ has occurred, and then compute $\delta_{i-2}$ and $\delta_{i-1}$ from Eq. (7),
8. If $i = 1$, control the error computed by Eq. (28) with $\bar{E}$,

$$|E_i| \leq \bar{E} \tag{35}$$

   If Eq. (35) is satisfied, accept the last response as the exact response and stop. Otherwise skip to step 12,
9. If Eqs. (25) and (22) are satisfied, skip to step 11,
10. If the $E_i$ obtained from Eq. (28) satisfies Eq. (35), accept the last response as the exact response and stop. Otherwise skip to step 12,
11. If the error computed by Eq. (11) satisfies Eq. (35), use Richardson's extrapolation (Eq. (31)) to enhance the reliability, accept the result as the exact response, and stop.
12. Save $\delta_{i-1}$ and return to step 4.

## 5. Generalizing to nonlinear regimes

In sections 2 and 3 it is first assumed that the time steps used in a time integration analysis are all equally sized. The results are then generalized to the cases where the time steps' sizes are unequal but undergo similar size reduction. For nonlinear analyses, further study and generalization is however essential due to the non-linearity iterations. In this regard, providing

$$\forall i \in \{0, 1, 2, \ldots\}: \qquad \frac{E_i}{E_{i+1}} \cong p = const. > 1, \quad |E_i| < \infty \tag{36}$$

where $i$ is the number of iterative time integration analysis also introduced in the procedure of section 4, and implementing mathematical induction, the definitions of $E_i$ and $\delta_i$ in Eqs. (6) and (7) results in

$$E_i = \frac{\delta_{i-1}}{p-1} \tag{37}$$

analogous to Eq. (11), and also after the following replacement

$$m^{q'} = p \tag{38}$$

for $q'$, Eqs. (25) and (22) will be re-derived in a straightforward manner. Consequently, after simple modifications in items 2 and 5 of the algorithm of section 4, this algorithm can be directly employed both in linear and nonlinear analyses. Hence convergence of the responses obtained from step by step time integration analysis is the only prerequisite for the method proposed in this paper. Nevertheless, in spite of the fact that the study of convergence is strictly needed for approximate numerical methods (Henrici 1962), and is indeed the main consideration when constructing new time integration methods, it cannot be easily analyzed and maintained in nonlinear problems (Cardona and Geradin 1989, Low 1991, Xie and Steven 1994, Xie 1996, Rashidi and Saadeghvaziri 1997, Farjoodi and Soroushian 2000, 2001). The main source of the shortcoming is inappropriate refinement of the time steps in which non-linearity is detected (Kardestuncer 1987, Hughes 1987, 2002). This problem can at times be obviated by the Event-To-Event method, only for SDOF systems (Farjoodi and Soroushian 2002, Bernal 1991). To overcome this deficiency several new methods are recently proposed (Kuhl and Crisfield 1999, Farjoodi and Soroushian 2001). Therefore, to evaluate the efficacy of the algorithm of section 4 in nonlinear problems, a simple elastic-plastic system is studied in section 7 in view of the recent methods proposed.

## 6. Computational cost

Computational cost depends on the required computer memory (RAM) and run time (Monro 1985, Jacob and Ebecken 1994). According to the algorithm of section 4, for time integration analysis of a special dynamic problem the computer memory corresponding to the conventional and suggested methods differ only in the need to compute (by linear interpolation) and save $\delta_{i-1}$ in the suggested method. As this difference is trivial for SDOF systems, and rapidly decreases for systems with higher number of DOFs, it can be ignored. Also, since time integration analysis is by itself time-consuming the difference of the run time required for linear interpolation and using Eqs. (25), (22), (11) and (35) instead of Eq. (35) is negligible. Therefore both for the conventional method and the algorithm presented in section 4, the computational cost might be expressed by

$$C = c_0 \cdot \left( \sum_{i=0}^{N} S_i \right) \tag{39}$$

In Eq. (39), $c_0$ is a constant, which depends on the structural system, the time integration method and computational facilities, $i$ is the number of the iterative time integration analysis initialized in the first step of the algorithm of section 4, $N$ is the maximum value of $i$ in the procedure of section 4, and $S_i$ is the total number of time steps in the $i^{th}$ time integration analysis. It should be also noted that due to the need for time step refinement in nonlinear analysis, Eq. (39) can not be simplified further. However since the algorithm of section 4 is completely independent of the integration's procedure, $S_i$ in Eq. (39) is identical for the conventional and suggested error estimation methods, and hence $N$ is indeed the only crucial parameter when comparing the computational costs of the conventional and proposed methods.

