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# Stochastic optimal control of coupled structures

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**Abstract.** The stochastic optimal nonlinear control of coupled adjacent building structures is studied based on the stochastic dynamical programming principle and the stochastic averaging method. The coupled structures with control devices under random seismic excitation are first condensed to form a reduced-order structural model for the control analysis. The stochastic averaging method is applied to the reduced model to yield Itô stochastic differential equations for structural modal energies as controlled diffusion processes. Then a dynamical programming equation for the energy processes is established based on the stochastic dynamical programming principle, and solved to determine the optimal nonlinear control law. The seismic response mitigation of the coupled structures is achieved through the structural energy control and the dimension of the optimal control problem is reduced. The seismic excitation spectrum is taken into account according to the stochastic dynamical programming principle. Finally, the nonlinear controlled structural response is predicted by using the stochastic averaging method and compared with the uncontrolled structural response to evaluate the control efficacy. Numerical results are given to demonstrate the response mitigation capabilities of the proposed stochastic optimal control method for coupled adjacent building structures.

**Key words:** building structure; random vibration; optimal control; stochastic averaging; stochastic dynamical programming.

## 1. Introduction

Interconnecting adjacent high-rise structures with control devices to mitigate the seismic or wind response has been an active research subject in recent years. The control devices can generate control forces by utilizing the relative motion of connected structures. The passive control of coupled adjacent tall structures connected with linear or nonlinear devices has been widely studied (Gurley *et al.* 1994, Iwanami *et al.* 1996, Luco and De Barros 1998, Xu *et al.* 1999, Sugino *et al.* 1999, Ni *et al.* 2001). The active or semi-active control of coupled adjacent tall structures under

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seismic excitation has been evolved (Mitsuta and Seto 1992, Seto 1994, Luco and Wong 1994, Yamada *et al*. 1994, Matsumoto *et al*. 1999, Christenson *et al*. 1999). The linear quadratic control method was applied to determine the control forces of coupled structures in those studies. However, the nonlinear optimal control method is more effective than linear one in reducing the seismic response (Yang *et al*. 1996, Agrawal and Yang 1996, Zhu and Ying 1999, Zhu *et al*. 1999, 2000, 2001). The seismic or wind excitation acting on building structures is random in nature. It is more reasonable to apply the stochastic dynamical programming principle (Fleming and Rishel 1975, Stengel 1986) and to take into account the random excitation spectrum features for the seismic or wind response control of coupled structures.

In the present paper, the stochastic optimal nonlinear control of coupled adjacent building structures under random seismic excitation is studied based on the stochastic dynamical programming principle and the stochastic averaging method (Zhu and Lin 1991, Zhu *et al.* 1997). A reduced-order model of coupled structures with an arbitrary number of stories and with connecting control devices at any floors is first formulated. Itô stochastic differential equations for modal energies of the coupled structures are derived by using the stochastic averaging method. Then a dynamical programming equation is obtained by applying the stochastic dynamical programming principle to the energy processes. The nonlinear optimal control law is determined by the dynamical programming equation. The controlled structural response to random seismic excitation is predicted by using the stochastic averaging method and compared with the uncontrolled structural response to evaluate the control efficacy. A numerical study is conducted to demonstrate the seismic response mitigation capabilities of the proposed stochastic optimal control method.

## 2. Equation of motion

Consider two adjacent high-rise building structures respectively with $n_1$ and $n_2$ ($n_1 \geq n_2$) stories, interconnected by control devices at $n_3$ ($n_3 \leq n_2$) floors as shown in Fig. 1. It is assumed that the

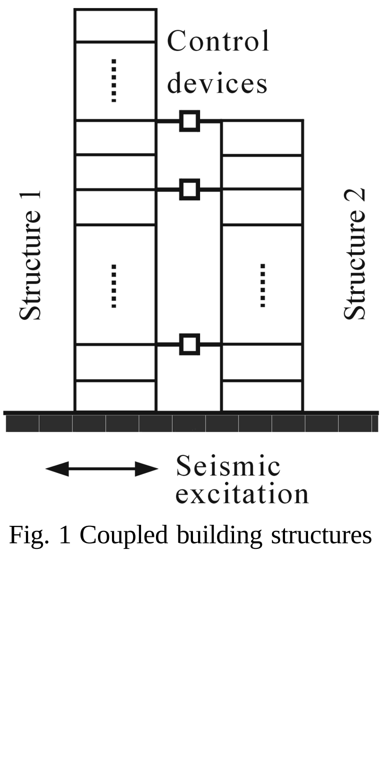


Fig. 1 Coupled building structures

coupled structures are subjected to a lateral ground acceleration excitation and the control forces are provided also in horizontal direction. The equations of motion of the shear-type coupled structures are of the form

$$M_1\ddot{X}_1 + C_1\dot{X}_1 + K_1X_1 = -\ddot{x}_g(t)M_1E_1 + P_1U \tag{1a}$$

$$M_2\ddot{X}_2 + C_2\dot{X}_2 + K_2X_2 = -\ddot{x}_g(t)M_2E_2 + P_2U \tag{1b}$$

where $X_i$ ($i$ = 1, 2) is the $n_i$-dimensional lateral displacement vector; $M_i$, $C_i$ and $K_i$ ($i$ = 1, 2) are the $n_i \times n_i$-dimensional symmetric positive-definite mass, damping and stiffness matrices of structure $i$ respectively; $E_i$ ($i$ = 1, 2) is the $n_i$-dimensional vector with unit elements; $U$ is the $n_3$-dimensional control force vector; and $P_i$ ($i$ = 1, 2) is the $n_i \times n_3$-dimensional matrix indicating the control devices placement. Note that the control forces of the coupled structures are exerted upon each other inversely. There exists the relation $P_1 = [0, -P_2^T]^T$. $\ddot{x}_g(t)$ represents the random ground acceleration excitation and for the seismic excitation, can be modeled as a non-white stationary random process with Kanai-Tajimi power spectrum density (Kanai 1957, Tajimi 1960). The random seismic excitation $\ddot{x}_g(t)$ is generated by

$$\ddot{x}_g = \omega_g^2 y + 2\zeta_g\omega_g\dot{y} \tag{2a}$$

$$\ddot{y} + 2\zeta_g\omega_g\dot{y} + \omega_g^2 y = \sigma\xi(t) \tag{2b}$$

where $y$ is the random response of filtering system (2b); $\xi(t)$ is a Gaussian white noise with unit intensity; $\sigma$ is the amplitude of the random excitation; $\omega_g$ and $\zeta_g$ are respectively the natural frequency and damping ratio of filtering system, which represent the characteristics of site soil.

