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Characteristics of solutions in softening plasticity and path criterion

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Abstract. Characteristics of solutions of softening plasticity are discussed in this article. The localized and non-localized solutions are obtained for a three-bar truss and their stability is evaluated with the aid of the second-order work. Beyond the bifurcation point, the single stable loading path splits into several post-bifurcation paths and the second-order work exhibits several competing minima. Among the multiple post-bifurcation equilibrium states, the localized solutions correspond to the minimum points of the second-order work, while the non-localized solutions correspond to the saddles and local maximum points. To determine the real post-bifurcation path, it is proposed that the structure should follow the path corresponding to the absolute minimum point of the second-order work. The proposal is further proved equivalent to Bazant's path criterion derived on a thermodynamics basis.

Key words: softening plasticity; strain localization; bifurcation; energy minimization; path criterion.

1. Introduction

Strain localization occurs in a wide variety of solids in different forms such as tension necking (Needleman 1972), shear bands (Rice 1976, Needleman and Tvergaard 1992), localized buckling modes (Tvergaard and Needleman 1980, Hunt *et al*. 1989, Goto *et al*. 1995) and interacting cracks (Bazant 1989, Horii 1993, Horii and Inoue 1997). From a mathematical viewpoint, strain localization is a bifurcation problem. At the bifurcation point, the fundamental path splits into several equilibrium paths (Maier 1971, Maier *et al*. 1973, Stavroulakis and Mistakidis 1995, Chen and Baker 2003a, 2003b).

A bifurcation in inelastic solids may be caused by non-linearity of either a geometrical or material nature. For the case of a geometrical non-linearity, at the bifurcation point the second-order work (or the tangential stiffness matrix) becomes singular. Through eigenvalue analysis of the tangential stiffness matrix, the critical condition is obtained and the post-bifurcation deformation mode is determined.

For the case of material non-linearity in which the bifurcation is induced by a discontinuous reduction of the material stiffness, however, the tangential stiffness matrix becomes indefinite without being singular (Petryk and Thermann 1992). Though the change of sign of the smallest

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eigenvalue detects a bifurcation, however, the corresponding eigenvector does not predict the post-bifurcation path (i.e., the post-bifurcation deformation mode). In order to determine the post-bifurcation path, many articles resort to an energy method.

Bazant (1988, 1989) for example, formulates in thermodynamic terms a maximum entropy criterion that can be expressed as a second-order energy increment under isothermal conditions. Runesson *et al.* (1989) similarly discover that the localized solutions in softening materials correspond to minima of the incremental energy.

In this paper, we adopt the concept of minimization of the second-order work as the criterion for post-bifurcation path. Our motivation is to reveal the relationship between the characteristics of solutions and the variation of the second-order work. First, the second-order work is formulated for a solid with material non-linearity. Then, the characteristics of solutions involving softening plasticity are discussed with reference to strain localization. The localized and non-localized solutions are obtained for a three-bar truss and the surface of the second-order work is drawn to evaluate the stability of the solutions. It is proposed that, among competing stable post-bifurcation equilibrium states, the structure follows the path corresponding to the absolute minimum point of the second-order work. Finally, the proposed criterion is further proved equivalent to Bazant's path criterion (Bazant 1988, Bazant and Cedolin 1991) derived on thermodynamics basis.

2. The second-order increment of work

Consider a problem with only material non-linearity. Suppose that $\Omega^{(N)}$ is an equilibrium state on the load-displacement curve. All the state variables, such as stress σ_{ij}, strain ε_{ij}, and displacement u_i, together with the loading history, are known up to the $\Omega^{(N)}$ state. At the $\Omega^{(N)}$ state, the volume V, occupied by the structure, and the volume force f_i are known; the traction $\bar{P}_i$ is prescribed on free boundary S_σ, and the displacement $\bar{u}_i$ is prescribed on constrained boundary S_u.

Now suppose there is a volume force increment Δf_i, a traction increment $\Delta\bar{P}_i$ on S_σ, and a displacement increment $\Delta\bar{u}_i$ on S_u. All the increments are infinitesimal. The problem is then to determine the displacement increment Δu_i, the strain increment $\Delta\varepsilon_{ij}$, and the stress increment $\Delta\sigma_{ij}$. Due to the infinitesimal increment Δu_i, the strain increment is

$$\Delta\varepsilon_{ij} = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}) \tag{1}$$

and the stress increment is

$$\Delta\sigma_{ij} = C_{ijkl}\Delta\varepsilon_{kl} \tag{2}$$

and the second-order increment of the internal work is

$$\Delta^2 U = \iiint_V \frac{1}{2}\Delta\varepsilon_{ij}C_{ijkl}\Delta\varepsilon_{kl}dV \tag{3}$$

where C_{ijkl} is the tangential stiffness tensor. The second-order increment of the work of the whole system is expressed as

$$\Delta^2\Pi = \iiint_V \frac{1}{2}\Delta\varepsilon_{ij}C_{ijkl}\Delta\varepsilon_{kl}dV - \iiint_V \Delta f_i \Delta u_i dV - \iint_{S_\sigma} \Delta\bar{P}_i \Delta u_i dS \tag{4}$$

