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Dynamic analysis of gradient elastic flexural beams

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Abstract. Gradient elastic flexural beams are dynamically analysed by analytic means. The governing equation of flexural beam motion is obtained by combining the Bernoulli-Euler beam theory and the simple gradient elasticity theory due to Aifantis. All possible boundary conditions (classical and non-classical or gradient type) are obtained with the aid of a variational statement. A wave propagation analysis reveals the existence of wave dispersion in gradient elastic beams. Free vibrations of gradient elastic beams are analysed and natural frequencies and modal shapes are obtained. Forced vibrations of these beams are also analysed with the aid of the Laplace transform with respect to time and their response to loads with any time variation is obtained. Numerical examples are presented for both free and forced vibrations of a simply supported and a cantilever beam, respectively, in order to assess the gradient effect on the natural frequencies, modal shapes and beam response.

Key words: beams; gradient elasticity; flexural vibrations; non-classical boundary conditions; free vibrations; forced vibrations.

1. Introduction

For certain materials, such as polymers, polycrystals or granular materials, the microstructure plays an important role in their mechanical behavior and has to be taken into account. These microstructural effects can be successfully taken into account in the framework of linear elastic macroscopic theories with the aid of higher-order gradient, micropolar and couple stress theories.

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For a literature review on these linear elastic microstructural theories one can consult Tiersten and Bleustein (1974), Lakes (1995) and Exadaktylos and Vardoulakis (2001). Solution of various boundary value problems of static and dynamic linear elasticity with microstructure has revealed that singularities of classical elasticity disappear, size effects are easily captured and wave dispersion is observed in cases where this was not possible in classical elasticity (e.g. Tiersten and Bluestein 1974, Lakes 1995, Vardoulakis and Sulem 1995, Exadaktylos and Vardoulakis 2001, Ru and Aifantis 1993, Altan *et al.* 1996, Chang and Gao 1997, Georgiadis and Vardoulakis 1998).

In particular, the problem of bending of beams under static loading has been studied by non-classical theories of elasticity mainly in order to explain test results, which could not be explained by classical elasticity theory. Thus Lakes (1983, 1986, 1995) investigated the dependence of the flexural rigidity of rods, made of various polymeric foams, upon specimen size both experimentally and by using the Cosserat (micropolar elasticity) theory. Vardoulakis *et al.* (1998) studied the effect of the beam length on the failure load and the variation of the beam curvature along the beam length both experimentally and on the basis of a gradient theory with surface energy for Timoshenko beams in flexure. Tsagrakis (2001) briefly considered the case of pure bending of elastic Bernoulli-Euler rods and verified the results of Lakes (1983, 1986) by using the simple gradient elasticity theory of Aifantis (Ru and Aifantis 1993, Altan *et al.* 1996) and a gradient elasticity theory with surface energy. Very recently, Papargyri-Beskou *et al.* (2002) studied analytically the problems of bending and buckling of Bernoulli-Euler beams on the basis of a simple theory of gradient elasticity with surface energy in a systematic and general way.

In this paper Bernoulli-Euler flexural beams are dynamically analysed by analytic means on the basis of the simple theory of gradient elasticity due to Aifantis (Ru and Aifantis 1993, Altan *et al.* 1996). The governing equation of flexural beam motion is derived both by combining the corresponding basic equations and by using a variational statement. All possible boundary conditions (classical and non-classical) are obtained with the aid of a variational statement. A wave propagation analysis easily reveals the existence of wave dispersion. Free and forced flexural vibrations of simple beams are also studied and the gradient effect on the natural frequencies and modal shapes as well as the transient beam response, respectively, is assessed with the aid of numerical examples dealing with a simply supported (for free vibrations) and a cantilever (for forced vibrations) beam.

This work represents the first study on the gradient effect in flexural beam dynamics. To be sure, dynamic analysis of gradient elastic beams in axial motion has been considered by Altan *et al.* (1996), Chang and Gao (1997) and more recently by Tsepoura *et al.* (2002) but, to the authors' best knowledge, dynamic analysis of gradient elastic beams in flexural motion has not been done before.

