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On an improved numerical method to solve the equilibrium problems of solids with bounded tensile strength that are subjected to thermal strain

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Abstract. In this paper we recall briefly the constitutive equations for solids subjected to thermal strain taking in account the bounded tensile stress of the material. In view to solve the equilibrium problem via the finite element method using the Newton Raphson procedure, we show that the tangent elasticity tensor is semi-definite positive. Therefore, in order to obtain a convergent numerical method, the constitutive equation needs to be modified. Specifically, the dependency of the stress by the anelastic deformation is made explicit by means of a parameter δ, varying from 0 to 1, that factorizes the elastic tensor. This parameterization, for δ near to 0, assures the positiveness of the tangent elasticity tensor and enforces the convergence of the numerical method. Some numerical examples are illustrated.

Key words: masonry; thermal strain; bounded tensile strength; finite element.

1. Introduction and motivation

A masonry-like material is a material that can not sustain tensile stress or its tensile strength is small. On the hypotheses of small strains, no tensile strength and a normality postulate, Del Piero (1989) proposed the constitutive equations for a masonry-like material. On the further assumption concerning the symmetry of the elastic tensor, the existence of the strain energy density was proved. Lucchesi *et al*. (1994) proposed a non-linear numerical method to solve the equilibrium problem for an isotropic body made of masonry-like material using the method of the finite element via Newton Raphson procedure. Furthermore, Lucchesi *et al.* (1995) extended this method to solve the equilibrium problems for materials in which the tensile strength is bounded.

The model of the no-tension material subjected to thermal loads was elaborated by Padovani *et al*. (2000) and a complete model concerning the no-tension materials in the framework of the thermodynamics and the thermoelasticity was presented by Lucchesi *et al*. (2000).

The aim of the present work is to extend the problem of the thermal loads acting on a masonry-like material, to the case of bounded tensile strength. Without some loss of generality, the dependence of the elastic moduli by the temperature is not made explicit.

In the framework of the numerical method, i.e., the finite element method, we will prove that the tangent elasticity tensor is not positive definite and therefore the numerical method is unstable *a*

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priori. Following a suggestion of Padovani (2000) we consider an approximated material depending by a parameter δ varying from 0 to 1. This is accomplished modifying the constitutive equation and making linear the dependence of the stress by the anelastic part of the deformation by means of the elasticity tensor factorized by the parameter δ. In this way, the modified constitutive law describes the behavior of a family of isotropic materials that, in absence of the thermal loads and for $\delta = 0$, coincides with that described by Lucchesi *et al*. (1995) whereas for $\delta = 1$ coincides with the one linearly elastic. Clearly, for $\delta \neq 0$, the constitutive equations inherent materials with bounded tensile strength or, in the limit case, with no-tensile strength, are verified only approximately. On the other hand, as we will show in the section 3, the assumption of $\delta \neq 0$ renders the constitutive law strictly monotone. In virtue of this assumption, the proposed numerical method possesses the indispensable requirements to be convergent.

The problem inherent to the slowness or the loss of the convergence associated to the numerical method based on the Newton Raphson procedure, is not new in the literature and it was recognized by Padovani (2000). In this paper, the author asserted the opportunity to consider an approximated material like-masonry in order to overcome the difficulties encountered during the solution of the equilibrium problem via the finite element method.

Furthermore, a detailed discussion on the numerical strategies to solve numerically the equilibrium problem of solids with no-tensile strength is illustrated in the paper of Alfano *et al.* (2000). In this work, the authors proposed a numerical strategy, named *enhanced tangent strategy,* based on the use of the tangent operator. In order to prevent the activation of zero-energy modes during the iterative process, it was considered a fictitious elastic stiffness at the Gauss points in which the elastic strain vanishes. The fictitious stiffness was assumed as the elastic stiffness scaled by an energy-parameterized coefficient that goes to zero as the convergence is attained.

In the cited work, it was not made mention of the fact that at the Gauss points, where there is a contemporary presence of the elastic-anelastic strains, the tangent operator is still not positive definite. This circumstance is reflected on the equations system that results ill conditioned and the convergence of the numerical method may be unavoidably compromised.