## 7. Illustrative examples

In this section, first a 2 DOF system is studied, once in relation to the error at a special time instant and then a peak response's error. The shortcomings of the conventional error estimation method are illustrated in the first part of this example. The accuracy and reliability of the proposed formulations are also demonstrated during the study of the first example. As for the accuracy, when approximate methods are implemented in practical analysis, the exact response is not at hand. Hence as implied in section 4, the additional accuracy obtained by Richardson's extrapolation is considered here merely for providing more reliability. Accordingly in the linear/nonlinear shear building of the second example after comparing the computed errors with the exact errors (which due to Richardson's extrapolation are different for the conventional and proposed methods) the computational cost is studied in view of the computed errors.

Unless specified, consistent S.I. units are used and the errors are all reported relatively after being divided by their exact values.

Furthermore for the second example and especially when the behavior is nonlinear, responses with sufficient exactness are obtained by continuing the iterative convergence analyses until the significant digits corresponding to the purpose of the analysis remain unchanged with a new analysis. Since the requirements of convergence are satisfied in advance, and the required number of significant digits is finite and sufficiently small, these responses are accepted as exact responses,

### *7.1 Two DOF undamped system (Bathe 1996)*

#### *7.1.1 Displacement of the second DOF at t = 1.68*

The following undamped dynamic linear system:

$$0 \le t < 6.72: \quad [M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad [K] = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}, \quad \{f_{ext}\} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

$$\{u(0)\} = \{\dot{u}(0)\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{40}$$

is analyzed with the average acceleration method and time step size $\Delta t_0 = 0.56$., equal to one-fifth of the second natural period. The analysis is then repeated several times each time after halving the time step sizes implemented in the previous analysis ($m = 2$). The displacements obtained from different analyses and for the second degree of freedom at $t = 1.68$ together with the corresponding actual errors are reported in Table 1. The errors estimated by the conventional method and those estimated by Eq. (11) (without applying the discussion of subsection 3.4 and Richardson's extrapolation) and subjected to Eqs. (25) and (22) are then reported in Table 2. According to Tables 1 and 2, the errors estimated by the conventional method (Eq. (28)) are generally far from the actual errors, whereas Eq. (11) generates results that are generally acceptable error estimations. Furthermore the errors estimated for $\Delta t = 0.28 (i = 1)$ (Table 2) demonstrate the unreliability of the conventional method i.e., the computed error is less than the actual error. However the inapplicability of Eq. (11) for $i = 2$, and its unreliability for $i = 3$ and $i = 5$ are later compensated by the discussion of subsection 3.4 and the Richardson's extrapolation technique. This is illustrated in Fig. 4, where the significant effect of Richardson's extrapolation in reducing the actual errors and enhancing the reliability is apparent.

#### *7.1.2 Maximum displacement of the first DOF*

Considering the error of maximum displacement at the first DOF, Tables 3 and 4, and Fig. 5 are obtained analogous to Tables 1 and 2 and Fig. 4, and once again demonstrate the accuracy of the suggested formulation and the reliability brought about by the Richardson's extrapolation technique.

Table 1 The displacements of the first example's second DOF at $t = 1.68$ generated by time integration analyses carried out with different time step sizes, together with the corresponding errors*

| Iteration number** | Time step ($\Delta t$) | Response Computed by Time Integration | Actual Error (%) |
|---|---|---|---|
| 0 | 0.56E0 | 0.536542655E+01 | 0.1416061E+1 |
| 1 | 0.28E0 | 0.533662142E+01 | 0.8715926E+0 |
| 2 | 0.14E0 | 0.530431989E+01 | 0.2610365E+0 |
| 3 | 0.7E-1 | 0.529411626E+01 | 0.6816980E-1 |
| 4 | 0.35E-1 | 0.529131743E+01 | 0.1526696E-1 |
| 5 | 0.175E-1 | 0.529073820E+01 | 0.4318487E-2 |
| 6 | 0.875E-2 | 0.529056688E+01 | 0.1080236E-2 |
| 7 | 0.4375E-2 | 0.529052402E+01 | 0.2701063E-3 |
| 8 | 0.21875E-2 | 0.529051330E+01 | 0.6747932E-4 |
| 9 | 0.109375E-2 | 0.529051062E+01 | 0.1682258E-4 |
| 10 | 0.546875E-3 | 0.529050995E+01 | 0.4158390E-5 |