The coupled structural response can be expressed using the substructuring concept and the modes of the corresponding uncoupled structures in the assumption that the higher-order mode effect is so slight as to be neglected. The first $m_i$ ($i$ = 1, 2) modes are taken for the response analysis of structure $i$ and are assembled into reduced mode matrix $\Phi_i$ normalized with respect to mass matrix $M_i$. The displacement response of the coupled structures can be represented as $X_i \cong \Phi_iQ_i$ based on the mode superposition method and then the coupled structural equations are represented by

$$\ddot{q}_{1i} + 2\zeta_{1i}\omega_{1i}\dot{q}_{1i} + \omega_{1i}^2 q_{1i} = -\beta_{1i}\ddot{x}_g(t) + v_{1i}, \qquad i = 1, 2, \ldots, m_1 \tag{3a}$$

$$\ddot{q}_{2i} + 2\zeta_{2i}\omega_{2i}\dot{q}_{2i} + \omega_{2i}^2 q_{2i} = -\beta_{2i}\ddot{x}_g(t) + v_{2i}, \qquad i = 1, 2, \ldots, m_2 \tag{3b}$$

where $q_{ji}$ ($j$ = 1, 2) is the $i$th element of modal displacement vector $Q_j$ of structure $j$; $\omega_{ji}$ and $\zeta_{ji}$ ($j$ = 1, 2) are the $i$th modal frequency and damping ratio respectively; $\beta_{ji} = \phi_{ji}^T M_j E_j$ ($j$ = 1, 2) is the coefficient of the $i$th modal excitation; and $v_{ji} = \phi_{ji}^T P_j U$ ($j$ = 1, 2) is the control force corresponding to the $i$th mode, in which $\phi_{ji}$ ($j$ = 1, 2) is the $i$th mode vector in mode matrix $\Phi_j$.

By combining Eqs. (3a,b) and (2a,b), the augmented matrix equation for the coupled structural and filtering system is obtained and rewritten in the following Itô differential form

$$dZ = (AZ + BU_v)dt + CdW(t) \tag{4}$$

where $W(t)$ is a unit Wiener process; $(2m_1 + 2m_2 + 2)$-dimensional state vector $Z$, $(2m_1 + 2m_2 + 2) \times (2m_1 + 2m_2 + 2)$-dimensional constant matrix $A$, $(2m_1 + 2m_2 + 2) \times (m_1 + m_2)$-dimensional constant matrix $B$, $(2m_1 + 2m_2 + 2)$-dimensional constant vector $C$ and $(m_1 + m_2)$-dimensional modal control force vector $U_v$ are respectively as

$$Z = [Q_1^T, \ \dot{Q}_1^T, \ Q_2^T, \ \dot{Q}_2^T, \ y, \ \dot{y}]^T \tag{5a}$$

$$A = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0 & I_1 \\ -\Phi_1^T K_1 \Phi_1 & -\Phi_1^T C_1 \Phi_1 \end{bmatrix} \tag{5b}$$

$$A_{22} = \begin{bmatrix} 0 & I_2 \\ -\Phi_2^T K_2 \Phi_2 & -\Phi_2^T C_2 \Phi_2 \end{bmatrix}, \quad A_{33} = \begin{bmatrix} 0 & 1 \\ -\omega_g^2 & -2\zeta_g \omega_g \end{bmatrix} \tag{5c}$$

$$A_{13} = \begin{bmatrix} 0 & 0 \\ -\omega_g^2 \Phi_1^T M_1 E_1 & -2\zeta_g \omega_g \Phi_1^T M_1 E_1 \end{bmatrix} \tag{5d}$$

$$A_{23} = \begin{bmatrix} 0 & 0 \\ -\omega_g^2 \Phi_2^T M_2 E_2 & -2\zeta_g \omega_g \Phi_2^T M_2 E_2 \end{bmatrix} \tag{5e}$$

$$B = \begin{bmatrix} 0 & I_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 & 0 \end{bmatrix}^T, \quad C = [0, 0, \ldots, 0, \sigma]^T \tag{5f}$$

$$U_v = [v_{11}, v_{12}, \ldots, v_{1m_1}, v_{21}, v_{22}, \ldots, v_{2m_2}]^T = [P_1^T \Phi_1, P_2^T \Phi_2]^T U \tag{5g}$$

in which $I_i$ ($i$ = 1, 2) is the $m_i \times m_i$-dimensional identity matrix.

## 3. Optimal control law

For the control analysis, a further reduced-order model of the coupled structures involving $m_3$ ($m_3 < m_1$) modes of structure 1 and $m_4$ ($m_4 < m_2$) modes of structure 2 can be obtained. The total energy $H_{jT}$ ($j$ = 1, 2) of the reduced-order model is represented by structural modal energies $H_{ji}$ ($i$ = 1, 2, …, $m_{j+2}$) based on the modal transformation, i.e.,

$$\begin{aligned} H_{jT} &= \frac{1}{2}(\dot{\bar{X}}_j^T M_j \dot{\bar{X}}_j + \bar{X}_j^T K_j \bar{X}_j) \cong \frac{1}{2}(\dot{\bar{Q}}_j^T \bar{\Phi}_j^T M_j \bar{\Phi}_j \dot{\bar{Q}}_j + \bar{Q}_j^T \bar{\Phi}_j^T K_j \bar{\Phi}_j \bar{Q}_j) \\ &= \frac{1}{2}(\dot{\bar{Q}}_j^T I_j \dot{\bar{Q}}_j + \bar{Q}_j^T \Omega_j \bar{Q}_j) = \sum_{i=1}^{m_{j+2}} H_{ji} \end{aligned} \tag{6}$$

where $\bar{X}_j$ is the displacement vector of the reduced-order model; $\bar{Q}_j = [q_{j1}, q_{j2}, \ldots, q_{jm_{j+2}}]^T$ is the reduced modal displacement vector; and $\bar{\Phi}_j = [\phi_{j1}, \phi_{j2}, \ldots, \phi_{jm_{j+2}}]$ is the reduced mode matrix. The seismic response control of the coupled structures can be achieved through the corresponding modal energy control. By applying the stochastic averaging method (Zhu and Lin 1991, Zhu *et al.* 1997) to the reduced structural model, the averaged Itô equation for the modal energies is obtained as follows

$$d\bar{H} = \left[\bar{m}(\bar{H}) + \frac{\partial\bar{H}}{\partial\dot{\bar{Q}}}\bar{U}_v\right]dt + \bar{\sigma}(\bar{H})d\bar{W}(t) \tag{7}$$

where the modal energy vector $\bar{H}$ of the reduced structural model, the reduced modal displacement vector $\bar{Q}$, the modal control force vector $\bar{U}_v$, the drift coefficient vector $\bar{m}(\bar{H})$, the diffusion coefficient matrix $\bar{\sigma}(\bar{H})$ and unit Wiener process vector $\bar{W}(t)$ are respectively as

$$\bar{H} = [\bar{H}_1^T, \bar{H}_2^T]^T = [H_{11}, H_{12}, \ldots, H_{1m_3}, H_{21}, H_{22}, \ldots, H_{2m_4}]^T \tag{8a}$$

$$\bar{Q} = [\bar{Q}_1^T, \bar{Q}_2^T]^T = [q_{11}, q_{12}, \ldots, q_{1m_3}, q_{21}, q_{22}, \ldots, q_{2m_4}]^T \tag{8b}$$

$$\bar{U}_v = [v_{11}, v_{12}, \ldots, v_{1m_3}, v_{21}, v_{22}, \ldots, v_{2m_4}]^T = [P_1^T\bar{\Phi}_1, P_2^T\bar{\Phi}_2]^T U \tag{8c}$$