If the body is stable up to the $\Omega^{(N)}$ state, the solution of Δu_i is unique, and the actual displacement increments Δu_i render the second-order work $\Delta^2\Pi$ an absolute minimum with respect to all kinematically admissible increments (Hill 1950).

3. Characteristics of the solutions of a three-bar truss

We are concerned with the situations after the fundamental solution loses its stability. Beyond the bifurcation point, the single stable loading path is liable to split into several post-bifurcation paths. In other words, for a set of loading increment, there may be several sets of possible displacement increments satisfying the equilibrium equations and the boundary conditions. In order to discuss the characteristics of solutions, the localized and non-localized solutions for a three-bar truss will be obtained and their stability is evaluated with the aid of the surface of the second-order work.

Consider a three-bar truss in the form of an equilateral triangle (Hunt and Baker 1995). A downward displacement ϖ is applied at the apex as shown in Fig. 1(a). For simplicity, all bars are assumed to have the same cross section, A, and the length, L. A bilinear constitutive law, as shown in Fig. 1(b), is adopted with equal stiffness but different peak strains for the bars in tension and compression.

3.1 Incremental formulation

A bar element is defined by two points i and j, as shown in Fig. 1(c); (a_i, b_i) and (a_j, b_j) represent the coordinates of points i and j, and (u_i, v_i) and (u_j, v_j) are the nodal displacements of points i and j. For infinitesimal displacements, the total strain ε of the bar element is expressed as

$$\varepsilon = \cos\theta \cdot \frac{u_j - u_i}{L} + \sin\theta \cdot \frac{v_j - v_i}{L} \tag{5}$$

with $\cos\theta = (a_j - a_i)/L$, $\sin\theta = (b_j - b_i)/L$, and $L = \sqrt{(a_j - a_i)^2 + (b_j - b_i)^2}$.

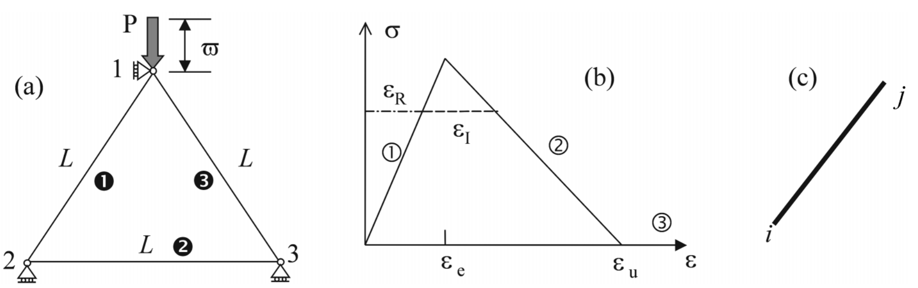


Fig. 1 (a) A three-bar truss, (b) Constitutive law, (c) Bar element

The total strain ε is composed of an elastic part ε_R and an inelastic part ε_I, as shown in Fig. 1(b), which can be expressed as

$$\begin{array}{llll} \textbf{Branch 1} & \varepsilon_R = \varepsilon & \varepsilon_I = 0 & \text{if} \quad \varepsilon < \varepsilon_e \\ \textbf{Branch 2} & \varepsilon_R = \varepsilon_e \dfrac{\varepsilon_u - \varepsilon}{\varepsilon_u - \varepsilon_e} & \varepsilon_I = \varepsilon_u \dfrac{\varepsilon - \varepsilon_e}{\varepsilon_u - \varepsilon_e} & \text{if} \quad \varepsilon_e \le \varepsilon < \varepsilon_u \end{array} \tag{6}$$

Suppose the solutions for the previous loading steps are known, for a set of loading increment, the corresponding displacement increments can be decided. The strain increment is related to the displacement increments by

$$\Delta\varepsilon = \cos\theta \cdot \frac{\Delta u_j - \Delta u_i}{L} + \sin\theta \cdot \frac{\Delta v_j - \Delta v_i}{L} \tag{7}$$