*Exact response: $u(t = 1.68) = 5.29050973$
**'*i*' in the procedure of section 4

Table 2 Error analysis in the first part of example one (%)

| Iteration Number* | Time Step ($\Delta t$) | Actual Error | $\delta_{i-1}$ | Conventional Method | $q'$ | Eq. (11) |
|---|---|---|---|---|---|---|
| 0 | 0.56E0 | 0.142E+1 | | | | |
| 1 | 0.28E0 | 0.872E+0 | 0.02880513 | 0.545E+0 | | |
| 2 | 0.14E0 | 0.261E+0 | 0.03230153 | 0.611E+0 | -0.165E+0 | -** |
| 3 | 0.7E-1 | 0.682E-1 | 0.01020363 | 0.193E+0 | 0.1663E+1 | 0.643E-1 |
| 4 | 0.35E-1 | 0.153E-1 | 0.00279883 | 0.529E-1 | 0.1866E+1 | 0.176E-1 |
| 5 | 0.175E-1 | 0.432E-2 | 0.00057923 | 0.110E-1 | 0.2273E+1 | 0.365E-2 |
| 6 | 0.875E-2 | 0.108E-2 | 0.00017132 | 0.324E-2 | 0.1757E+1 | 0.108E-2 |
| 7 | 0.4375E-2 | 0.270E-3 | 0.00004286 | 0.810E-3 | 0.1999E+1 | 0.270E-3 |
| 8 | 0.21875E-2 | 0.675E-4 | 0.00001072 | 0.203E-3 | 0.1999E+1 | 0.675E-4 |
| 9 | 0.109375E-2 | 0.168E-4 | 0.00000268 | 0.507E-4 | 0.2000E+1 | 0.169E-4 |
| 10 | 0.546875E-3 | 0.416E-5 | 0.00000067 | 0.127E-4 | 0.2000E+1 | 0.422E-5 |

*'*i*' in the procedure of section 4
**One of the Eqs. (25) or (22) is not satisfied.

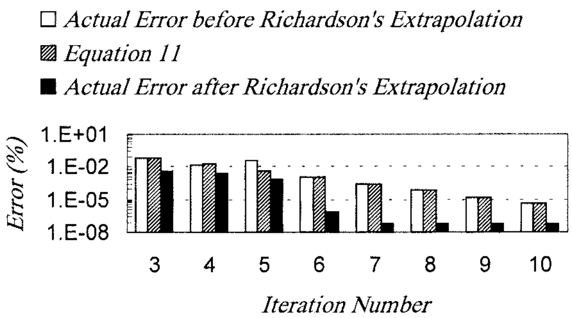


Fig. 4 Actual error and reliability for the results of Richardson's extrapolation in the first part of example one

Table 3 The maximum displacement of the first example's first DOF generated by time integration analyses carried out with different time step sizes, together with the corresponding errors*

| Iteration Number** | Time Step ($\Delta t$) | $u_{max}$ by Time Integration | $t_i(u_{max} = u(t_i))$ | Actual Error (%) |
|---|---|---|---|---|
| 0 | 0.56E0 | 0.286238689E+01 | 0.2800000E+1 | 0.6215470E+1 |
| 1 | 0.28E0 | 0.300350878E+01 | 0.2520000E+1 | 0.1591698E+1 |
| 2 | 0.14E0 | 0.304371402E+01 | 0.2520000E+1 | 0.3726833E+0 |
| 3 | 0.7E-1 | 0.304903813E+01 | 0.2520000E+1 | 0.9995433E-1 |
| 4 | 0.35E-1 | 0.305104616E+01 | 0.2520000E+1 | 0.3416241E-1 |
| 5 | 0.175E-1 | 0.305180374E+01 | 0.2502500E+1 | 0.9340914E-2 |
| 6 | 0.875E-2 | 0.305202239E+01 | 0.2511250E+1 | 0.2176837E-2 |
| 7 | 0.4375E-2 | 0.305207071E+01 | 0.2506875E+1 | 0.5934952E-3 |
| 8 | 0.21875E-2 | 0.3052084909E+01 | 0.2509063E+1 | 0.1284039E-3 |
| 9 | 0.109375E-2 | 0.3052087532E+01 | 0.2507969E+1 | 0.4246272E-4 |
| 10 | 0.546875E-3 | 0.3052088634E+01 | 0.2508516E+1 | 0.6356303E-5 |