$$\bar{m}(\bar{H}) = [\bar{m}_1^T, \bar{m}_2^T]^T = [m_{11}, m_{12}, \ldots, m_{1m_3}, m_{21}, m_{22}, \ldots, m_{2m_4}]^T \tag{8d}$$

$$\bar{\sigma}(\bar{H}) = \mathrm{diag}\{\bar{\sigma}_1, \bar{\sigma}_2\} = \mathrm{diag}\{\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1m_3}, \sigma_{21}, \sigma_{22}, \ldots, \sigma_{2m_4}\} \tag{8e}$$

$$\bar{W}(t) = [\bar{W}_{11}, \bar{W}_{12}, \ldots, \bar{W}_{1m_3}, \bar{W}_{21}, \bar{W}_{22}, \ldots, \bar{W}_{2m_4}]^T \tag{8f}$$

with

$$H_{1i} = (\dot{q}_{1i}^2 + \omega_{1i}^2 q_{1i}^2)/2, \qquad H_{2i} = (\dot{q}_{2i}^2 + \omega_{2i}^2 q_{2i}^2)/2 \tag{9a}$$

$$m_{1i}(H_{1i}) = -2\zeta_{1i}\omega_{1i}H_{1i} + \frac{1}{2}\beta_{1i}^2 S_g(\omega_{1i}) \tag{9b}$$

$$m_{2i}(H_{2i}) = -2\zeta_{2i}\omega_{2i}H_{2i} + \frac{1}{2}\beta_{2i}^2 S_g(\omega_{2i}) \tag{9c}$$

$$\sigma_{1i}^2(H_{1i}) = \beta_{1i}^2 H_{1i} S_g(\omega_{1i}), \qquad \sigma_{2i}^2(H_{2i}) = \beta_{2i}^2 H_{2i} S_g(\omega_{2i}) \tag{9d}$$

$$S_g(\omega) = \sigma^2 \frac{\omega_g^4 + 4\zeta_g^2\omega_g^2\omega^2}{(\omega_g^2 - \omega^2)^2 + 4\zeta_g^2\omega_g^2\omega^2} \tag{9e}$$

It is assumed that all states such as displacements and velocities or modal energies of the reduced structural model can be determined exactly by measurement. The optimal control problem is independent of state observation (Fleming and Rishel 1975, Stengel 1986). Eq. (6) implies that modal energy $\bar{H}$ is a controlled vector diffusion process. For the stochastic optimal control of the modal energy processes, the performance index in finite time interval is of the form

$$J = E\left[\int_{t_0}^{T} L(\overline{H}(\tau), U(\tau))d\tau + \Psi(\overline{H}(T))\right] \tag{10}$$

where $E[\cdot]$ denotes the expectation operator; $L$ is a continuous differentiable convex function; and $\Psi$ is a terminal state function. In the case of infinite time interval, the performance index becomes

$$J = \lim_{T\to\infty}\frac{1}{T}\int_{0}^{T} L(\overline{H}(\tau), U(\tau))d\tau \tag{11}$$

According to the stochastic dynamical programming principle (Fleming and Rishel 1975, Stengel 1986), the following dynamical programming equations are obtained

$$\frac{\partial V}{\partial t} = -\min_{U}\left\{L(\overline{H}, U) + \left(\frac{\partial V}{\partial \overline{H}}\right)^T\left[\overline{m}(\overline{H}) + \frac{\partial \overline{H}}{\partial \dot{\overline{Q}}}\overline{U}_v\right] + \frac{1}{2}\mathrm{tr}\left[\frac{\partial^2 V}{\partial \overline{H}^2}\overline{\sigma}(\overline{H})\overline{\sigma}^T(\overline{H})\right]\right\} \tag{12}$$

for controlled system (7) with finite time-interval performance index (10), or

$$\lambda = \min_{U}\left\{L(\overline{H}, U) + \left(\frac{\partial V}{\partial \overline{H}}\right)^T\left[\overline{m}(\overline{H}) + \frac{\partial \overline{H}}{\partial \dot{\overline{Q}}}\overline{U}_v\right] + \frac{1}{2}\mathrm{tr}\left[\frac{\partial^2 V}{\partial \overline{H}^2}\overline{\sigma}(\overline{H})\overline{\sigma}^T(\overline{H})\right]\right\} \tag{13}$$

for controlled system (7) with infinite time-interval performance index (11), where $\mathrm{tr}[\cdot]$ denotes the trace operator of square matrix; $V$ is a value function of $\overline{H}$ corresponding to the optimal control force; and $\lambda$ is a constant.

The stochastic optimal control law can be determined by minimizing the right-hand side of dynamical programming Eq. (12) or (13). Let function $L$ be of the form

$$L = g(\overline{H}) + \overline{U}_v^T R\overline{U}_v = g(\overline{H}_1, \overline{H}_2) + U^T R_p U \tag{14}$$

where $g$ is a continuous differentiable convex function; $R$ is a positive-definite symmetric weight matrix

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}, \qquad \begin{aligned} R_p &= [P_1^T\overline{\Phi}_1, P_2^T\overline{\Phi}_2]R[P_1^T\overline{\Phi}_1, P_2^T\overline{\Phi}_2]^T \\ &= P_1^T\overline{\Phi}_1 R_1\overline{\Phi}_1^T P_1 + P_2^T\overline{\Phi}_2 R_2\overline{\Phi}_2^T P_2 \end{aligned} \tag{15}$$

in which $R_1$ and $R_2$ are the $m_3 \times m_3$-dimensional and $m_4 \times m_4$-dimensional positive-definite symmetric constant matrices corresponding to structure 1 and structure 2 respectively. If the dimension of the reduced structural model is not less than the number of control devices, i.e., $m_3 + m_4 \geq n_3$ and the rank of matrix $[P_1^T\overline{\Phi}_1, P_2^T\overline{\Phi}_2]_{n_3\times(m_3+m_4)}$ is equal to $n_3$ for a certain placement of the control devices, $R_p$ is a positive-definite symmetric matrix. Then the optimal control force is

$$U^* = -\frac{1}{2}R_p^{-1}\left(P_1^T\overline{\Phi}_1\frac{\partial \overline{H}_1}{\partial \dot{\overline{Q}}_1}\frac{\partial V}{\partial \overline{H}_1} + P_2^T\overline{\Phi}_2\frac{\partial \overline{H}_2}{\partial \dot{\overline{Q}}_2}\frac{\partial V}{\partial \overline{H}_2}\right) \tag{16}$$

which acts as a generalized nonlinear damping force since the partial derivative of the $i$th mode energy $H_{ji}$ ($j$ = 1, 2) with respect to modal velocity $\dot{q}_{ji}$ is just the corresponding modal velocity $\dot{q}_{ji}$.