The strain increment $\Delta\varepsilon$ is decomposed into an elastic part $\Delta\varepsilon_R$ and an inelastic part $\Delta\varepsilon_I$, i.e.,

$$\Delta\varepsilon = \Delta\varepsilon_R + \Delta\varepsilon_I \tag{8}$$

For unloading, no matter the material point is on which branch of the constitutive law, we have

$$\Delta\varepsilon_R = \Delta\varepsilon \qquad \Delta\varepsilon_I = 0 \tag{9}$$

For loading, the elastic part $\Delta\varepsilon_R$ and inelastic part $\Delta\varepsilon_I$ are:

$$\begin{array}{llll} \textbf{Branch 1} & \Delta\varepsilon_R = \Delta\varepsilon & \Delta\varepsilon_I = 0 & \text{if} \quad \varepsilon < \varepsilon_e \\ \textbf{Branch 2} & \Delta\varepsilon_R = -\dfrac{\varepsilon_e}{\varepsilon_u - \varepsilon_e}\Delta\varepsilon & \Delta\varepsilon_I = \dfrac{\varepsilon_u}{\varepsilon_u - \varepsilon_e}\Delta\varepsilon & \text{if} \quad \varepsilon_e \le \varepsilon < \varepsilon_u \end{array} \tag{10}$$

It should be noted that, in this instance, loading means an increase of strain with a decrease of stress.

The stress increment is related to the elastic strain increment by

$$\Delta\sigma = E_e \cdot \Delta\varepsilon_R \tag{11}$$

From Fig. 1(b), a simple relation between the elastic positive stiffness E_e and the post-peak negative stiffness E_s can be obtained

$$E_s = -\frac{\varepsilon_e}{\varepsilon_u - \varepsilon_e} E_e \tag{12}$$

By making use of (10), (11), and (12), the stress increment is obtained

$$\Delta\sigma = \begin{cases} E_e \cdot \Delta\varepsilon & \text{for unloading and elastic loading} \\ E_s \cdot \Delta\varepsilon & \text{for plastic softening} \end{cases} \tag{13}$$

The second-order work per unit volume is

$$\Delta^2 \tilde{U} = \frac{1}{2}\Delta\sigma \cdot \Delta\varepsilon \tag{14}$$

and the total increment the second-order work of the truss model is

$$\Delta^2 U = \sum AL\Delta^2 \tilde{U} \tag{15}$$

3.2 The localized and non-localized solutions of a three-bar truss

First, we must determine the bifurcation point and the bifurcation conditions. Consider the boundary conditions $u_1 = 0$, $v_1 = -\varpi$, $v_2 = 0$, $v_3 = 0$, then from (5), the strain-displacement relations for the three bars are obtained

$$\varepsilon_1 = -\frac{1}{2}\cdot\frac{u_2}{L} - \frac{\sqrt{3}}{2}\cdot\frac{\varpi}{L} \qquad \varepsilon_2 = \frac{u_3 - u_2}{L} \qquad \varepsilon_3 = \frac{1}{2}\cdot\frac{u_3}{L} - \frac{\sqrt{3}}{2}\cdot\frac{\varpi}{L} \tag{16}$$

If all the three elements are in the elastic stage and have the same elastic Young's modulus, the equilibrium conditions for the corner points 2 and 3 are written as

$$\frac{AE_e\varepsilon_1}{2} + AE_e\varepsilon_2 = 0 \qquad \frac{AE_e\varepsilon_3}{2} + AE_e\varepsilon_2 = 0 \tag{17}$$

With the substitution of (16) into (17), the displacements are obtained

$$u_2 = -\frac{\sqrt{3}}{9}\varpi \qquad u_3 = \frac{\sqrt{3}}{9}\varpi \tag{18}$$

Then from (16), the strains are obtained

$$\varepsilon_1 = -\frac{4\sqrt{3}}{9}\cdot\frac{\varpi}{L} \qquad \varepsilon_2 = \frac{2\sqrt{3}}{9}\cdot\frac{\varpi}{L} \qquad \varepsilon_3 = -\frac{4\sqrt{3}}{9}\cdot\frac{\varpi}{L} \tag{19}$$