*Exact response: $u_{max} = u(t = 2.508428179) = 0.305208883E+01$ computed by analytical methods
**'*i*' in the procedure of section 4

Table 4 Error analysis in the second part of example one (%)

| Iteration Number* | Time Step ($\Delta t$) | Actual Error | $u_i$ at $u_i = u_{max}$ / $u_{i-1}$ at $u_i = u_{max}$ | $\delta_{i-1}$ | Conventional Method | $q'$ | Eq. (11) |
|---|---|---|---|---|---|---|---|
| 0 | 0.56E0 | 0.622E+1 | 0.286238687E+1 | | | | |
| 1 | 0.28E0 | 0.159E+1 | 0.300350878E+1<br>0.267964142E+1 | –0.3238674E+0 | 0.462E+1 | | |
| 2 | 0.14E0 | 0.373E+0 | 0.304071420E+1<br>0.300350813E+1 | –0.37203423E-1 | 0.122E+1 | 0.312E+1 | 0.406E+0 |
| 3 | 0.7E-1 | 0.100E+0 | 0.304903813E+1<br>0.304071420E+1 | –0.83239302E-2 | 0.272E+0 | 0.216E+1 | 0.909E-1 |
| 4 | 0.35E-1 | 0.342E-1 | 0.305104616E+1<br>0.304903813E+1 | –0.20080283E-2 | 0.658E-1 | 0.205E+1 | 0.219E-1 |
| 5 | 0.175E-1 | 0.934E-2 | 0.305180374E+1<br>0.305038204E+1 | –0.14216954E-2 | 0.248E-1 | 0.498E+0 | 0.155E-1 |
| 6 | 0.875E-2 | 0.218E-2 | 0.305202239E+1<br>0.305167357E+1 | –0.34882017E-3 | 0.716E-2 | 0.203E+1 | 0.381E-2 |
| 7 | 0.4375E-2 | 0.593E-3 | 0.305207071E+1<br>0.305198268E+1 | –0.88035302E-4 | 0.158E-3 | 0.199E+1 | 0.961E-3 |
| 8 | 0.21875E-2 | 0.128E-3 | 0.305208491E+1<br>0.305206300E+1 | –0.21903540E-4 | 0.465E-3 | 0.201E+1 | 0.239E-3 |
| 9 | 0.109375E-2 | 0.425E-4 | 0.305208753E+1<br>0.305208204E+1 | –0.54885395E-5 | 0.859E-4 | 0.200E+1 | 0.599E-4 |
| 10 | 0.546875E-3 | 0.636E-5 | 0.305208863E+1<br>0.305208726E+1 | –0.13710000E-5 | 0.361E-4 | 0.200E+1 | 0.150E-4 |

*'$i$' in the procedure of section 4

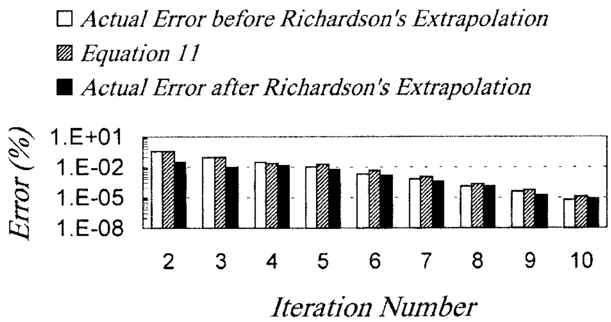


Fig. 5 Actual error and reliability for the results of Richardson's extrapolation in the second part of example one

## 7.2 Eight storey 2-D shear building

### 7.2.1 The first floor's displacement at t = 3.0, in the linear case

In this example, an eight-storey 2-D shear building (Fig. 6) is studied. Although in Fig. 6, the columns have an elastic-fully-plastic behavior, the nonlinear behavior will be considered only in the second part of the example. The strong motion excitation i.e., $\ddot{u}_g$, is also left out in this part. The modal damping is 5% and the other properties of the structural model are mentioned in the first