2. The mechanical model and the modified constitutive law

In this section we shortly begin to show the constitutive assumption for the materials with bounded tensile strength (Lucchesi *et al.* 1995) that are subjected to the thermal strains.

We denote by Lin the space of the second order tensors equipped by the inner product $\boldsymbol{A} \cdot \boldsymbol{B} = tr(\boldsymbol{A}^T\boldsymbol{B}),\ \boldsymbol{A}, \boldsymbol{B} \in Lin$ whereas we denote by Sym, Sym$^+$ and Sym$^-$ the subsets of Lin constituted by the symmetric, symmetric positive semi-definite and symmetric negative semi-definite tensors, respectively.

If the material is isotropic and the temperature variation $\Delta\theta$ is small, we can assume that the thermal strain $\boldsymbol{E}^t$ due to $\Delta\theta$ is:

$$\boldsymbol{E}^t = \beta(\theta)\mathbf{1} = \alpha_t \Delta\theta \mathbf{1} \tag{1}$$

where $\beta(\theta)$ is the thermal expansion, α_t is the linear coefficient of the thermal expansion and $\mathbf{1}$ is the identity tensor.

Following a suggestion of Lucchesi *et al*. (1995) and Padovani (2000), from a kinematical point of view, we assume that the infinitesimal strain tensor $\boldsymbol{E}$, minus the thermal part $\boldsymbol{E}^t$, may be decomposed into an elastic part $\boldsymbol{E}^e$ and into an anelastic part $\boldsymbol{E}^a$ that is positive semi-definite:

$$\boldsymbol{E} - \boldsymbol{E}^t = \boldsymbol{E}^e + \boldsymbol{E}^a \tag{2}$$

In the framework of the no-tension materials or bounded tension materials, the Cauchy stress tensor $\boldsymbol{T}$ depends only on the elastic part of the deformation $\boldsymbol{E}^e$ (Del Piero 1989):

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^t - \boldsymbol{E}^a) \\ \boldsymbol{C} &= 2\mu\boldsymbol{I} + \lambda\mathbf{1} \otimes \mathbf{1} \end{aligned} \tag{3}$$

where $\boldsymbol{C}$ is the isotropic forth order tensor of the elasticity and $\boldsymbol{I}$ is the forth order identity tensor over the elements of Sym.

We assume that the Lame's moduli μ and λ do not depend on the temperature and that they satisfy the inequalities:

$$\begin{aligned} &\mu > 0 \\ &2\mu + 3\lambda > 0 \end{aligned} \tag{4}$$

Moreover, we denote by σ the tensile strength of the material and we assume that:

$$\begin{aligned} &(\boldsymbol{T} - \sigma\mathbf{1}) \in \text{Sym}^- \\ &(\boldsymbol{T} - \sigma\mathbf{1}) \cdot \boldsymbol{E}^a = 0 \end{aligned} \tag{5}$$

where Eq. $(5)_1$ is the limitation on the normal stress and Eq. $(5)_2$ is the normality condition.

Following the scheme of the proof shown by Lucchesi *et al.* (1995) it is possible to demonstrate that $\boldsymbol{T}$ and $\boldsymbol{E}^a$ are coaxial and by the isotropic properties of the elastic tensor $\boldsymbol{C}$, $\boldsymbol{T}$ and $\boldsymbol{E}^e$ are also coaxial. Finally, by Eqs. (1) and (2), the stress tensor $\boldsymbol{T}$ is coaxial with $\boldsymbol{E}$ and $\boldsymbol{E}^t$.

Using the representation theorem for the isotropic function, there exist three scalar function β_0, β_1 and β_2 of the principal invariants of $\boldsymbol{E}$ such that:

$$\boldsymbol{T} = (\beta_0\mathbf{1} + \beta_1\boldsymbol{E} + \beta_2\boldsymbol{E}^2) \tag{6}$$

The Eqs. (1), (2), (3) and (5) define the response of an isotropic non-linear material with bounded tensile strength that is subjected to the thermal loads. The elastic behavior, in the uniaxial stress state is shown in Fig. 1.