Substituting the optimal control force (16) into Eq. (12) or (13) and averaging terms involving the control forces yield the final dynamical programming equation, for example, in the case of infinite time interval as

$$\lambda = g(\bar{H}_1, \bar{H}_2) + \left(\frac{\partial V}{\partial \bar{H}_1}\right)^T \bar{m}_1(\bar{H}_1) + \left(\frac{\partial V}{\partial \bar{H}_2}\right)^T \bar{m}_2(\bar{H}_2) + \frac{1}{2}\bar{m}_{u1}^T(\bar{H})\frac{\partial V}{\partial \bar{H}_1}$$
$$+ \frac{1}{2}\bar{m}_{u2}^T(\bar{H})\frac{\partial V}{\partial \bar{H}_2} + \frac{1}{2}\mathrm{tr}\left[\frac{\partial^2 V}{\partial \bar{H}_1^2}\bar{\sigma}_1(\bar{H}_1)\bar{\sigma}_1^T(\bar{H}_1)\right] + \frac{1}{2}\mathrm{tr}\left[\frac{\partial^2 V}{\partial \bar{H}_2^2}\bar{\sigma}_2(\bar{H}_2)\bar{\sigma}_2^T(\bar{H}_2)\right] \tag{17}$$

where

$$\bar{m}_{u1}(\bar{H}) = -\frac{1}{2}\begin{Bmatrix} \phi_{11}^T P_u \phi_{11} H_{11}\dfrac{\partial V}{\partial H_{11}} \\ \phi_{12}^T P_u \phi_{12} H_{12}\dfrac{\partial V}{\partial H_{12}} \\ \vdots \\ \phi_{1m_3}^T P_u \phi_{1m_3} H_{1m_3}\dfrac{\partial V}{\partial H_{1m_3}} \end{Bmatrix}, \qquad \bar{m}_{u2}(\bar{H}) = -\frac{1}{2}\begin{Bmatrix} \phi_{21}^T P_w \phi_{21} H_{21}\dfrac{\partial V}{\partial H_{21}} \\ \phi_{22}^T P_w \phi_{22} H_{22}\dfrac{\partial V}{\partial H_{22}} \\ \vdots \\ \phi_{2m_4}^T P_w \phi_{2m_4} H_{2m_4}\dfrac{\partial V}{\partial H_{2m_4}} \end{Bmatrix} \tag{18a}$$

$$P_u = P_1 R_p^{-1} P_1^T, \qquad P_w = P_2 R_p^{-1} P_2^T \tag{18b}$$

The value function $V$ can be obtained from solving Eq. (17) and then the optimal control force $U^*$ is determined as a function of modal energy $\bar{H}$ or modal displacement $\bar{Q}$ and modal velocity $\dot{\bar{Q}}$. Suppose that function $g$ is of the form

$$g(\bar{H}_1, \bar{H}_2) = s_0 + \sum_{i=1}^{m_3} s_{1i}^a H_{1i} + \sum_{i=1}^{m_4} s_{2i}^a H_{2i} + \sum_{i=1}^{m_3} s_{1i}^b H_{1i}^2 + \sum_{i=1}^{m_4} s_{2i}^b H_{2i}^2$$
$$+ \sum_{i=1}^{m_3} s_{1i}^c H_{1i}^3 + \sum_{i=1}^{m_4} s_{2i}^c H_{2i}^3 + \sum_{i \neq j}^{m_3} s_{1ij}^b H_{1i} H_{1j} + \sum_{i \neq j}^{m_4} s_{2ij}^b H_{2i} H_{2j}$$
$$+ \sum_{i,j=1}^{m_3, m_4} s_{3ij}^b H_{1i} H_{2j} + O(H_{i_1 j_1} H_{i_2 j_2} H_{i_3 j_3}) \tag{19}$$

where $s_{1ij}^b = s_{1ji}^b, s_{2ij}^b = s_{2ji}^b$ and the weight coefficients are non-negative. Then a polynomial solution of the value function is obtained as follows

$$V(\bar{H}_1, \bar{H}_2) = \sum_{i=1}^{m_3} p_{1i}^a H_{1i} + \sum_{i=1}^{m_4} p_{2i}^a H_{2i} + \sum_{i=1}^{m_3} p_{1i}^b H_{1i}^2 + \sum_{i=1}^{m_4} p_{2i}^b H_{2i}^2$$
$$+ \sum_{i \neq j}^{m_3} p_{1ij}^b H_{1i} H_{1j} + \sum_{i \neq j}^{m_4} p_{2ij}^b H_{2i} H_{2j} + \sum_{i,j=1}^{m_3, m_4} p_{3ij}^b H_{1i} H_{2j} \tag{20}$$

where $p_{1ij}^b = p_{1ji}^b$ and $p_{2ij}^b = p_{2ji}^b$. The weight coefficients in the value function are determined by Eq. (17) for certain coefficients $s_{1i}^a, s_{2i}^a, s_{1i}^c, s_{2i}^c, s_{1ij}^b, s_{2ij}^b$ and $s_{3ij}^b$.

## 4. Response prediction

The stochastic-optimally controlled structures are nonlinear due to the control forces. The random response of the nonlinear coupled structures can be predicted by using the stochastic averaging method. Substituting the optimal control force (16) into the coupled structural Eq. (4) and applying the averaging method to it yield the following averaged Itô equation for the modal energies

$$dH = [m(H) + m_u(\bar{H})]dt + \sigma(H)d\tilde{W}(t) \tag{21}$$

where the structural modal energy vector $H$, the drift coefficient vectors $m(H)$ and $m_u(\bar{H})$ involving the control forces, the diffusion coefficient matrix $\sigma(H)$, and unit Wiener process vector $\tilde{W}(t)$ are respectively as

$$H = [H_1^T, H_2^T]^T = [H_{11}, H_{12}, \ldots, H_{1m_1}, H_{21}, H_{22}, \ldots, H_{2m_2}]^T \tag{22a}$$

$$m(H) = [m_1^T, m_2^T]^T = [m_{11}, m_{12}, \ldots, m_{1m_1}, m_{21}, m_{22}, \ldots, m_{2m_2}]^T \tag{22b}$$

$$\begin{aligned} m_u(\bar{H}) &= [m_{u1}^T, m_{u2}^T]^T = [\bar{m}_{u1}^T, 0, \bar{m}_{u2}^T, 0]^T \\ &= [m_{u11}, m_{u12}, \ldots, m_{u1m_1}, m_{u21}, m_{u22}, \ldots, m_{u2m_2}]^T \end{aligned} \tag{22c}$$

$$\sigma(H) = \mathrm{diag}\{\sigma_1, \sigma_2\} = \mathrm{diag}\{\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1m_1}, \sigma_{21}, \sigma_{22}, \ldots, \sigma_{2m_2}\} \tag{22d}$$

$$\tilde{W}(t) = [\tilde{W}_{11}, \tilde{W}_{12}, \ldots, \tilde{W}_{1m_1}, \tilde{W}_{21}, \tilde{W}_{22}, \ldots, \tilde{W}_{2m_2}]^T \tag{22e}$$

Note that the averaged Itô equation of the uncontrolled structures corresponding to Eq. (21) is separable. In vector $m_u(\bar{H})$ of averaged Itô equation of the controlled structures, the first $m_3$ elements of $m_{u1}$ and first $m_4$ elements of $m_{u2}$ are represented by Eq. (18a) and the other elements of $m_{u1}$ and $m_{u2}$ are equal to zeros. Thus the optimal control forces affect only the controlled reduced mode processes in the sense of stochastic averaging. Since $R_p$ is a positive-definite symmetric matrix as stated previously, $\phi_{ji}^T P_j R_p^{-1} P_j^T \phi_{ji}$ $(i = 1, 2, ..., m_{j+2};\ j = 1, 2)$ is non-negative. And $g(\bar{H})$ is taken so that $\partial V/\partial H_{ji} \geq 0$ $(i = 1, 2, ..., m_{j+2};\ j = 1, 2)$, for example, the derivative of value function (20) with non-negative coefficients. Then the controlled response process of the coupled structures is stabilized based on the averaged Itô Eq. (21) with (18).