2. Governing equation and boundary conditions for beam in flexural motion

Consider a straight prismatic beam of length L subjected to a dynamic lateral load $q(x, t)$ distributed along the longitudinal axis x of the beam with t denoting time, as shown in Fig. 1. The cross section A of the beam is characterized by the two axes y (vertical) and z (horizontal) with the former one being its axis of symmetry. Thus, the bending plane coincides with the yx plane. According to the one-dimensional gradient elasticity theory due to Aifantis (Ru and Aifantis 1993, Altan *et al.* 1996) written with the notation of Vardoulakis and Sulem (1995), the Cauchy, double and total stresses τ, μ and σ, respectively, are given by

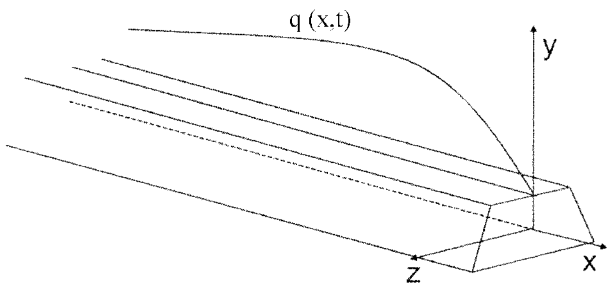

Fig. 1 Geometry and loading of a straight prismatic beam

$$\tau = E\varepsilon \tag{1}$$

$$\mu = g^2 E\varepsilon' \tag{2}$$

$$\sigma = \tau - \mu' = E(\varepsilon - g^2\varepsilon'') \tag{3}$$

where ε represents the axial strain of the beam. In Eqs. (1)-(3) the constant g (gradient coefficient) respresents material length, E is the elastic modulus and primes indicate differentiaition with respect to x.

Conditions of dynamic equilibrium imply that

$$\int_A \sigma dA = 0, \qquad \int_A \sigma y dA = -M \tag{4}$$

$$\frac{dM}{dx} = V, \qquad \frac{dV}{dx} = -q - m\ddot{v} \tag{5}$$

where M and V denote bending moment and shear force, respectively, m represents the mass per unit length of the beam, $v = v(x, t)$ is the lateral deflection of the beam and overdots indicate differentiation with respect to time. Furthermore, according to the Bernoulli-Euler hypothesis one has

$$\varepsilon = -y\frac{d^2v}{dx^2} \tag{6}$$

Introduction of Eqs. (3) and (6) into Eqs. (4) renders the latter ones in the form

$$E(-v'' + g^2 v^{IV})\int_A y dA = 0 \tag{7}$$

$$E(-v'' + g^2 v^{IV})\int_A y^2 dA = -M \tag{8}$$

Eq. (7) is satisfied for $\int_A y dA = 0$, indicating that the x axis is centroidal, while Eq. (8) takes the form

$$EI(v'' - g^2 v^{IV}) = M \tag{9}$$

where $I = \int_A y^2 dA$ stands for the moment of intertia of the cross-section of the beam about its z axis. Finally, differentiation of Eq. (9) twice with respect to x and use of Eq. (5) result into the governing equation of flexural motion of the beam of the form

$$EI(v^{IV} - g^2 v^{VI}) + m\ddot{v} = -q \tag{10}$$

The above governing equation of motion can also be obtained with the aid of the variational (Hamilton's) principle

$$\delta \int_{t_0}^{t_1} (U - K - W) dt = 0 \tag{11}$$

where U is the strain energy, K is the kinetic energy, W is the total work done by external forces, t_0 and t_1 denote time instants and δ indicates variation. For the present one dimensional case, the strain energy takes the form (Vardoulakis and Sulem 1995)

$$U = \frac{1}{2}\int_A \int_0^L (\tau\varepsilon + \mu\varepsilon') dx\, dA \tag{12}$$

which, in view of Eqs. (1), (2) and (6), becomes

$$U = \frac{1}{2}\int_0^L EI[(v'')^2 + g^2 (v''')^2] dx \tag{13}$$

According to the calculus of variations, the variation of an integral of the type $H = \int_0^L F(v'', v''') dx$ is obtained through the relation (Lanczos 1970)

$$\delta H = \int_0^L \left[\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial v''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial v'''}\right)\right] \delta v dx + $$