In the uniaxial behavior, it is trivial to observe that, for $e > e^e$, the response function is not invertible and that the derivative of the stress respect to the total strain is zero. In the framework of the three-dimensional case this means that there are strain directions for which the derivative of the stress, i.e., the tangent elasticity tensor, is zero. In other words, the tangent elasticity tensor is positive semi-definite.

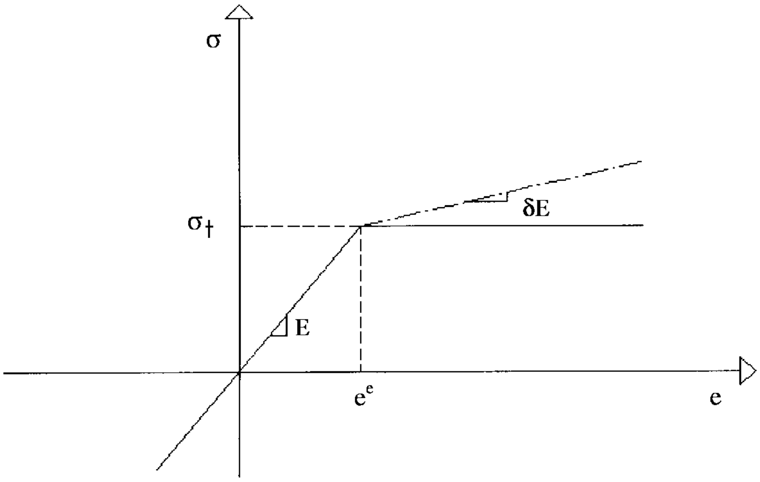

Fig. 1 The uniaxial stress-strain law for a material with bounded tensile strength

Looking at the plot in Fig. 1, we stipulate that for $e > e^e$ the stress $\boldsymbol{T}$ depends also on the anelastic part of the strain $\boldsymbol{E}^a$:

$$\boldsymbol{T} = \boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^t - \boldsymbol{E}^a) + \delta \boldsymbol{C}\boldsymbol{E}^a \tag{7}$$

where δ is a parameter that varies from 0 to 1. Using Eq. (2) we obtain:

$$\boldsymbol{T} = (1-\delta)\boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^t - \boldsymbol{E}^a) + \delta \boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^t) \tag{8}$$

under the conditions:

$$[\boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^t - \boldsymbol{E}^a) - \sigma \mathbf{1}] \in \mathrm{Sym}^-$$

$$[\boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^t - \boldsymbol{E}^a) - \sigma \mathbf{1}] \cdot \boldsymbol{E}^a = 0 \tag{9}$$

It appears obvious that the condition $(\boldsymbol{T} - \sigma\mathbf{1}) \in \mathrm{Sym}^-$ is verified only approximately and that it depends on the choice of the parameter δ. Choosing δ very close to zero, the effect on the material response would be small. In fact, let us suppose $\boldsymbol{E}^e = \mathbf{0}$, that is $\boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^t - \boldsymbol{E}^a) = \sigma\mathbf{1}$. Thus, by Eq. (7), $\boldsymbol{T} = \sigma\mathbf{1} + \delta\boldsymbol{C}\boldsymbol{E}^a$ and the stress increases as the anelastic strain $\boldsymbol{E}^a$ grows according to the choice of the parameter δ.

On the other hand, we will show that in the case of the plane stress and by the hypothesis (4), the tangent elasticity tensor is positive definite for $\delta > 0$.

Notice that in the case $\delta = 0$ and $\sigma = 0$, the material behavior coincides with that described by Padovani *et al.* (2000) whereas, for $\delta = 1$, the material is simply linearly elastic.