This is the elastic solution. If the three elements yield at the same time, from (19), it is obvious that their elastic limit strains must satisfy $\varepsilon_{e1} = \varepsilon_{e3} = 2\varepsilon_{e2}$. Assume the material parameters of each bar are: $E_e = 3.2\times10^{10}$, $\varepsilon_{e1} = \varepsilon_{e3} = 2.5\times10^{-4}$, $\varepsilon_{e2} = 1.25\times10^{-4}$, $\varepsilon_{u1} = \varepsilon_{u3} = 2.5\times10^{-3}$, $\varepsilon_{u2} = 2.5\times10^{-3}$. When $\varpi = 9L\varepsilon_{e1}/4\sqrt{3} = 3.2476\times10^{-4}$, from (19) we know that all the three bars reach their elastic limit strains, i.e., $\varepsilon_1 = -\varepsilon_{e1}$, $\varepsilon_2 = \varepsilon_{e2}$, $\varepsilon_3 = -\varepsilon_{e3}$. Bifurcation happens at this point. If ϖ has an increment $\Delta\varpi$ (suppose it is positive), each bar has the opportunity of loading and the opportunity of elastic unloading. (In this instance, "loading" means that, as the strain increases, the material enters the plastic stage, and the stress decreases rather than increases as in the hardening case.) From (7), the relations between the strain increments and the displacement increments are obtained

$$\Delta\varepsilon_1 = -\frac{1}{2}\cdot\frac{\Delta u_2}{L} - \frac{\sqrt{3}}{2}\cdot\frac{\Delta\varpi}{L} \qquad \Delta\varepsilon_2 = \frac{\Delta u_3 - \Delta u_2}{L} \qquad \Delta\varepsilon_3 = \frac{1}{2}\cdot\frac{\Delta u_3}{L} - \frac{\sqrt{3}}{2}\cdot\frac{\Delta\varpi}{L} \tag{20}$$

The equilibrium conditions of the corner points 2 and 3 are

$$\frac{A\Delta\sigma_1}{2} + A\Delta\sigma_2 = 0 \qquad \frac{A\Delta\sigma_3}{2} + A\Delta\sigma_2 = 0 \tag{21}$$

Since the previous step is a bifurcation point, for an infinitesimal loading increment, each element may be elastically unloaded or plastically loaded. For each case, the tangent stiffness is different, denoted by E_t, which is equal to either E_e for unloading or E_s for loading. Thus, the equilibrium conditions (21) become

$$\frac{E_{t1}\Delta\varepsilon_1}{2} + E_{t2}\Delta\varepsilon_2 = 0 \qquad \frac{E_{t3}\Delta\varepsilon_3}{2} + E_{t2}\Delta\varepsilon_2 = 0 \tag{22}$$

With the substitution of (20) into (22), the displacement increments are obtained:

$$\Delta u_2 = \frac{4E_{t2}(E_{t3}-E_{t1}) - E_{t1}E_{t3}}{E_{t1}E_{t3} + 4E_{t2}E_{t3} + 4E_{t1}E_{t2}} \cdot \sqrt{3}\Delta\varpi$$

$$\Delta u_3 = \frac{4E_{t2}(E_{t3}-E_{t1}) + E_{t1}E_{t3}}{E_{t1}E_{t3} + 4E_{t2}E_{t3} + 4E_{t1}E_{t2}} \cdot \sqrt{3}\Delta\varpi \tag{23}$$

Then, the strain increments are

$$\Delta\varepsilon_1 = \frac{-4E_{t2}E_{t3}}{E_{t1}E_{t3} + 4E_{t2}E_{t3} + 4E_{t1}E_{t2}} \cdot \frac{\sqrt{3}\Delta\varpi}{L}$$

$$\Delta\varepsilon_2 = \frac{2E_{t1}E_{t3}}{E_{t1}E_{t3} + 4E_{t2}E_{t3} + 4E_{t1}E_{t2}} \cdot \frac{\sqrt{3}\Delta\varpi}{L}$$

$$\Delta\varepsilon_2 = \frac{-4E_{t2}E_{t1}}{E_{t1}E_{t3} + 4E_{t2}E_{t3} + 4E_{t1}E_{t2}} \cdot \frac{\sqrt{3}\Delta\varpi}{L} \tag{24}$$

Introduce the displacement rates, $\dot{u}_2, \dot{u}_3$ for example $\dot{u}_2 = \lim\limits_{\Delta\varpi\to 0}\frac{\Delta u_2}{\Delta\varpi}$, the strain rate $\dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{\varepsilon}_3$, and $\ddot{U} = \lim\limits_{\Delta\varpi\to 0}\frac{\Delta^2 U}{(\Delta\varpi)^2}$. From (13), (14), and (15), we have

$$\ddot{U} = \sum\frac{1}{2}ALE_t\dot{\varepsilon}^2 \tag{25}$$

From (20), the strain rate $\dot{\varepsilon}_1, \dot{\varepsilon}_2,$ and $\dot{\varepsilon}_3$, are obtained:

$$\dot{\varepsilon}_1 = -\frac{1}{2}\cdot\frac{\dot{u}_2}{L} - \frac{\sqrt{3}}{2}\cdot\frac{1}{L} \qquad \dot{\varepsilon}_2 = \frac{\dot{u}_3 - \dot{u}_2}{L} \qquad \dot{\varepsilon}_3 = \frac{1}{2}\cdot\frac{\dot{u}_3}{L} - \frac{\sqrt{3}}{2}\cdot\frac{1}{L} \tag{26}$$