$$+ \left[\left[-\frac{d}{dx}\left(\frac{\partial F}{\partial v''}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial v'''}\right)\right]\delta v\right]_0^L + \left[\left[\frac{\partial F}{\partial v''} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'''}\right)\right]\delta v'\right]_0^L +$$

$$+ \left[\frac{\partial F}{\partial v'''}\delta v''\right]_0^L \tag{14}$$

In the present case $H = U$ of Eq. (13) and the Lagrangian function F is

$$F = \frac{EI}{2}[(v'')^2 + g^2 (v''')^2] \tag{15}$$

Eqs. (14) and (15) help to express δU in the form

$$\delta U = \int_0^L EI(v^{IV} - g^2 v^{VI}) \delta v dx + [EI(g^2 v^V - v''')\delta v]_0^L +$$

$$[EI(v'' - g^2 v^{VI})\delta v']_0^L + [EIg^2 v''' \delta v'']_0^L \tag{16}$$

On the other hand, the variation of the work done by the external force q, the boundary shear force V and the boundary classical and double moments M and M_d, respectively, reads

$$\delta W = -\int_0^L q\delta v dx - [V\delta v]_0^L + [M\delta v']_0^L + [M_d dv'']_0^L \tag{17}$$

Finally, the variation of the kinetic energy of the beam has the form

$$\delta\int_{t_0}^{t_1} K dt = \delta\int_{t_0}^{t_1}\int_0^L \frac{1}{2}m(\dot{v})^2 dx dt = \int_0^L \delta\int_{t_0}^{t_1} m(\dot{v})^2 dt dx \tag{18}$$

which after integration by parts with respect to time becomes

$$\delta\int_{t_0}^{t_1} K dt = -\int_{t_0}^{t_1}\int_0^L m\ddot{v}\delta v dx dt + \int_0^L m[\dot{v}\delta v]_{t_0}^{t_1} dx \tag{19}$$

Thus, in view of Eqs. (16), (17) and (19), Eq. (11) takes the form

$$\begin{aligned}&\int_{t_0}^{t_1}\int_0^L [EI(v^{IV} - g^2 v^{VI}) + m\ddot{v} + q]\delta v dx dt + \\ &+ \int_{t_0}^{t_1}[\{V - EI(v''' - g^2 v^{V})\}\delta v]_0^L dt - \int_{t_0}^{t_1}[\{M - EI(v'' - g^2 v^{IV})\}\delta v']_0^L dt - \\ &- \int_{t_0}^{t_1}[\{M_d - EIg^2 v'''\}\delta v'']_0^L dt + \int_0^L m[\dot{v}\delta v]_{t_0}^{t_1} dx = 0\end{aligned} \tag{20}$$

The above variational equation implies that each integrand of Eq. (20) must be equal to zero. This leads to the governing equation of motion identical to Eq. (10) derived with the aid of the basic equations and also to all possible boundary conditions (classical and non-classical) of the problem. Thus, the boundary conditions satisfy the equations

$$\begin{aligned}&[V(L,t) - EI(v'''(L,t) - g^2 v^{V}(L,t))]\delta v(L,t) - [V(0,t) - EI(v'''(0,t) - g^2 v^{V}(0,t)]\delta v(0,t) = 0 \\ &[M(L,t) - EI(v''(L,t) - g^2 v^{IV}(L,t))]\delta v'(L,t) - [M(0,t) - EI(v''(0,t) - g^2 v^{IV}(0,t))]\delta v'(0,t) = 0 \\ &[M_d(L,t) - EIg^2 v'''(L,t)]\delta v''(L,t) - [M_d(0,t) - EIg^2 v'''(0,t)]\delta v''(0,t) = 0\end{aligned} \tag{21}$$

while the end time conditions satisfy the equation

$$\dot{v}(x, t_1)\delta v(x, t_1) - \dot{v}(x, t_0)\delta v(x, t_0) = 0 \tag{22}$$