We conclude this section observing that in order to calculate the derivative of $\boldsymbol{T}$ respect to $\boldsymbol{E}$, it is necessary to use the representation theorem of the isotropic functions. By the coaxiality property between the stress tensor $\boldsymbol{T}$ and the strain tensor $\boldsymbol{E}$ and by Eq. (8), we obtain:

$$\boldsymbol{T} = (1-\delta)(\beta_0\mathbf{1} + \beta_1\boldsymbol{E} + \beta_2\boldsymbol{E}^2) + \delta(\gamma_0\mathbf{1} + \gamma_1\boldsymbol{E}) \tag{10}$$

where the coefficients β_i, $i = 0, 1, 2$ depend in a non-linear way on the eigenvalues of $\boldsymbol{E}$ and on the thermal expansion $\beta(\theta)$, respectively. Moreover, the coefficients γ_0 and γ_1 are defined by:

$$\gamma_0 = \lambda tr[\boldsymbol{E} - \beta(\theta)\mathbf{1}] - 2\mu\beta(\theta)$$

$$\gamma_1 = 2\mu \tag{11}$$

3. The two-dimensional case

In this section, we consider the case of the plane stress. Here we denote by (e_1, e_2) the eigenvalues of $\boldsymbol{E}$ such that $e_1 \le e_2$ and (a_1, a_2) are the eigenvalues of $\boldsymbol{E}^a$ that we assume non-negative.

Setting:

$$\alpha = \frac{\lambda}{\mu} \ge 0$$

$$\varepsilon = \frac{\sigma}{\mu} \ge 0$$

$$\eta = \frac{\sigma^0}{\mu} \tag{12}$$

where:

$$\sigma^0 = 2\mu\left[\beta(\theta) + \frac{2\alpha}{2+\alpha}\beta(\theta)\right] \tag{13}$$

the constitutive law (8) is:

$$\boldsymbol{T} = (1-\delta)2\mu\left[(\boldsymbol{E} - \boldsymbol{E}^a) + \frac{\alpha}{2+\alpha}tr(\boldsymbol{E} - \boldsymbol{E}^a)\mathbf{1}\right] + \delta 2\mu\left[\boldsymbol{E} + \frac{\alpha}{2+\alpha}tr(\boldsymbol{E})\mathbf{1}\right] - \sigma^0\mathbf{1} \tag{14}$$

In this case, the isotropic function (10) reduces to:

$$\boldsymbol{T} = (1-\delta)(\beta_0\mathbf{1} + \beta_1\boldsymbol{E}) + \delta(\gamma_0\mathbf{1} + \gamma_1\boldsymbol{E}) \tag{15}$$

Denoting by I_1 and I_2 the invariants of $\boldsymbol{E}$, i.e.:

$$I_1 = tr(\boldsymbol{E}) = e_1 + e_2$$

$$I_2 = \boldsymbol{E}\cdot\boldsymbol{E} = e_1^2 + e_2^2 \tag{16}$$

the coefficients γ_0 and γ_1 are expressed by:

$$\gamma_0 = \frac{2\lambda}{2+\alpha}I_1 - \sigma^0$$

$$\gamma_1 = 2\mu \tag{17}$$

Therefore, the derivative of (15) assumes the compact form:

$$D_E\boldsymbol{T} = (1-\delta)\left[\frac{\partial\beta_0}{\partial I_1}\mathbf{1}\otimes\mathbf{1} + 2\frac{\partial\beta_0}{\partial I_2}(\mathbf{1}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes\mathbf{1}) + 2\frac{\partial\beta_1}{\partial I_2}\boldsymbol{E}\otimes\boldsymbol{E} + \beta_1\boldsymbol{I}\right] + 2\mu\delta\left(\frac{\alpha}{2+\alpha}\mathbf{1}\otimes\mathbf{1} + \boldsymbol{I}\right) \quad (18)$$

Moreover, setting:

$$\begin{aligned} \alpha_1 &= (1-\delta)\frac{\partial\beta_0}{\partial I_1} + \delta\frac{2\lambda}{(2+\alpha)} \\ \alpha_2 &= (1-\delta)2\frac{\partial\beta_0}{\partial I_2} \\ \alpha_3 &= (1-\delta)2\frac{\partial\beta_1}{\partial I_2} \\ \alpha_4 &= (1-\delta)\beta_1 + \delta 2\mu \end{aligned} \quad (19)$$

we obtain the engineering components of the tangent elasticity matrix:

$$\begin{aligned} D_{11} &= \alpha_1 + 2\alpha_2 E_{11} + \alpha_3 E_{11}^2 + \alpha_4 \\ D_{12} &= \alpha_1 + \alpha_2(E_{11} + E_{22}) + \alpha_3 E_{11}E_{22} \\ D_{13} &= \alpha_2 E_{12} + \alpha_3 E_{11}E_{12} \\ D_{22} &= \alpha_1 + 2\alpha_2 E_{22} + \alpha_3 E_{22}^2 + \alpha_4 \\ D_{23} &= \alpha_2 E_{12} + \alpha_3 E_{22}E_{12} \\ D_{33} &= \alpha_3 E_{12}^2 + \frac{\alpha_4}{2} \end{aligned} \quad (20)$$

where E_{ij} are the components of $\boldsymbol{E}$ with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.

The Eq. $(9)_2$, projected in the strain principal reference frame, is split into a system of two equations, namely:

$$\left\{\left[2(e_1 - a_1) + \frac{2\alpha}{2+\alpha}(e_1 + e_2 - a_1 - a_2)\right] - (\varepsilon + \eta)\right\}a_1 = 0 \quad (21)$$

$$\left\{\left[2(e_2 - a_2) + \frac{2\alpha}{2+\alpha}(e_1 + e_2 - a_1 - a_2)\right] - (\varepsilon + \eta)\right\}a_2 = 0 \quad (22)$$

where a_1, a_2 are the principal anelastic strains. The condition $a_1 = a_2 = 0$ defines a subset of Sym in which the behavior of the material is linearly elastic and the condition $(5)_1$ determines it. Specifically, in the domain:

$$\Re_1 = \{\boldsymbol{E}\in \text{Sym};\ 2\alpha e_2 + 4(1+\alpha)e_1 - (\varepsilon+\eta)(2+\alpha) \le 0,\ 2\alpha e_1 + 4(1+\alpha)e_2 - (\varepsilon+\eta)(2+\alpha) \le 0\} \quad (23)$$

the behavior of the material is linearly elastic and the following relations hold:

$$
\begin{aligned}
t_1 &= \frac{2\mu}{2+\alpha}[2(1+\alpha)e_1 + \alpha e_2] - \sigma^0 \\
t_2 &= \frac{2\mu}{2+\alpha}[2(1+\alpha)e_2 + \alpha e_1] - \sigma^0 \\
\beta_0 &= \gamma_0 = \frac{2\lambda}{2+\alpha} I_1 - \sigma^0 \\
\beta_1 &= \gamma_1 = 2\mu \\
\alpha_1 &= \frac{2\lambda}{(2+\alpha)} \\
\alpha_2 &= \alpha_3 = 0 \\
\alpha_4 &= 2\mu
\end{aligned} \tag{24}
$$

where by t_1 and t_2 we denote the principal stress of $\boldsymbol{T}$.

The coefficients β_α, $\alpha = 0, 1$, are determined equating Eq. $(24)_{1,2}$ to the components of (15) in the principal reference frame constituted by the eigenvectors of $\boldsymbol{E}$.

The condition $a_1 \neq 0$ and $a_2 \neq 0$ defines a domain dominated by the anelastic deformations. The Eqs. (21) and (22) determine the value of a_1 and a_2 and the condition expressed by Eq. $(5)_1$ determines the domain:

$$
\Re_2 = \left\{ \boldsymbol{E} \in \mathrm{Sym};\ \left(e_1 - \frac{(\varepsilon+\eta)(2+\alpha)}{2(2+3\alpha)}\right) \geq 0,\ \left(e_2 - \frac{(\varepsilon+\eta)(2+\alpha)}{2(2+3\alpha)}\right) > 0 \right\}
$$