The Fokker-Planck-Kolmogorov (FPK) equation associated with the averaged Itô Eq. (21) can be established. The stationary FPK equation is

$$\begin{aligned} &\sum_{i=1}^{m_1} \frac{\partial}{\partial H_{1i}} \left\{ \left[ -m_{1i}(H_{1i}) + \frac{1}{2}\phi_{1i}^T P_u \phi_{1i} H_{1i} \frac{\partial V}{\partial H_{1i}} \right] p + \frac{1}{2}\frac{\partial}{\partial H_{1i}}[\sigma_{1i}^2(H_{1i})p] \right\} \\ &+ \sum_{i=1}^{m_2} \frac{\partial}{\partial H_{2i}} \left\{ \left[ -m_{2i}(H_{2i}) + \frac{1}{2}\phi_{2i}^T P_w \phi_{2i} H_{2i} \frac{\partial V}{\partial H_{2i}} \right] p + \frac{1}{2}\frac{\partial}{\partial H_{2i}}[\sigma_{2i}^2(H_{2i})p] \right\} = 0 \end{aligned} \tag{23}$$

A stationary probability density is obtained as follows

$$p(H_1, H_2) = C_p \exp\{-\varphi(H_1, H_2)\} \tag{24}$$

where $C_p$ is a normalization constant; the probability potential $\varphi$ is represented by

$$\varphi(H_1, H_2) = \int_0^{(H_1, H_2)} \sum_{i=1}^{m_1} \frac{\partial\varphi}{\partial H_{1i}} dH_{1i} + \sum_{i=1}^{m_2} \frac{\partial\varphi}{\partial H_{2i}} dH_{2i} \tag{25a}$$

$$\frac{\partial\varphi}{\partial H_{1i}} = \frac{\partial\sigma_{1i}^2/\partial H_{1i} - 2m_{1i} + \phi_{1i}^T P_u \phi_{1i} H_{1i} \partial V/\partial H_{1i}}{\sigma_{1i}^2} \tag{25b}$$

$$\frac{\partial\varphi}{\partial H_{2i}} = \frac{\partial\sigma_{2i}^2/\partial H_{2i} - 2m_{2i} + \phi_{2i}^T P_w \phi_{2i} H_{2i} \partial V/\partial H_{2i}}{\sigma_{2i}^2} \tag{25c}$$

The mean square (MS) modal displacement and modal velocity are obtained from Eq. (24) as

$$E[q_{jk}^2] = \frac{1}{\omega_{jk}^2} \int_0^{+\infty} H_{jk}\, p(H_1, H_2) dH_1 dH_2 \tag{26a}$$

$$E[\dot{q}_{jk}^2] = \int_0^{+\infty} H_{jk}\, p(H_1, H_2) dH_1 dH_2 \tag{26b}$$

Then the MS displacement, interstorey drift and base shear of the controlled coupled structures are obtained by using the modal transformation as follows

$$E[x_{ji}^2] = \sum_{k=1}^{m_j} (\phi_{jk}^i)^2 E[q_{jk}^2] \tag{27a}$$

$$E[(x_{ji} - x_{j,i-1})^2] = \sum_{k=1}^{m_j} (\phi_{jk}^i - \phi_{jk}^{i-1})^2 E[q_{jk}^2] \tag{27b}$$

$$\begin{aligned} & E\left[\left(\sum_{i=1}^{n_j} m_{jii}(\ddot{x}_{ji} + \ddot{x}_g) + (-1)^{j+1} \sum_{i=1}^{n_3} u_i^*\right)^2\right] \\ &= E[(E_j^T M_j(\ddot{X}_j + \ddot{x}_g E_j) - E_j^T P_j U^*)^2] \\ &= \sum_{k=1}^{m_j} \left(\sum_{i=1}^{n_j} m_{jii} \phi_{jk}^i\right)^2 (\omega_{jk}^4 E[q_{jk}^2] + 4\zeta_{jk}^2 \omega_{jk}^2 E[\dot{q}_{jk}^2]) \end{aligned} \tag{27c}$$

while the MS optimal control force is

$$\begin{aligned} E[u_i^{*2}] = \frac{1}{4} R_{pi}^{-1} \Bigg[ & P_1^T \left( \sum_{k=1}^{m_3} \phi_{1k} \phi_{1k}^T E\left[\dot{q}_{1k}^2 \left(\frac{\partial V}{\partial H_{1k}}\right)^2\right] \right) P_1 \\ & + P_2^T \left( \sum_{k=1}^{m_4} \phi_{2k} \phi_{2k}^T E\left[\dot{q}_{2k}^2 \left(\frac{\partial V}{\partial H_{2k}}\right)^2\right] \right) P_2 \Bigg] R_{pi}^{-T} \end{aligned} \tag{28}$$

where $\phi_{jk}^{i}$ is the $i$th element of mode vector $\phi_{jk}$ of structure $j$; $m_{jii}$ is the $i$th diagonal element of mass matrix $M_j$; $R_{pi}^{-1}$ is the $i$th row vector of inverse matrix $R_p^{-1}$ and

$$E\left[\dot{q}_{jk}^{2}\left(\frac{\partial V}{\partial H_{jk}}\right)^{2}\right] = \int_{0}^{+\infty} H_{jk}\left(\frac{\partial V}{\partial H_{jk}}\right)^{2} p(H_1, H_2)dH_1 dH_2 \tag{29}$$

The response statistics of the uncontrolled structures can be obtained in the same way by making the optimal control forces vanishing.

To evaluate the optimal control efficacy, two performance criteria are used as follows (Zhu *et al.* 1999, 2000, 2001)

$$K = \frac{\mathrm{RMS}(\mathrm{response}_u) - \mathrm{RMS}(\mathrm{response}_c)}{\mathrm{RMS}(\mathrm{response}_u)} \tag{30a}$$

$$\mu = \frac{K}{\sum_{i=1}^{n_3} \mathrm{RMS}(u_i^{*}) \Big/ \left[\sigma\left(\sum_{i=1}^{n_1} m_{1ii} + \sum_{i=1}^{n_2} m_{2ii}\right)\right]} \tag{30b}$$

where RMS(·) denotes the root mean square value; the subscripts $u$ and $c$ denote the uncontrolled and controlled structures respectively. The ratio $K$ measures the relative response reduction of the controlled and uncontrolled structures, and the ratio $\mu$ measures the relative response reduction per control force or control efficiency. The higher $K$ and $\mu$ indicate the control method with more response mitigation capabilities.