For a given set of $\dot{u}_2, \dot{u}_3,$ from (26) the strain rates $\dot{\varepsilon}_1, \dot{\varepsilon}_2, \dot{\varepsilon}_3$ are decided, and then $\ddot{U}(\dot{u}_2, \dot{u}_3)$ is obtained from (25).

By use of (23), the solutions are written in a rate form as:

$$\dot{u}_2 = \frac{4E_{t2}(E_{t3}-E_{t1}) - E_{t1}E_{t3}}{E_{t1}E_{t3} + 4E_{t2}E_{t3} + 4E_{t1}E_{t2}} \cdot \sqrt{3} \qquad \dot{u}_3 = \frac{4E_{t2}(E_{t3}-E_{t1}) + E_{t1}E_{t3}}{E_{t1}E_{t3} + 4E_{t2}E_{t3} + 4E_{t1}E_{t2}} \cdot \sqrt{3} \tag{27}$$

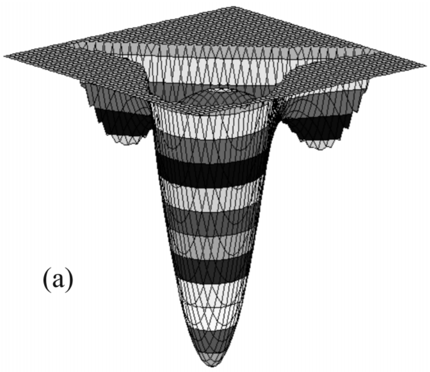
In this instance, only the distributed solution and three localized solutions are discussed. Consider two cases: (1) $E_e = 3.2\times10^{10}$, $\varepsilon_{e1} = \varepsilon_{e3} = 2.5\times10^{-4}$, $\varepsilon_{u1} = \varepsilon_{u3} = 2.5\times10^{-3}$, $\varepsilon_{e2} = 1.25\times10^{-4}$, $\varepsilon_{u2} = 2.5\times10^{-3}$; (2) $E_e = 3.2\times10^{10}$, $\varepsilon_{e1} = \varepsilon_{e3} = 2.5\times10^{-4}$, $\varepsilon_{u1} = \varepsilon_{u3} = 2.5\times10^{-3}$, $\varepsilon_{e2} = 1.25\times10^{-4}$, $\varepsilon_{u2} = 1.25\times10^{-3}$. For the distributed solution, all the three bars soften simultaneously beyond the bifurcation point, hence, $E_{t1} = E_{s1}$, $E_{t2} = E_{s2}$, $E_{t3} = E_{s3}$. For each localized solution, only one bar softens. Moreover, there are partially localized solutions, for which two bars soften and the other elastically unloads. The distributed solution and three localized solutions are listed in Table 1.

Table 1 The solutions for the three-bar truss

Case	Stiffness		Distributed	Localized 1	Localized 2	Localized 3
(1)	$E_e = 3.2 \times 10^{10}$	$\dot{u}_2$	–0.3616	2.291	–2.992	–2.179
	$E_{s1} = E_{s3} = -3.556 \times 10^9$	$\dot{u}_3$	0.3616	2.179	2.992	–2.291
	$E_{s2} = -1.684 \times 10^9$	$\ddot{U}$	–2.11 E+08	–6.193 E+08	–1.745 E+09	–6.193 E+08
(2)	$E_e = 3.2 \times 10^{10}$	$\dot{u}_2$	–0.1925	2.291	–15.59	–2.179
	$E_{s1} = E_{s3} = -3.556 \times 10^9$	$\dot{u}_3$	0.1925	2.179	15.59	–2.291
	$E_{s2} = -3.556 \times 10^9$	$\ddot{U}$	–237000	–6.193 E+08	–1.920 E+10	–6.193 E+08

3.3 The surface of the second-order work

The stability of the solutions can be evaluated with the aid of the second-order work. Stability is closely related to boundary conditions, and the distinction between stability under load and displacement control was discussed by Chen and Baker (2003c). For displacement control, from (4) we know $\Delta^2\Pi = \Delta^2 U$. Hence, the surface of the second-order work is shown by plotting $\ddot{U}(\dot{u}_2, \dot{u}_3)$ with $\dot{u}_2$ and $\dot{u}_3$ as variables. For the case (1), the surface of the second-order work near the extremum points is shown in Fig. 2. The surface has a local maximum point that corresponds to the distributed solution and three minimum points that correspond to the three localized solutions. The saddles that correspond to the three partially localized solutions cannot be seen clearly. For the case (2), Table 1 shows that the extremum points are far from each other and the difference between the corresponding values of the second-order work is immense. Hence, an illustration of the surface of the second-order work that contains all the solution points cannot clearly show the local topology around the extremum points. Thus, in Fig. 3, the topologies around each solution point are drawn separately. Obviously, the distributed solution is a local maximum point of the second-order work, and the localized solution 1 and 2 are the minimum points.