$$
\begin{aligned}
a_1 &= e_1 - \frac{(\varepsilon+\eta)(2+\alpha)}{2(2+3\alpha)} \\
a_2 &= e_2 - \frac{(\varepsilon+\eta)(2+\alpha)}{2(2+3\alpha)} \\
t_1 &= (1-\delta)\sigma + \delta\left\{\frac{2\mu}{2+\alpha}[2(1+\alpha)e_1 + \alpha e_2] - \sigma^0\right\} \\
t_2 &= (1-\delta)\sigma + \delta\left\{\frac{2\mu}{2+\alpha}[2(1+\alpha)e_2 + \alpha e_1] - \sigma^0\right\} \\
&\beta_0 = \sigma,\quad \gamma_0 = \frac{2\lambda}{2+\alpha} I_1 - \sigma^0,\quad \beta_1 = 0,\ \gamma_1 = 2\mu \\
&\alpha_1 = \delta\frac{2\lambda}{(2+\alpha)},\quad \alpha_2 = \alpha_3 = 0,\quad \alpha_4 = 2\delta\mu
\end{aligned} \tag{25}
$$

Let us denote by $\Delta t_i(\delta) = t_i(\delta) - t_i(0)$, $i = 1, 2$ the *principal extra stress*. Thus, we have:

$$\Delta t_1(\delta) = \delta\left\{\frac{2\mu}{2+\alpha}[2(1+\alpha)e_1 + \alpha e_2] - (\sigma^0 + \sigma)\right\}$$

$$\Delta t_2(\delta) = \delta\left\{\frac{2\mu}{2+\alpha}[2(1+\alpha)e_2 + \alpha e_1] - (\sigma^0 + \sigma)\right\} \tag{26}$$

In the particular case, defined by $\alpha = 0$, i.e., when the Poisson modulus vanishes, we obtain

$$\Delta t_1(\delta) = \delta[2\mu e_1 - (\sigma^0 + \sigma)]$$

$$\Delta t_2(\delta) = \delta[2\mu e_2 - (\sigma^0 + \sigma)] \tag{27}$$

where the role played by δ results evident.

Finally, setting $a_1 = 0$ and $a_2 \neq 0$ we find the domain $\Re_3$ and the values of the anelastic strains. For this domain we will show explicitly the calculation. We set:

$$\Re_3 = \left\{\boldsymbol{E} \in \text{Sym};\ e_1 - \frac{(\varepsilon+\eta)(2+\alpha)}{2(2+3\alpha)} < 0,\ 2\alpha e_1 + 4(1+\alpha)e_2 - (\varepsilon+\eta)(2+\alpha) > 0\right\} \tag{28}$$

$$a_1 = 0, \qquad a_2 = e_2 + \frac{\alpha}{2(1+\alpha)}e_1 - \frac{(\varepsilon+\eta)(2+\alpha)}{4(1+\alpha)} \tag{29}$$

where, in virtue of Eq. (22) we obtain the anelastic deformation a_2 and Eq. $(29)_2$. Therefore, using Eq. (14) jointly to Eq. $(5)_1$, we obtain Eq. $(28)_1$.

Furthermore, denoting by $\varphi = \mu(2+3\alpha)/(1+\alpha) = E_m$, the elasticity modulus of the masonry, by the aid of Eqs. (14) and (29), we compute the principal stress:

$$t_1 = (1-\delta)\left(\varphi e_1 + \frac{\alpha(\sigma+\sigma^0)}{2(1+\alpha)} - \sigma^0\right) + \delta\left\{\frac{2\mu}{2+\alpha}[2(1+\alpha)e_1 + \alpha e_2] - \sigma^0\right\}$$

$$t_2 = (1-\delta)\sigma + \delta\left\{\frac{2\mu}{2+\alpha}[2(1+\alpha)e_2 + \alpha e_1] - \sigma^0\right\} \tag{30}$$

The calculation of the coefficients β_0 and β_1 is performed solving the following system:

$$\beta_0 + \beta_1 e_1 = \varphi e_1 + \frac{\alpha(\sigma+\sigma^0)}{2(1+\alpha)} - \sigma^0$$

$$\beta_0 + \beta_1 e_2 = \sigma \tag{31}$$

obtained equating Eq. $(25)_3$ to Eq. (15).