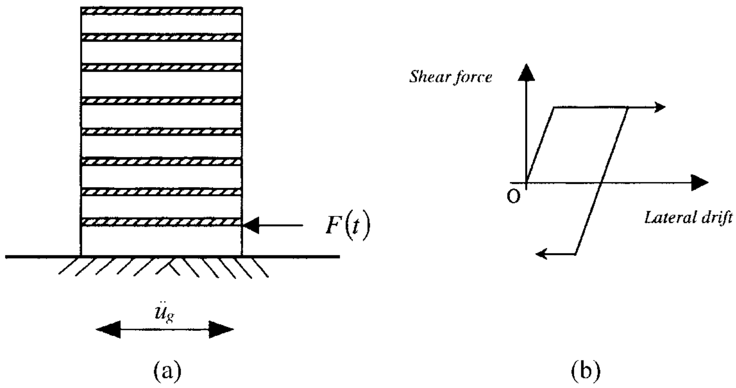


Fig. 6 The structural system studied in example two (a) Structural model, (b) The schematic force-displacement diagram for a typical column

Table 5 Structural properties in example two

| Level | Mass (Tons) | Stiffness (KN/m) | Yield Limit (m) |
|---|---|---|---|
| 1 | 518 | 8.6E5 | 0.36 |
| 2 | 517 | 8.4E5 | 0.37 |
| 3 | 516 | 8.2E5 | 0.38 |
| 4 | 515 | 7.0E5 | 0.40 |
| 5 | 514 | 6.8E5 | 0.41 |
| 6 | 513 | 6.6E5 | 0.42 |
| 7 | 512 | 6.4E5 | 0.43 |
| 8(top) | 511 | 6.2E5 | 0.44 |

three columns of Table 5. The excitation is a vehicle collision to the first floor and is modeled below:

$$F(t) = \begin{cases} 0 & t < 0 \\ 1000 \cdot (1 - t)\text{tons} & 0 \le t < 1.0 \\ 0 & 1.0 \le t \end{cases} \tag{41}$$

The integration method is selected to be the linear acceleration version of the Newmark method. The objective is to calculate the displacement of the first floor at $t = 3.0$ and $m$ is selected to be 2. Time integration analysis is carried out with time steps all equal to 0.01 (slightly more than one tenth of the least natural period), followed with iterative analyses with smaller time steps. The exact response that is utilized for controlling the efficiency of the method is 0.5844014625E-4. For the conventional error estimation method, the responses computed for the first floor's displacements at $t = 3.0$ together with the corresponding computed errors, actual errors and computational costs are set out in Table 6. (For the sake of simplicity $c_0$ is assumed equal to 1.0E-3). Analogous responses obtained by the algorithm of section 4 are reported in Table 7. According to Tables 6 and 7, the computed errors are all greater than the actual errors; thus the conventional and proposed method are both reliable in this example. Also according to the last column in Tables 6 and 7, the computational costs of the two methods are identical for a special number of analyses i.e., iteration

Table 6 Error control for the first floor displacement at $t = 3.0$, in the first part of example two* (conventional method)

| Iteration Number** | Time Step ($\Delta t$) | Response Computed by Time Integration | Computed Error (%) | Actual Error (%) | Computational Cost*** |
|---|---|---|---|---|---|
| 0 | 0.1E-2 | 0.5981616415E-3 | | 0.235E+1 | 0.30E-1 |
| 1 | 0.5E-3 | 0.5878709787E-3 | 0.176E+1 | 0.594E+0 | 0.90E+0 |
| 2 | 0.25E-3 | 0.5852698644E-3 | 0.445E+0 | 0.149E+0 | 0.21E+1 |
| 3 | 0.125E-3 | 0.5846186336E-3 | 0.111E+0 | 0.372E-1 | 0.45E+1 |
| 4 | 0.625E-4 | 0.5844557600E-3 | 0.279E-1 | 0.929E-2 | 0.93E+1 |
| 5 | 0.3125E-4 | 0.5844150371E-3 | 0.697E-2 | 0.232E-2 | 0.19E+2 |
| 6 | 0.15625E-4 | 0.5844048561E-3 | 0.174E-2 | 0.581E-3 | 0.38E+2 |
| 7 | 0.78125E-5 | 0.5844023109E-3 | 0.436E-3 | 0.145E-3 | 0.77E+2 |
| 8 | 0.390625E-5 | 0.5844016746E-3 | 0.109E-3 | 0.363E-4 | 0.15E+3 |
| 9 | 0.1953125E-5 | 0.5844015155E-3 | 0.272E-4 | 0.907E-5 | 0.31E+3 |
| 10 | 0.9765625E-6 | 0.5844014757E-3 | 0.681E-5 | 0.226E-5 | 0.61E+3 |