For example, if one assumes the four classical boundary conditions to be $v(0, t)$, $v(L, t)$, $v'(0, t)$ and $v'(L, t)$ prescribed and the corresponding non-classical ones to be $v''(0, t)$ and $v''(L, t)$ prescribed, then $\delta v(0, t) = \delta v(L, t) = 0$, $\delta v'(0, t) = \delta v'(L, t) = 0$, $\delta v''(0, t) = \delta v''(L, t) = 0$ and Eqs. (21) are all satisfied. One can also observe from (21) that, when dealing with the classical boundary conditions, either v or $V = EI(v''' - g^2 v^{V})$ and v' or $M = EI(v'' - g^2 v^{IV})$ at the boundaries of the beam have to be specified, while when dealing with the non-classical boundary conditions, either v'' or $M_d = EIg^2 v'''$ at the beam boundaries have to be specified. Concerning now Eq. (22), one can observe that if, e.g., $v(x, t_0)$ and $v(x, t_1)$ are prescribed, then $\delta v(x, t_0) = \delta v(x, t_1) = 0$ and Eq. (22) is satisfied.

3. Flexural wave propagation

Consider a gradient elastic beam, infinitely long and governed by the equation of motion (10) with $q = 0$, i.e. by the equation

$$v^{IV} - g^2 v^{VI} + (1/\alpha^2)\ddot{v} = 0 \tag{23}$$

where

$$\alpha^2 = EI/m \tag{24}$$

In order to study the propagation of harmonic waves one assumes a solution of the form

$$v = A \exp[i(\gamma x - \omega t)] \tag{25}$$

where A is the amplitude, $i = \sqrt{-1}$, γ the wave number and ω the circular frequency of propagation. Substitution of solution (25) into Eq. (23) leads to a frequency-wavenumber relation of the form

$$\gamma^4 - g^2\gamma^6 - (\omega^2/\alpha^2) = 0 \tag{26}$$

Since the phase velocity $c = \omega/\gamma$, one can obtain from (26) the phase velocity - wave number relation or dispersion relation in the form

$$c = \pm\alpha\gamma\sqrt{1 - g^2\gamma} \tag{27}$$

Fig. 2 shows the positive branch of the dispersion curve (27) for various values of the gradient coefficient g, together with the classical dispersion curve (for $g = 0$), which is a straight line. It is observed that for the classical case, the phase velocity increases without limit for increasing wavenumber or shorter wavelength, which is certainly an anomaly. This anomaly disappears for the gradient case where the phase velocity increases for increasing wavenumber but not without limit, demonstrating the positive effect of the gradient theory.

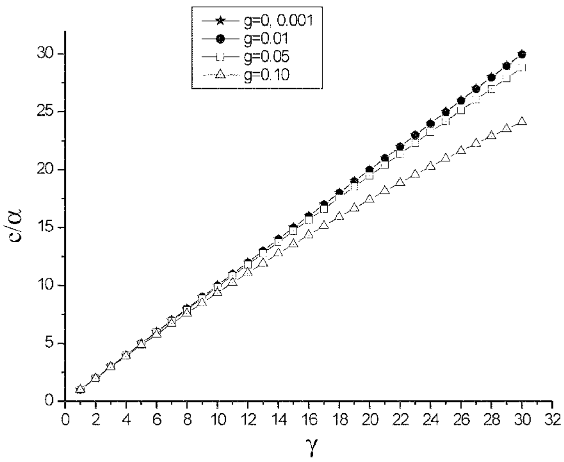

Fig. 2 Dispersion curves (phase velocity-wavenumber relation) for various values of g

4. Free flexural vibrations

The problem of free flexural vibrations of finite beams consists of determining the natural frequencies and modal shapes of these beams. Consider the free flexural motion of a beam described by Eq. (23). Assuming a solution of the form

$$v(x, t) = v_o(x)e^{i\omega t} \tag{28}$$

one obtains from (23) the equation

$$v_o^{IV} - g^2 v_o^{VI} - (\omega^2/\alpha^2)v_o = 0 \tag{29}$$

which has a solution of the form

$$v_o(x) = \sum_{i=1}^{6} C_i e^{\lambda_i x} \tag{30}$$

where the, in general complex, exponents λ_i are the roots of the algebraic equation

$$\lambda^4 - g^2\lambda^6 - (\omega^2/\alpha^2) = 0 \tag{31}$$

and C_i are constants of integration to be determined with the aid of the boundary conditions.