Recalling that $e_{1/2} = \dfrac{I_1 \mp \sqrt{2I_2 - I_1^2}}{2}$ we get:

$$\beta_0 = \frac{\varphi}{2}\frac{(I_1^2 - I_2)}{\sqrt{2I_2 - I_1^2}} + \frac{\varphi\varepsilon}{4(2+3\alpha)}\left(3\alpha + 2 - \frac{(2+\alpha)I_1}{\sqrt{2I_2 - I_1^2}}\right) - \frac{\sigma^0(2+\alpha)}{4(1+\alpha)}\left(1 + \frac{I_1}{\sqrt{2I_1 - I_1^2}}\right)$$

$$\beta_1 = -\frac{\varphi}{2}\frac{(I_1 - \sqrt{2I_2 - I_1^2})}{\sqrt{2I_2 - I_1^2}} + \frac{\varphi\varepsilon(2+\alpha)}{2(2+3\alpha)\sqrt{2I_2 - I_1^2}} + \frac{\sigma^0(2+\alpha)}{2(1+\alpha)\sqrt{2I_2 - I_1^2}}$$

$$\gamma_0 = \frac{2\lambda}{2+\alpha}I_1 - \sigma^0$$

$$\gamma_1 = 2\mu \tag{32}$$

Finally, deriving Eq. (32) respect to the invariants of $\boldsymbol{E}$, we obtain the coefficients of the tangent elasticity matrix collected in Eq. (20):

$$\alpha_1 = (1-\delta)\left[\frac{\varphi}{2}\frac{I_1(3I_2 - I_1^2) - \frac{\varepsilon(2+\alpha)}{(2+3\alpha)}I_2}{(2I_2 - I_1^2)^{3/2}} - \frac{\sigma^0(2+\alpha)I_2}{2(1+\alpha)(2I_2 - I_1^2)^{3/2}}\right] + \delta\frac{2\lambda}{(2+\alpha)}$$

$$\alpha_2 = (1-\delta)\left[\varphi\frac{-I_2 + \frac{\varepsilon(2+\alpha)}{2(2+3\alpha)}I_1}{(2I_2 - I_1^2)^{3/2}} + \frac{\sigma^0(2+\alpha)I_1}{2(1+\alpha)(2I_2 - I_1^2)^{3/2}}\right]$$

$$\alpha_3 = (1-\delta)\left[\varphi\frac{I_1 - \frac{\varepsilon(2+\alpha)}{(2+3\alpha)}}{(2I_2 - I_1^2)^{3/2}} - \frac{\sigma^0(2+\alpha)}{(1+\alpha)(2I_2 - I_1^2)^{3/2}}\right]$$

$$\alpha_4 = (1-\delta)\left[\frac{\varphi}{2}\frac{(-I_1 + \sqrt{2I_2 - I_1^2}) + \frac{\varepsilon(2+\alpha)}{(2+3\alpha)}}{\sqrt{2I_2 - I_1^2}} + \frac{\sigma^0(2+\alpha)}{2(1+\alpha)\sqrt{2I_2 - I_1^2}}\right] + 2\mu\delta \tag{33}$$

In this domain, the *principal extra stresses* assume the form:

$$\Delta t_1(\delta) = \delta\left\{\frac{2\mu}{2+\alpha}[2(1+\alpha)e_1 + \alpha e_2] - \varphi e_1 - \frac{\alpha(\sigma + \sigma^0)}{2(1+\alpha)}\right\}$$

$$\Delta t_2(\delta) = \delta\left\{\frac{2\mu}{2+\alpha}[2(1+\alpha)e_2 + \alpha e_1] - (\sigma^0 + \sigma)\right\} \tag{34}$$

Here, it is interesting to consider the case $\alpha = 0$. Thus, recalling the expression of φ, Eq. (34) reduce to

$$\Delta t_1 = 0$$

$$\Delta t_2(\delta) = \delta[2\mu e_2 - (\sigma^0 + \sigma)] \tag{35}$$

that is, the expression of t_1 in Eq. (30) is exact.