*Exact response: $u(t = 3.0) = 0.5844014625\text{E-}3$
**'$i$' in the procedure of section 4
*** $c_0 \cdot \sum_{j=0}^{i} S_j$

Table 7 Error control for the first floor displacement at $t = 3.0$, in the first part of example two* (proposed method)

| Iteration Number** | Time Step ($\Delta t$) | Response Computed by Time Integration and when applicable Extrapolation | Computed Error (%) | | | Actual Error (%) | Computational Cost*** |
|---|---|---|---|---|---|---|---|
| | | | $\delta_{i-1}$ | $q'$ | $E_i$ | | |
| 0 | 0.1E-2 | 0.5981616415E-3 | | | | 0.235E+1 | 0.30E-1 |
| 1 | 0.5E-3 | 0.5878709787E-3 | 0.1029066E-4 | | 0.176E+1 | 0.594E+0 | 0.90E+0 |
| 2 | 0.25E-3 | 0.5844028263E-3 | 0.2601114E-5 | 0.19813E+1 | 0.148E+0 | 0.233E-3 | 0.21E+1 |
| 3 | 0.125E-3 | 0.5844015567E-3 | 0.6512278E-6 | 0.19979E+1 | 0.372E-1 | 0.161E-4 | 0.45E+1 |
| 4 | 0.625E-4 | 0.5844014688E-3 | 0.1628736E-6 | 0.19994E+1 | 0.930E-2 | 0.108E-5 | 0.93E+1 |
| 5 | 0.3125E-4 | 0.5844014628E-3 | 0.4072290E-7 | 0.19998E+1 | 0.232E-2 | 0.513E-7 | 0.19E+2 |
| 6 | 0.15625E-4 | 0.5844014624E-3 | 0.1018100E-7 | 0.20000E+1 | 0.581E-3 | 0.114E-7 | 0.38E+2 |
| 7 | 0.78125E-5 | 0.5844014625E-3 | 0.2545200E-8 | 0.20000E+1 | 0.145E-3 | 0.00E+0 | 0.77E+2 |
| 8 | 0.390625E-5 | 0.5844014625E-3 | 0.6363000E-9 | 0.20000E+1 | 0.363E-4 | 0.00E+0 | 0.15E+3 |
| 9 | 0.1953125E-5 | 0.5844014625E-3 | 0.1591000E-9 | 0.19998E+1 | 0.908E-5 | 0.570E-8 | 0.31E+3 |
| 10 | 0.9765625E-6 | 0.5844014624E-3 | 0.398000E-10 | 0.19991E+1 | 0.227E-5 | 0.114E-7 | 0.61E+3 |

*Exact response: $u(t = 3.0) = 0.5844014625\text{E-}3$
**'$i$' in the procedure of section 4
*** $c_0 \cdot \sum_{j=0}^{i} S_j$

number, $i$. Nevertheless after an arbitrary number of iterations, the computed/actual errors in Table 7 are less than or equal to the computed/actual errors in Table 6. Therefore it can be concluded that in this example the suggested method is computationally more efficient. For more clarification, the computational costs necessary to arrive at different acceptable error levels are compared in Fig. 7. As noted before, the results shown in Fig. 7 are derived merely based on the errors computed by the

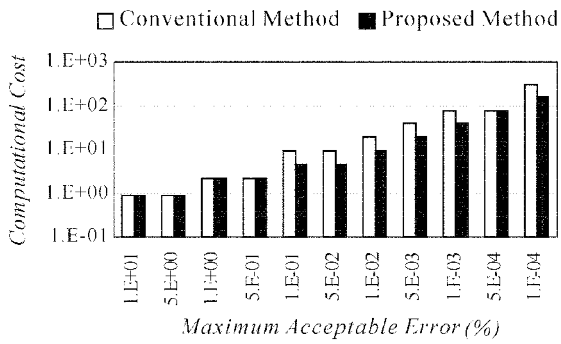


Fig. 7 Computational cost for the first part of example two

conventional and proposed methods and not on the generally unavailable exact responses and errors, i.e., Eq. (6). Hence, taking into account the logarithmic scaling of the vertical axis of Fig. 7, the higher efficiency of the proposed method is demonstrated in this example.