For example, for the case of a simply supported beam of length L the classical boundary conditions are

$$v_0(0) = v_0(L) = 0, \tag{32}$$

$$M(0) = M(L) = 0, \tag{33}$$

while for the non-classical ones a possible choice is

$$v_0''(0) = v_0''(L) = 0 \tag{34}$$

The above boundary conditions satisfy Eq. (21) written in the frequency domain according to (28). Indeed Eqs. (32) and (34) imply that $\delta v_o(0) = \delta v_o(L) = 0$ and $\delta v_o''(0) = \delta v_o''(L) = 0$, respectively and thus Eqs. $(21)_1$ and $(21)_3$ are satisfied, while Eqs. (33) not only imply the satisfaction of Eq. $(21)_2$ but in view of (34) lead to the conditions

$$v_0^{IV}(0) = v_0^{IV}(L) = 0 \tag{35}$$

Thus the six boundary conditions, exclusively in terms of displacements, are given by Eqs. (32), (34) and (35).

Applying the above boundary conditions to Eq. (30) and the equations

$$v_o''(x) = \sum_{i=1}^{6} \lambda_i^2 C_i e^{\lambda_i x} \tag{36}$$

$$v_0^{IV}(x) = \sum_{i=1}^{6} \lambda_i^4 C_i e^{\lambda_i x} \tag{37}$$

one obtains the following system of equations in matrix form to be solved for the constants C_i:

$$[A(\omega)]\{C\} = \{0\} \tag{38}$$

In the above, $\{C\} = \{C_1, C_2, \ldots, C_6\}^T$ and for $i = 1, 2, \ldots 6$,

$$\begin{aligned} A_{1i} &= 1, & A_{2i} &= e^{\lambda_i L} \\ A_{3i} &= \lambda_i^2, & A_{4i} &= \lambda_i^2 e^{\lambda_i L} \\ A_{5i} &= \lambda_i^4, & A_{6i} &= \lambda_i^4 e^{\lambda_i L} \end{aligned} \tag{39}$$

where the $\lambda_i = \lambda_i(\omega)$ are the six roots of Eq. (31) and no summation is implied over repeated indices.

In order for the system (38) to have non-zero solution for $\{C\}$, the condition

$$\det[A(\omega)] = 0 \tag{40}$$

should be satisfied. Eq. (40) is the frequency equation and provides all the natural frequencies of the gradient elastic simply supported beam. The solution of Eq. (40) is accomplished numerically using complex arithmetic due to the fact that λ_i's are, in general, complex. Thus, for a sequence of values of ω, one computes the corresponding values of the real function $D(\omega) = \ln|\det[A(\omega)]|$ and those values of ω for which $D(\omega)$ attains minimum values, are the natural frequencies. For more details on this approach, one can look at the book by Kitahara (1985).

Once the natural frequencies $\omega_k(k = 1, 2, \ldots, \infty)$ have been obtained from (40), one can go again to Eq. (38) and assuming $C_{1k} = 1$ to solve it for the remaining $C_{2k}, C_{3k}, \ldots, C_{6k}$ constants, each time for a different ω_k. Thus, with the aid of Eq. (30) one can obtain the k modal shapes corresponding to the k natural frequencies in the form

$$v_{ok}(x) = \sum_{i=1}^{6} C_{ik} e^{\lambda_{ik} x} \tag{41}$$

where there is no summation over the repeated index k and $\lambda_{ik} = \lambda_i(\omega_k)$.

Table 1 presents the first five normalized natural frequencies of a gradient elastic simply supported beam for various values of the normalized gradient coefficient g/L. The normalization of the gradient elastic natural frequencies ω_k^g is done by dividing them with the classical natural frequencies $\omega_k^c = (k^2\pi^2/L^2)\alpha$, where $k = 1, 2, \ldots., 5$. It is observed that natural frequencies increase for increasing values of the gradient coefficient for all the modes. This is explained as follows: Increasing values of the gradient coefficient decrease the static beam deflections (Papargyri-Beskou *et al.* 2002) and hence increase its stiffness leading to increased values of natural frequencies. Finally, it is also observed from Table 1 that for the same gradient coefficient there is a frequency increase from lower to higher modes with this increase becoming larger for higher values of the gradient coefficient.