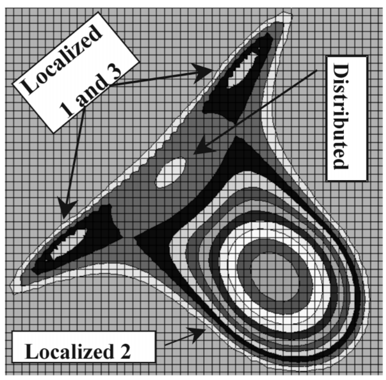


Fig. 2 The energy variation of the three-bar truss (for Case 1). (a) The energy profile close to the extremum points; (b) The contour plot of the locality close to the extremum points

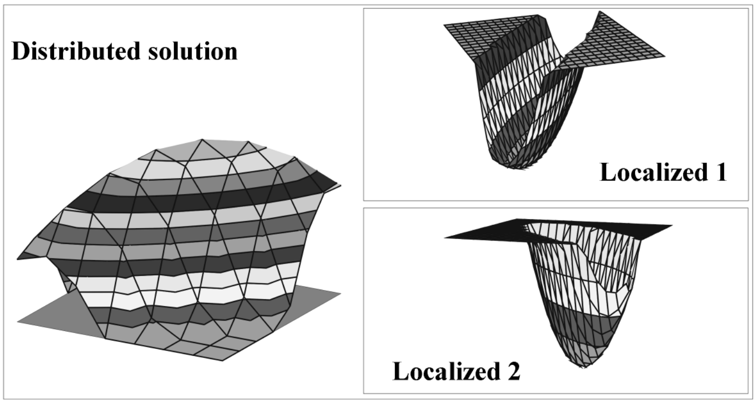


Fig. 3 The energy variation of the three-bar truss close to the extremum points (for Case 2)

Figs. 2 and 3 indicate that the localized solutions, which correspond to local minimum points, are stable and the distributed solution, which corresponds to the local maximum point, is unstable. At the bifurcation point, the stable fundamental solution (i.e., the distributed solution) starts to lose its stability. Beyond the bifurcation point, the single stable loading path splits into several post-bifurcation paths, and the surface of the second-order work exhibits three competing minima.

To determine which post-bifurcation path that the structure actually follows, Bazant (1989, 1991) proposes the concept of "stable path" and establishes a criterion for the "stable path" (i.e., the actual post-bifurcation path that the structure follows) on thermodynamics basis. The criterion states that the "stable path" maximizes the internally produced entropy among all the possible post-bifurcation equilibrium paths. By ignoring the temperature changes, the internally produced entropy can be expressed in terms of the second-order work. For displacement control, the second-order work must be minimized; while for load control it must be maximized (Bazant and Cedolin 1991, p658). For mixed control, a mixed free energy is introduced; the real path minimizes the second-order increment of the mixed free energy.

Obviously, Bazant's general principle cannot be applied directly to determine the real post-bifurcation path. Following Petryk and Thermann (1992), in this article, it is supposed that the real post-bifurcation path corresponds to the absolute minimum point of the second-order work. Hence, only the localized solution 2 for the three-bar truss is the real post-bifurcation equilibrium state that the structure follows. In the next section, the proposal will be proved equivalent to Bazant's path criterion derived on thermodynamics basis.

4. Path criterion

For brevity, in the following discussion we turn to the discrete model, though it is not difficult to continue this discussion by a lengthy variational derivation.