### 7.2.2 The top drift at t = 10.0, in the nonlinear case

As a more practical example, the nonlinear behavior of the previous example is studied by considering the last column of Table 5, and replacing the excitation with the North-South component of the El Centro strong motion (Chopra 1995). The histories of the fourth floor's displacement and the top drift are depicted as representatives for the system's behavior in Fig. 8. Non-linearity is implied in Fig. 8(a), and the oscillatory behavior is indicated in Fig. 8(b). To study the purpose of the analysis, which is the top drift at $t$ = 10.0, the integration method is selected to be the central difference method. Parameter $m$ and $\Delta t_0$ are set at 2 and 0.01 respectively, and non-linearity is detected and localized by the Fractional Time Stepping method (Nau 1993, Mahin and Lin 1983). The non-linearity detecting/localizing tolerances are set at 0.01 for the first analysis, and in order to preserve convergence are then modified according to the recent research (Farjoodi and Soroushian 2001). The exact response used for error analysis is 0.3836666482E-1. After maintaining convergence for the responses of the nonlinear system, the conventional and proposed methods are implemented, and the results of error analysis are set out in Tables 8 and 9 and Fig. 9. Similar to the previous example the results obtained once again clearly confirm the superiority of the proposed method.

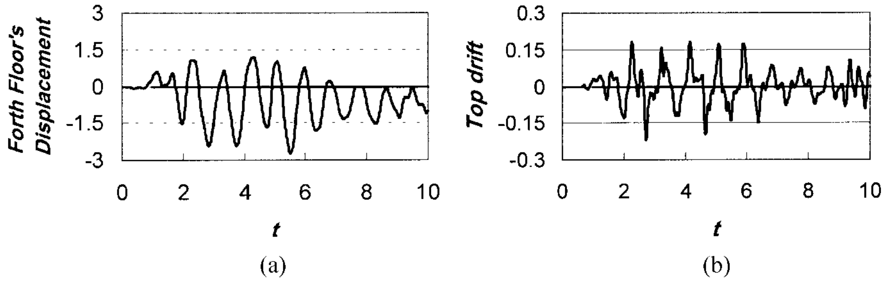


Fig. 8 Representatives for the behavior of the system studied in the second part of example two (a) Forth floor's displacement, (b) Top drift

Table 8 Error control for the top drift at $t = 10.0$, in the second part of example two* (conventional method)

| Iteration Number** | Time Step ($\Delta t$) | $S_i$ | Response Computed by Time Integration | Computed Error (%) | Actual Error (%) | Computational Cost *** |
|---|---|---|---|---|---|---|
| 0 | 0.1E-2 | 1131 | 0.38116694E-01 | | 0.652E+0 | 0.11E+1 |
| 1 | 0.5E-3 | 2770 | 0.38277916E-01 | 0.420E+0 | 0.231E+0 | 0.39E+1 |
| 2 | 0.25E-3 | 5344 | 0.38340286E-01 | 0.163E+0 | 0.688E-1 | 0.92E+1 |
| 3 | 0.125E-3 | 9463 | 0.38359819E-01 | 0.509E-1 | 0.178E-1 | 0.19E+2 |
| 4 | 0.625E-4 | 17759 | 0.38364935E-01 | 0.133E-1 | 0.451E-2 | 0.36E+2 |
| 5 | 0.3125E-4 | 33736 | 0.38366231E-01 | 0.338E-2 | 0.113E-2 | 0.70E+2 |
| 6 | 0.15625E-4 | 65782 | 0.38366552E-01 | 0.837E-3 | 0.294E-3 | 0.14E+3 |
| 7 | 0.78125E-5 | 129847 | 0.38366637E-01 | 0.222E-3 | 0.725E-4 | 0.27E+3 |
| 8 | 0.390625E-5 | 257823 | 0.38366655E-01 | 0.469E-4 | 0.256E-4 | 0.52E+3 |