## 5. Numerical results

A numerical study is conducted on the stochastic optimal control of coupled adjacent building structures consisting of a 20-storey building and a 10-storey building with a few control devices. The mass of each floor is $1.6 \times 10^6$ kg; the interstorey stiffness is $1.2 \times 10^{10}$ N/m; and the modal damping ratio is taken to be 0.02. The spectral parameters of random seismic excitation are taken as $\sigma^2 = 0.6$ m$^2$/s$^3$, $\omega_g = 19$ rad/s and $\zeta_g = 0.2$ unless otherwise mentioned. The numbers of structural modes used for response analysis $m_1 = 6$ and $m_2 = 4$ while the control mode numbers $m_3 = 3$ and $m_4 = 2$. The weight coefficients of control forces and modal energies are $R_1 = \mathrm{diag}\{10, 10, 8\}$, $R_2 = \mathrm{diag}\{3, 2\}$, $S_{1i}^{a} = 0$, $S_{2i}^{a} = 0$, $[S_{1i}^{c}] = [0.08, 0.1, 0.04]$, $S_{2i}^{c} = [0.08, 0.04]$, $S_{1ij}^{b} = 0$, $S_{2ij}^{b} = 0$ and $S_{3ij}^{b} = 0$. Some numerical results are displayed in Figs. 2-6 and in Tables 1-4.

Fig. 2 shows the performance criteria $K$ and $\mu$ of displacements and interstorey drifts of the coupled structures by using the proposed control method when the control device connects the two adjacent buildings only at the 10th floor level. About 60% displacement response reduction ($K$) with 0.85 efficiency ($\mu$) at the middle of taller building and 55% response reduction with 0.80 efficiency for shorter building are achieved. The interstorey drift is relative to the corresponding displacement response and then only the numerical results of interstorey drifts are given in the following.

The effect of seismic excitation features on the control efficacy is studied with the control device at the 10th floor level. Fig. 3 illustrates the relative response reduction $K$ and control efficiency $\mu$ of

RMS interstorey drifts under different excitation intensity $\sigma$. With the increase of intensity $\sigma$, the response reduction capability is enhanced while the efficiency is decreased. Fig. 4 shows the relative response reduction and control efficiency of RMS interstorey drifts under different dominant excitation frequency $\omega_g$. It is observed that the response reduction or mitigation capability increases as the dominant frequency $\omega_g$ is close to the structural natural frequency, even though the efficiency has a little decrease.

The effect of control device placement and number on the control efficacy is eventually studied. Fig. 5 shows the relative response reduction and control efficiency of RMS interstorey drifts when a single control device is placed at the 10th floor, the 8th floor or the 6th floor. It is seen that the seismic response mitigation capability of the control device at the 10th floor level is better than at the others. The result means the optimum position of control devices close to the floor level of the largest amplitude of dominant structural modes. Fig. 6 shows the relative response reduction and control efficiency of RMS interstorey drifts under different control device number and placement (three cases: one control device at the 10th floor; two control devices at the 10th and 8th floors; three control devices at the 10th, 8th and 6th floors respectively). It is found that the response reduction capability does not increase with using more control devices at lower floors, due to the interaction among the control devices. A similar observation is made for the RMS base shears as given in Tables 1-4.

Table 1 Relative reduction $K$ and efficiency $\mu$ of RMS base shears under different $\sigma$

| $\sigma^2$ | Taller building | | Shorter building | |
|---|---|---|---|---|
| | $K$ | $\mu$ | $K$ | $\mu$ |
| 0.6 | 0.796 | 1.10 | 0.808 | 1.12 |
| 0.9 | 0.830 | 1.00 | 0.842 | 1.01 |

Table 2 Relative reduction $K$ and efficiency $\mu$ of RMS base shears under different $\omega_g$

| $\omega_g$ | Taller building | | Shorter building | |
|---|---|---|---|---|
| | $K$ | $\mu$ | $K$ | $\mu$ |
| 8 | 0.925 | 0.51 | 0.562 | 0.31 |
| 11 | 0.882 | 0.88 | 0.819 | 0.82 |
| 14 | 0.850 | 0.82 | 0.876 | 0.84 |
| 19 | 0.796 | 1.10 | 0.808 | 1.12 |
| 23 | 0.785 | 1.23 | 0.767 | 1.20 |

Table 3 Relative reduction $K$ and efficiency $\mu$ of RMS base shears for various control device position

| Position of control device | Taller building | | Shorter building | |
|---|---|---|---|---|
| | $K$ | $\mu$ | $K$ | $\mu$ |
| 10th floor | 0.796 | 1.10 | 0.808 | 1.12 |
| 8th floor | 0.779 | 1.04 | 0.787 | 1.05 |
| 6th floor | 0.734 | 0.91 | 0.761 | 0.95 |

Table 4 Relative reduction $K$ and efficiency $\mu$ of RMS base shears for various control device number

| Control devices | | Taller building | | Shorter building | |
|---|---|---|---|---|---|
| Number | Position | $K$ | $\mu$ | $K$ | $\mu$ |
| 1 | 10th floor | 0.796 | 1.10 | 0.808 | 1.12 |
| 2 | 10th & 8th floors | 0.810 | 0.79 | 0.816 | 0.80 |
| 3 | 10th, 8th & 6th floors | 0.768 | 0.09 | 0.756 | 0.09 |

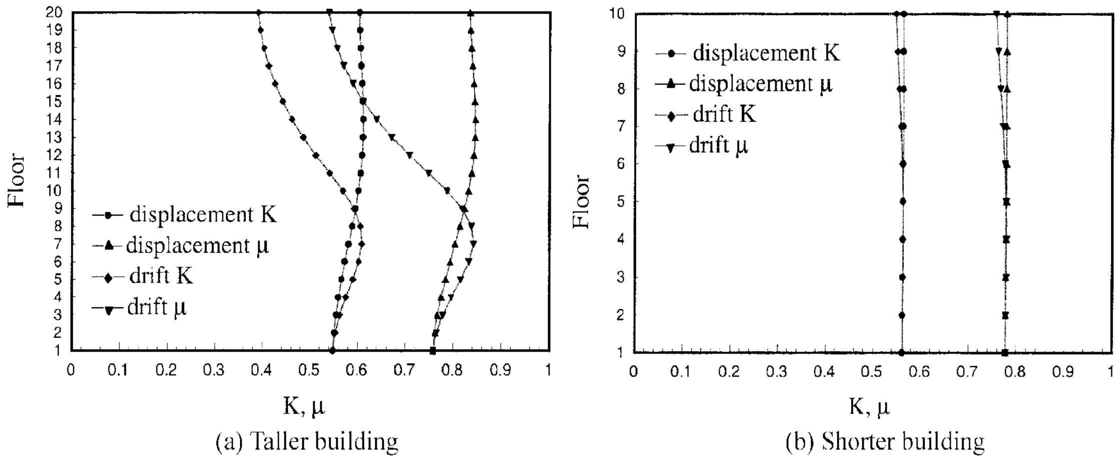


Fig. 2 Relative reduction $K$ and efficiency $\mu$ of RMS displacements and interstorey drifts