Let $\mathbf{q}$ be the nodal displacement vector, it consists of two parts, the constrained part $\mathbf{q}_c$ (resulting from $\bar{u}_i$), and the free part $\mathbf{q}_f$, i.e., $\mathbf{q} = \begin{bmatrix} \mathbf{q}_f \\ \mathbf{q}_c \end{bmatrix}$. The nodal forces are represented by vector $\mathbf{P} = \begin{bmatrix} \mathbf{P}_c \\ \mathbf{P}_f \end{bmatrix}$; here

$\mathbf{P}_c$, which corresponds to $\mathbf{q}_f$, results from the integral of the volume force f_i, and the traction $\bar{P}_i$, on S_σ, and $\mathbf{P}_f$ represents the reactions of the constraints $\mathbf{q}_c$. Let $\Delta\mathbf{q} = \begin{bmatrix} \Delta\mathbf{q}_f \\ \Delta\mathbf{q}_c \end{bmatrix}$ and $\Delta\mathbf{P} = \begin{bmatrix} \Delta\mathbf{P}_c \\ \Delta\mathbf{P}_f \end{bmatrix}$ and be the displacement increment and the force increment respectively, and $\mathbf{K} = \begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fc} \\ \mathbf{K}_{cf} & \mathbf{K}_{cc} \end{bmatrix}$ be the tangential stiffness matrix. From the symmetry, we know $\mathbf{K}_{cf} = \mathbf{K}_{fc}^T$. Then from (4) we have

$$\Delta^2\Pi = \frac{1}{2}\begin{bmatrix} \Delta\mathbf{q}_f^T & \Delta\mathbf{q}_c^T \end{bmatrix} \begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fc} \\ \mathbf{K}_{cf} & \mathbf{K}_{cc} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{q}_f \\ \Delta\mathbf{q}_c \end{bmatrix} - \Delta\mathbf{P}_c^T\Delta\mathbf{q}_f \tag{28}$$

The possible equilibrium states are decided by

$$\delta(\Delta^2\Pi) = 0 \tag{29}$$

Since $\Delta\mathbf{q}_c$ is fixed and $\Delta\mathbf{q}_f$ is unknown, thus only $\Delta\mathbf{q}_f$ in (28) can have a variation of $\delta(\Delta\mathbf{q}_f)$. Hence, from (29) we have the equilibrium equation

$$\mathbf{K}_{ff}\Delta\mathbf{q}_f + \mathbf{K}_{fc}\Delta\mathbf{q}_c = \Delta\mathbf{p}_c \tag{30}$$

from which we have

$$\Delta\mathbf{q}_f = \mathbf{K}_{ff}^{-1}[\Delta\mathbf{P}_c - \mathbf{K}_{fc}\Delta\mathbf{q}_c] \tag{31}$$

Then we can obtain the reaction increment $\Delta\mathbf{P}_f$ that associates with $\Delta\mathbf{q}_c$:

$$\Delta\mathbf{P}_f = \mathbf{K}_{cf}\Delta\mathbf{q}_f + \mathbf{K}_{cc}\Delta\mathbf{q}_c \tag{32}$$

By combining (30) and (32), we have

$$\begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fc} \\ \mathbf{K}_{cf} & \mathbf{K}_{cc} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{q}_f \\ \Delta\mathbf{q}_c \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{P}_c \\ \Delta\mathbf{P}_f \end{bmatrix} \tag{33}$$

With substitution of (33) into (28), we obtain

$$\Delta^2\Pi = \frac{1}{2}(\Delta\mathbf{P}_f^T\Delta\mathbf{q}_c - \Delta\mathbf{P}_c^T\Delta\mathbf{q}_f) \tag{34}$$

Let $\hat{W}$ denote the work done by the external agency through the loading history. It includes the work done by the external force $\mathbf{P}_c$ on $\mathbf{q}_f$ and the work done by the reaction $\mathbf{P}_f$ on the prescribed displacements $\mathbf{q}_c$. Its second-order increment is

$$\Delta^2\hat{W} = \frac{1}{2}(\Delta\mathbf{P}_c^T\Delta\mathbf{q}_f + \Delta\mathbf{P}_f^T\Delta\mathbf{q}_c) \tag{35}$$

Assume that the considered state $\Omega^{(N)}$ is a bifurcation point. At this point, the stiffness does not change smoothly. For the sake of clarity, consider two possible post-bifurcation paths. For a set of prescribed increments ($\Delta\mathbf{P}_c$, $\Delta\mathbf{q}_c$), we have two equilibrium solutions, $(\Delta\mathbf{P}_f^1, \Delta\mathbf{q}_f^1)$ and $(\Delta\mathbf{P}_f^2, \Delta\mathbf{q}_f^2)$. For the different post-bifurcation path, the tangential stiffness matrix is different, so the bifurcation conditions are expressed as

$$\begin{bmatrix} \mathbf{K}_{ff}^1 & \mathbf{K}_{fc}^1 \\ \mathbf{K}_{cf}^1 & \mathbf{K}_{cc}^1 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{q}_f^1 \\ \Delta\mathbf{q}_c \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{P}_c \\ \Delta\mathbf{P}_f^1 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{K}_{ff}^2 & \mathbf{K}_{fc}^2 \\ \mathbf{K}_{cf}^2 & \mathbf{K}_{cc}^2 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{q}_f^2 \\ \Delta\mathbf{q}_c \end{bmatrix} = \begin{bmatrix} \Delta\mathbf{P}_c \\ \Delta\mathbf{P}_f^2 \end{bmatrix} \tag{36}$$