*Exact response: $u(t = 10.0) = 0.3836666482\text{E-}1$
**'$i$' in the procedure of section 4
***$c_0 \cdot \sum_{j=0}^{i} S_j$

Table 9 Error control for the top drift at $t = 10.0$, in the second part of example two* (proposed method)

| Iteration Number** | Time Step ($\Delta t$) | $S_i$ | Response Computed by Time Integration and when applicable Extrapolation | Computed Error (%) | | | Actual Error (%) | Computa-tional Cost *** |
|---|---|---|---|---|---|---|---|---|
| | | | | $\delta_{i-1}$ | $q'$ | $E_i$ | | |
| 0 | 0.1E-2 | 1131 | 0.3811669400E-1 | | | | 0.652E+0 | 0.11E+1 |
| 1 | 0.5E-3 | 2770 | 0.38277916E-01 | –0.161222E-3 | | 0.420E+0 | 0.231E+0 | 0.39E+1 |
| 2 | 0.25E-3 | 5344 | 0.3836107600E-1 | –0.623700E-4 | 0.13701E+1 | 0.542E-1 | 0.146E-1 | 0.92E+1 |
| 3 | 0.125E-3 | 9463 | 0.3836633000E-1 | –0.195330E-4 | 0.16749E+1 | 0.170E-1 | 0.873E-3 | 0.19E+2 |
| 4 | 0.625E-4 | 17759 | 0.3836664033E-1 | –0.511600E-5 | 0.19328E+1 | 0.444E-2 | 0.638E-4 | 0.36E+2 |
| 5 | 0.3125E-4 | 33736 | 0.3836666300E-1 | –0.129600E-5 | 0.19810E+1 | 0.113E-2 | 0.474E-5 | 0.70E+2 |
| 6 | 0.15625E-4 | 65782 | 0.3836665900E-1 | –0.321000E-6 | 0.20134E+1 | 0.279E-3 | 0.152E-4 | 0.14E+3 |
| 7 | 0.78125E-5 | 129847 | 0.3836666533E-1 | –0.850000E-7 | 0.19170E+1 | 0.738E-4 | 0.134E-5 | 0.27E+3 |
| 8 | 0.390625E-5 | 257823 | 0.3836666100E-1 | –0.180000E-7 | 0.22395E+1 | 0.156E-4 | 0.996E-5 | 0.52E+3 |

*Exact response: $u(t = 10.0) = 0.3836666482\text{E-}1$
**'$i$' in the procedure of section 4
***$c_0 \cdot \sum_{j=0}^{i} S_j$

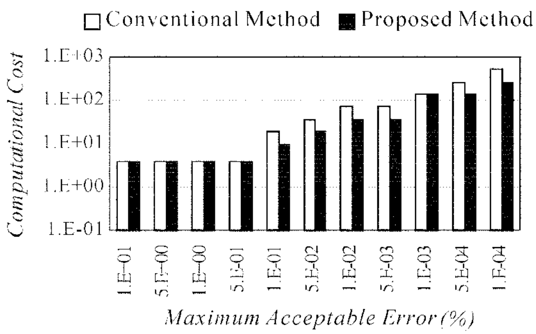


Fig. 9 Computational cost for the second part of example two

## 8. Conclusions

By ignoring the round off error and after a thorough attention to the concept of convergence, a considerably adequate formulation is proposed for the errors occurring in time integration analyses. In addition, based on the derived formulation and with the aid of Richardson's extrapolation, a computationally efficient method for attaining considerably reliable responses from time integration analyses is presented here. Furthermore it is demonstrated that:

1. The conventional method for controlling the errors of time integration analyses (Clough and Penzien 1993) might be unreliable.
2. Compared to the conventional method the presented method has a more rational and yet simple basis.
3. When Eqs. (25) and (22) are satisfied, the proposed method is more reliable compared to the conventional method. Otherwise both methods are identical.
4. By reducing the number of required time integration analyses, (or allowing the use of larger time steps in the first time integration), the new method is at least as computationally efficient as the conventional method.

Besides, since the achieved results are solely based on the convergence of the responses, Richardson's extrapolation and negligible round-off errors, it is reasonable to apply the proposed method to other approximate structural analysis methods e.g. FEM and BEM, and even numerical analysis methods in other areas. More research on this subject is however essential and is recommended.

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