Table 1 First five normalized natural frequencies of gradient elastic simply supported beam for various values of *g/L*

g/L	0.005	0.010	0.050	0.100
ω_1^g/ω_1^c	1.0001	1.0005	1.0122	1.0483
ω_2^g/ω_2^c	1.0005	1.0019	1.0481	1.1810
ω_3^g/ω_3^c	1.0010	1.0044	1.1054	1.3740
ω_4^g/ω_4^c	1.0019	1.0078	1.1809	1.6058
ω_5^g/ω_5^c	1.0030	1.0122	1.2715	1.8621

5. Forced flexural vibrations

The problem of forced flexural vibrations of finite beams consists of determining their response to any lateral load of any time variation. Consider the forced flexural motion of a beam described by Eq. (10). This equation will be solved with the aid of Laplace transform with respect to time, which for a function $f(x, t)$ is defined as $\bar{f}(x, s)$ and given by

$$\bar{f}(x, s) = \int_0^\infty f(x, t)e^{-st}dt \tag{42}$$

where s is the, in general complex, Laplace transform parameter.

Application of the above Laplace transform on Eq. (10) under zero initial conditions results in

$$\bar{v}^{IV} - g^2\bar{v}^{VI} + (s^2/\alpha^2)\bar{v} = -(1/EI)\bar{q} \tag{43}$$

Eq. (43) has a solution of the form

$$\bar{v}(x, s) = \bar{v}_h(x, s) + \bar{v}_p(x, s) \tag{44}$$

where $\bar{v}_h(x, s)$ is the solution of the homogeneous part of Eq. (43), i.e., of the equation

$$\bar{v}^{IV} - g^2\bar{v}^{VI} + (s^2/\alpha^2)\bar{v} = 0 \tag{45}$$

and $\bar{v}_p(x, s)$ is a particular solution of Eq. (43). Eq. (45) has a solution of the form

$$\bar{v}_h(x, s) = \sum_{i=1}^{6} K_i e^{\rho_i x} \tag{46}$$

where the, in general complex, exponents ρ_i are the roots of the algebraic equation

$$\rho^4 - g^2\rho^6 + (s^2/\alpha^2) = 0 \tag{47}$$

and K_i are constants of integration to be determined with the aid of the boundary conditions.

For example, consider the case of a cantilever beam of length L with its fixed end at $x = 0$, under a vertical concentrated load of magnitude P_o suddenly applied at the free end of the beam. For this case the classical boundary conditions are

$$\bar{v}(0, s) = 0, \qquad \bar{v}'(0, s) = 0 \tag{48}$$

$$\bar{M}(L, s) = 0, \qquad \bar{V}(L, s) = P_o/s \tag{49}$$

while for the non-classical ones a possible choice is

$$\bar{v}''(0, s) = 0, \qquad \bar{v}'''(L, s) = 0 \tag{50}$$

One can easily prove that the above boundary conditions satisfy Eq. (21) in the Laplace transformed domain. One can also observe that in this case $q(x, t) = 0$ and thus $\bar{v}(x, s) = \bar{v}_h(x, s)$ given by Eq. (46). In order to express all boundary conditions in terms of the deformation, Eqs. (49) are replaced by

$$\bar{v}''(L, s) - g^2\bar{v}^{IV}(L, s) = 0$$

$$-g^2 EI\bar{v}^{V}(L, s) = P_o/s \tag{51}$$

where use has been made in the expression for $\bar{V}(L, s)$ of the fact that $\bar{v}'''(L, s) = 0$ as described in $(50)_2$.

Applying the boundary conditions (48), (50) and (51) to Eq. (46) and the equations

$$\bar{v}'(x, s) = \sum_{i=1}^{6} \rho_i K_i e^{\rho_i x}, \qquad \bar{v}''(x, s) = \sum_{i=1}^{6} \rho_i^2 K_i e^{\rho_i x}$$

$$\bar{v}'''(x, s) = \sum_{i=1}^{6} \rho_i^3 K_i e^{\rho_i x}, \qquad \bar{v}^{IV}(x, s) = \sum_{i=1}^{6} \rho_i^4 K_i e^{\rho_i x}$$

$$\bar{v}^{V}(x, s) = \sum_{i=1}^{6} \rho_i^5 K_i e^{\rho_i x} \tag{52}$$

one obtains the following system of equations in matrix form to be solved for the constants K_i:

$$[B(s)]\{K\} = \{F(s)\} \tag{53}$$

In the above, $\{K\} = \{K_1, K_2, \ldots, K_6\}^T$, $\{F(s)\} = \{0, 0, 0, 0, 0, P_o/s\}^T$ and for $i = 1, 2, \ldots, 6$,

$$B_{1i} = 1, \qquad B_{2i} = \rho_i, \qquad B_{3i} = \rho_i^2$$

$$B_{4i} = \rho_i^3 e^{\rho_i L}, \qquad B_{5i} = \rho_i^2 e^{\rho_i L}(1 - g^2\rho_i^2)$$

$$B_{6i} = -g^2 EI\rho_i^5 e^{\rho_i L} \tag{54}$$

where the $\rho_i = \rho_i(s)$ are the six roots of Eq. (47) and no summation is implied over repeated indices.

Eq. (53) is solved for a sequence of values of s to determine the constants of integration K_i to be used in Eq. (46) to obtain the response $\bar{v}(x, s)$ in the Laplace transformed domain. A numerical inversion of this transformed response at any point x provides the time domain response of the beam. The sequence of values of s is determined by the particular algorithm of Laplace transform

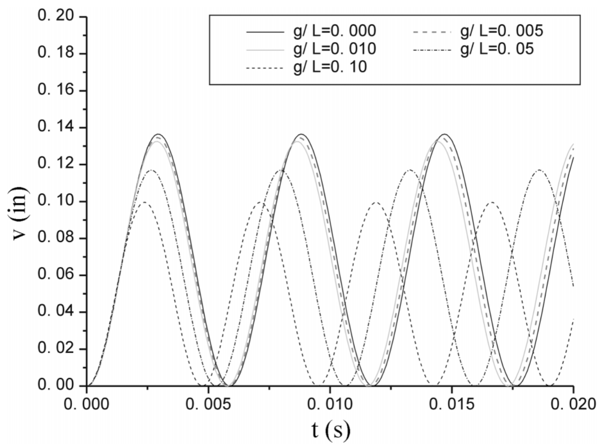

Fig. 3 Free end deflection of gradient elastic cantilever beam versus time for various values of *g/L*

inversion used. For every value of s one first solves Eq. (47) numerically to determine the roots ρ_i and then Eq. (53) to numerically determine the constants K_i. In this work the algorithm of Durbin (1974) is employed for the numerical inversion of Laplace transform because of its high accuracy (Narayanan and Beskos 1982). This algorithm works with complex values of s and combines both finite Fourier cosine and sine transforms.

Consider a numerical example with geometrical and material parameters taken from Beskos and Michael (1984), where a numerical solution for the classical case is available for comparison purposes. The data are as follows: L = 4 ft (= 1.219 m), I = 144 in^4 (= 5993.73 cm^4), $E = 3 \times 10^4$ ksi (= 20.685×10^4 MPa), m = 1.268 lbsec^2/ft^2 (= 60.709 Kg/m) and P_o = 8 k (= 35584 N). Fig. 3 shows the free end beam deflection versus time for various values of the normalized gradient coefficient g/L. It is observed that increasing values of g/L decrease the maximum values of the deflection and shift them to smaller time values of occurrence. The former observation is in agreement with the static results (Papargyri-Beskou *et al.* 2002), while the latter one is a consequence of the increase of natural frequencies for increasing values of g/L, observed in the previous section.

6. Conclusions

On the basis of the previous discussion the following conclusions can be drawn:

1) Using a simple theory of gradient elasticity due to Aifantis, the governing equation of flexural motion as well as all possible boundary conditions (classical and non-classical ones) for a Bernoulli-Euler beam have been derived with the aid of a variational principle.
2) Flexural harmonic wave propagation analysis in an infinitely long beam leads to a dispersion relation, which, unlike the classical case, does not imply increase of the phase velocity without limit for increasing wavenumber, thereby avoiding the anomaly of classical elasticity.
3) Free vibration analysis of finite beams reveals an increase of the natural frequencies for increasing values of the gradient coefficient for all the natural modes and a frequency increase from lower to higher modes for the same value of the gradient coefficient with this increase becoming larger for higher values of this coefficient.

4) Forced transient vibration analysis of finite beams reveals a decrease of the maximum values of the response and a shift of them to smaller time values of occurrence for increasing values of the gradient coefficient.

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