Unlike the bifurcation caused by geometrical non-linearity, the determinant of the stiffness matrix does not vanish; the bifurcation happens due to the stiffness matrix being different for different post-bifurcation path. For the two assumed solutions, $\Delta^2\Pi$ has the values

$$\Delta^2\Pi^{(1)} = \frac{1}{2}[(\Delta\mathbf{P}_f^1)^T\Delta\mathbf{q}_c - \Delta\mathbf{P}_c{}^T\Delta\mathbf{q}_f^1] \qquad \Delta^2\Pi^{(2)} = \frac{1}{2}[(\Delta\mathbf{P}_f^2)^T\Delta\mathbf{q}_c - \Delta\mathbf{P}_c{}^T\Delta\mathbf{q}_f^2] \tag{37}$$

Assume that the first solution corresponds to the absolute minimum of $\Delta^2\Pi$ with respect to all kinematically admissible displacement increments, i.e.

$$\Delta^2\Pi^{(1)} \le \Delta^2\Pi^{(2)} \tag{38}$$

We need prove that this solution satisfies Bazant's path criterion, i.e., it minimizes the second-order work for displacement control, and maximizes it for load control.

- *Displacement Control*: In this case, $\Delta\mathbf{P}_c = \mathbf{0}$, from (34) we have

$$\Delta^2\Pi = \frac{1}{2}\Delta\mathbf{P}_f{}^T\Delta\mathbf{q}_c \tag{39}$$

On the other hand, from (35) we have the second-order increment of the external work

$$\Delta^2\hat{W} = \frac{1}{2}\Delta\mathbf{P}_f{}^T\Delta\mathbf{q}_c = \Delta^2\Pi \tag{40}$$

In this case, the real solution minimizes the second-order increment of the external work.

- *Force Control*: In this case, $\Delta\mathbf{q}_c = \mathbf{0}$, from (34) we have

$$\Delta^2\Pi = -\frac{1}{2}\Delta\mathbf{P}_c{}^T\Delta\mathbf{q}_f \tag{41}$$

From (35), the second-order increment of the external work is

$$\Delta^2\hat{W} = \frac{1}{2}\Delta\mathbf{P}_c{}^T\Delta\mathbf{q}_f = -\Delta^2\Pi \tag{42}$$

Obviously, the real solution maximizes the second-order increment of the external work.

• *Mixed Control*: In these circumstances, Bazant (see Bazant and Cedolin 1991, p654) introduced the mixed Helmholtz and Gibbs free energy Z, the second-order increment of which is,

$$\Delta^2 Z = \frac{1}{2}(\Delta \mathbf{P}_f^{\ T} \Delta \mathbf{q}_c - \Delta \mathbf{P}_c^{\ T} \Delta \mathbf{q}_f) \tag{43}$$

Obviously $\Delta^2 Z = \Delta^2 \Pi$, hence, the first solution, which minimizes $\Delta^2 \Pi$, will minimize the second-order increment of Z.

As a result, the solution, which is assumed to absolutely minimize $\Delta^2 \Pi$, will meet Bazant's path criterion. Hence, absolute minimization of $\Delta^2 \Pi$ can be used to seek the real solution among all the possible post-bifurcation solutions.

5. Conclusions

Softening plasticity involves multiple equilibrium paths. As the fundamental path loses its stability, a bifurcation occurs. Beyond the bifurcation point, the single stable loading path splits into several post-bifurcation paths and the second-order work exhibits several competing minima. By making use of a three-bar truss, the surface of the second-order work close to the bifurcation point has been drawn. The stability of the post-bifurcation equilibrium paths is discussed with the aid of the second-order work. Based on the analyses above, the following conclusions are reached:

(1) Among the multiple post-bifurcation equilibrium states, the localized solutions, which correspond to the local minimum points, are stable; the distributed solution and the partially-localized solutions, which correspond to the local maximum point and saddles, are unstable;

(2) There are multiple stable post-bifurcation equilibrium states. Among competing stable post-bifurcation equilibrium states, it is proposed that the structure should follow the path corresponding to the absolute minimum point of the second-order work;

(3) The proposal has been proved equivalent to Bazant's path criterion derived on a thermodynamics basis.

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