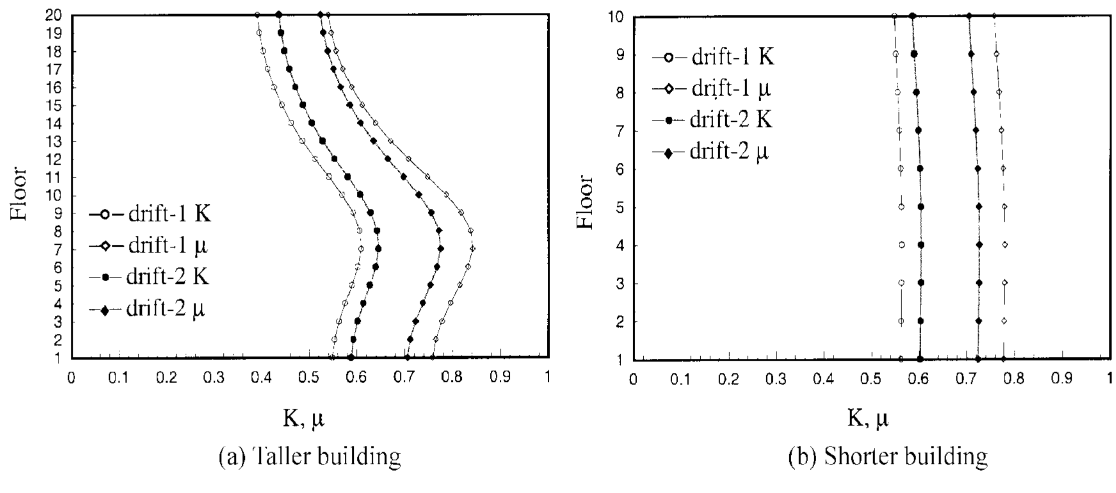


Fig. 3 Relative reduction $K$ and efficiency $\mu$ of RMS interstorey drifts under different $\sigma$ (drift-1: $\sigma^2$ = 0.6; drift-2: $\sigma^2$ = 0.9)

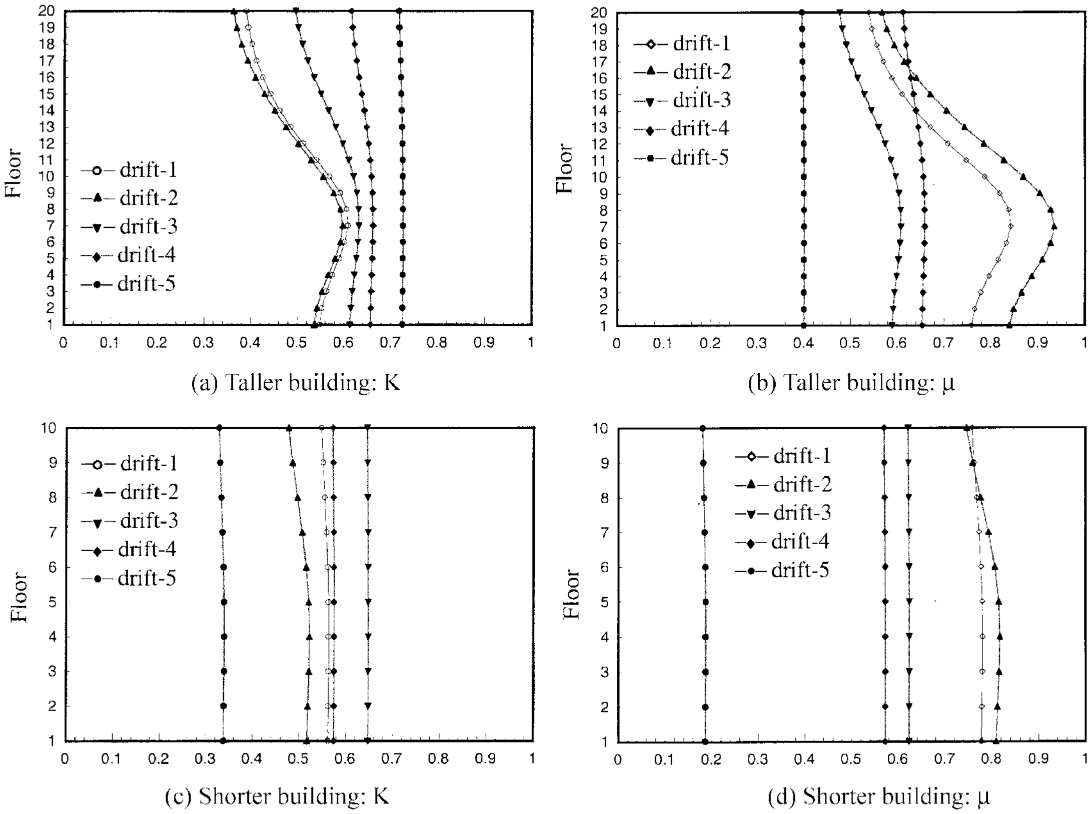


Fig. 4 Relative reduction $K$ and efficiency $\mu$ of RMS interstorey drifts under different $\omega_g$ (drift-1: $\omega_g = 19$; drift-2: $\omega_g = 23$; drift-3: $\omega_g = 14$; drift-4: $\omega_g = 11$; drift-5: $\omega_g = 8$)

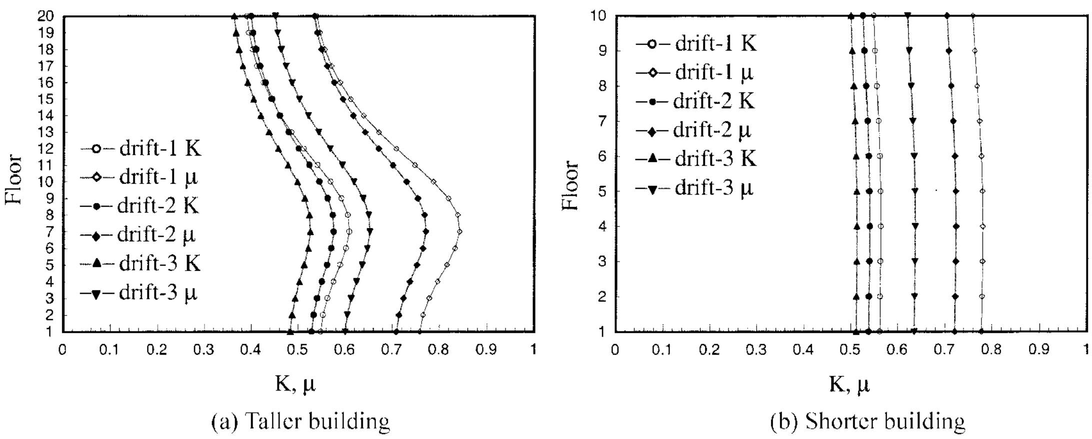


Fig. 5 Relative reduction $K$ and efficiency $\mu$ of RMS interstorey drifts for various control device position (drift-1: at the 10th floor; drift-2: at the 8th floor; drift-3: at the 6th floor)

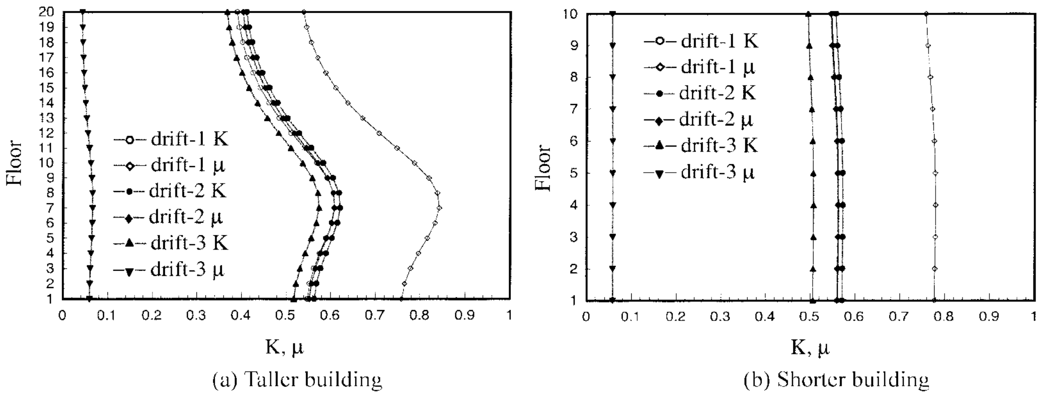


Fig. 6 Relative reduction $K$ and efficiency $\mu$ of RMS interstorey drifts for various control device number (drift-1: 1 device at the 10th floor; drift-2: 2 devices at the 10th and 8th floors; drift-3: 3 devices at the 10th, 8th and 6th floors)

## 6. Conclusions

The stochastic optimal nonlinear control of coupled adjacent building structures under random seismic excitation has been studied based on the stochastic dynamical programming principle and the stochastic averaging method. The proposed control method has the following advantages: (a) the random seismic excitation spectrum is taken into account according to the stochastic dynamical programming principle; (b) the structural energy control instead of usual state control is conducted and then the dimension of the optimal control problem is reduced based on the stochastic averaging method; (c) the optimal control force is a generalized nonlinear damping force which can be provided by active or semi-active dampers; (d) it is applicable to coupled structures with an arbitrary number of stories and with connecting control devices at any floors. The numerical study has drawn the following points: (a) the seismic response mitigation of coupled structures can be achieved by using only a few connecting control devices at properly selected floors; (b) the response reduction capability can increase with seismic excitation intensity and dominant frequency approaching the structural natural frequency; (c) the proposed stochastic optimal control method for coupled structures is more effective and efficient.

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## References

Agrawal, A.K. and Yang, J.N. (1996), “Optimal polynomial control of seismically excited linear structures”, *J.*

*Engrg. Mech.*, ASCE, **122**, 753-761.

Christenson, R.E., Spencer, Jr. B.F. and Johnson, E.A. (1999), "Coupled building control using active and smart damping strategies", *Optimization and Control in Civil and Struct. Engrg.*, Civil-Comp Press, Edinburgh, 187-195.

Fleming, W.H. and Rishel, R.W. (1975), *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York.

Gurley, K., Kareem, A., Bergman, L.A., Johnson, E.A. and Klein, R.E. (1994), "Coupling tall buildings for control of response to wind", *Struct. Safety Reliability*, Balkema, Rotterdam, 1553-1560.

Iwanami, K., Suzuki, K. and Seto, K. (1996), "Vibration control method for parallel structures connected by damper and spring", *JSME Int. J.*, Series C, **39**, 714-720.

Kanai, K. (1957), "Seismic-empirical formula for the seismic characteristics of the ground", *Bull. Earthquake Res. Institute*, University of Tokyo, Tokyo, 309-325.

Luco, J.E. and De Barros, F.C.P. (1998), "Optimal damping between two adjacent elastic structures", *Earthq. Engrg. Struct. Dyn.*, **27**, 649-659.

Luco, J.E. and Wong, H.L. (1994), "Control of the seismic response of adjacent structures", *Proc. of the 1st World Conf. on Structural Control*, Los Angeles, California, 21-30.

Matsumoto, Y., Doi, F. and Seto, K. (1999), "Vibration control of multiple building structures connected with active bridges", *Proc. of the 2nd World Conf. on Structural Control*, John Wiley & Sons, Chichester, 599-607.

Mitsuta, S. and Seto, K. (1992), "Active vibration control of structures arranged in parallel", *Proc. of the 1st Int. Conf. on Motion and Vibration Control*, Japan, 146-151.

Ni, Y.Q., Ko, J.M. and Ying, Z.G. (2001), "Random seismic response analysis of adjacent building coupled with non-linear hysteretic dampers", *J. Sound Vib.*, **246**, 403-417.

Seto, K. (1994), "Vibration control method for flexible structures arranged in parallel", *Proc. of the 1st World Conf. on Structural Control*, Los Angeles, California, 62-71.

Stengel, R.F. (1986), *Stochastic Optimal Control: Theory and Application*, John Wiley & Sons, New York.

Sugino, S., Sakai, D., Kundu, S. and Seto, K. (1999), "Vibration control of parallel structures connected with passive devices designed by GA", *Proc. of the 2nd World Conf. on Structural Control*, John Wiley & Sons, Chichester, 329-337.

Tajimi, H. (1960), "A statistics method of determining the maximum response of a building structure during an earthquake", *Proc. of the 2nd World Conf. on Earthquake Engineering*, Tokyo-Kyoto, 781-798.

Xu, Y.L., He, Q. and Ko J.M. (1999), "Dynamic response of damper-connected adjacent buildings under earthquake excitation", *Engrg. Struct.*, **21**, 135-148.

Yamada, Y., Ikawa, N., Yokoyama, H. and Tachibana, E. (1994), "Active control of structures using the joining member with negative stiffness", *Proc. of the 1st World Conf. on Structural Control*, Los Angeles, California, 41-49.

Yang, J.N., Agrawal, A.K. and Chen, S. (1996), "Optimal polynomial control for seismically excited non-linear and hysteretic structures", *Earthq. Engrg. Struct. Dyn.*, **25**, 1211-1230.

Zhu, W.Q., Huang, Z.L. and Yang, Y.Q. (1997), "Stochastic averaging of quasi-integrable Hamiltonian systems", *J. Appl. Mech.*, ASME, **64**, 975-984.

Zhu, W.Q. and Lin, Y.K. (1991), "Stochastic averaging of energy envelope", *J. Engrg. Mech.*, ASCE, **117**, 1890-1905.

Zhu, W.Q. and Ying, Z.G. (1999), "Optimal nonlinear feedback control of quasi-Hamiltonian systems", *Sci. China*, Series A, **42**, 1213-1219.

Zhu, W.Q., Ying, Z.G., Ni, Y.Q. and Ko, J.M. (2000), "Optimal nonlinear stochastic control of hysteretic systems", *J. Engrg. Mech.*, ASCE, **126**, 1027-1032.

Zhu, W.Q., Ying, Z.G. and Soong, T.T. (1999), "Optimal nonlinear feedback control of structures under random loading", *Stochastic Struct. Dyn.*, Balkema, Rotterdam, 141-148.

Zhu, W.Q., Ying, Z.G. and Soong, T.T. (2001), "An optimal nonlinear feedback control strategy for randomly excited structural systems", *Nonlinear Dyn.*, **24**, 31-51.

Zhu, W.Q., Ying, Z.G. and Suzuki, Y. (2000), "Optimal nonlinear feedback control of building structures with AMD under non-white random ground acceleration excitation", *Adv. Struct. Dyn.*, Elsevier, 1429-1436.