As announced in the previous section, we prove that, for $\delta > 0$, the elasticity matrix defined by Eq. (20) is positive definite. We begin to observe that, (Ogden 1997, Appendix), the positiveness of the elastic tensor is equivalent to require:

$$\left\{\begin{array}{ll} (a) & [J] = \left[\dfrac{\partial t_\alpha}{\partial e_\beta}\right], \ \alpha, \beta = 1, 2 \ \textit{is positive definite} \\ (b) & \dfrac{t_1 - t_2}{e_1 - e_2} > 0 \end{array}\right\} \tag{36}$$

Using Eqs. (24), (25) and (30), we write (36) for the three regions $\Re_i$:

$$\text{If} \quad \boldsymbol{E} \in \Re_1, \quad \left\{\begin{array}{l} [J] = \dfrac{2\mu}{2+\alpha}\begin{bmatrix} 2(1+\alpha) & \alpha \\ \alpha & 2(1+\alpha) \end{bmatrix} \\ \dfrac{t_1 - t_2}{e_1 - e_2} = \dfrac{4\mu}{2+\alpha} \end{array}\right\}$$

$$\text{If} \quad \boldsymbol{E} \in \Re_2, \quad \left\{\begin{array}{l} [J] = \dfrac{2\mu\delta}{2+\alpha}\begin{bmatrix} 2(1+\alpha) & \alpha \\ \alpha & 2(1+\alpha) \end{bmatrix} \\ \dfrac{t_1 - t_2}{e_1 - e_2} = \delta\dfrac{4\mu}{2+\alpha} \end{array}\right\}$$

$$\text{If} \quad \boldsymbol{E} \in \Re_3, \quad \left\{\begin{array}{l} [J] = \begin{bmatrix} (1-\delta)\varphi + \delta\dfrac{4\mu(1+\alpha)}{2+\alpha} & \delta\dfrac{2\mu\alpha}{2+\alpha} \\ \delta\dfrac{2\mu\alpha}{2+\alpha} & \delta\dfrac{4\mu(1+\alpha)}{2+\alpha} \end{bmatrix} \\ \dfrac{t_1 - t_2}{e_1 - e_2} = \dfrac{(1-\delta)\varphi}{(e_1 - e_2)}\left[e_1 - \dfrac{(\varepsilon+\eta)(2+\alpha)}{2(2+3\alpha)}\right] + \delta\dfrac{4\mu}{2+\alpha} > 0 \end{array}\right\} \tag{37}$$

If we assume $\mu > 0$ and $\lambda \geq 0$, we see that the conditions (a) and (b) of (37) hold simultaneously only in the region $\Re_1$. They are true in the regions $\Re_2$ and $\Re_3$ only when $\delta > 0$.

We conclude this section observing that if (36) holds, then, for the convexity of Sym, (14) is strictly monotone in the interior points of, $\Re_i$, $i = 1, 2, 3$ i.e.:

$$(\boldsymbol{T}^* - \boldsymbol{T}) \cdot (\boldsymbol{E}^* - \boldsymbol{E}) > 0, \quad \boldsymbol{E}^* \neq \boldsymbol{E} \in \text{Sym} \tag{38}$$

that is an equivalent condition to assure that the energy is a strictly convex function in the regions $\Re_i$, $i = 1, 2, 3$ of Sym.

Finally, defining the derivative of (15) respect to $\boldsymbol{E}$:

$$D_{\boldsymbol{E}}\boldsymbol{T} = (1-\delta)D_{\boldsymbol{E}}(\beta_0\mathbf{1} + \beta_1\boldsymbol{E}) + \delta(\gamma_0\mathbf{1}\otimes\mathbf{1} + \gamma_1\boldsymbol{I}) \tag{39}$$

we note that the first part of (39) is positive semi-definite whereas the second part is positive definite if $\delta > 0$. Therefore, if $\delta > 0$, then $D_{\boldsymbol{E}}\boldsymbol{T}$ is positive definite.

Only incidentally, here we recall the procedure proposed by Alfano *et al.* (2000) that is at the base of the *enhanced tangent strategy.*

In the ith structural iteration, for every Gauss point belonging to the finite element, the elasticity tensor is so evaluated:

$$\boldsymbol{C}^{(i)} = \begin{cases} \boldsymbol{C} & \text{if} \quad \boldsymbol{E} \in \Re_1 \\ \rho\boldsymbol{C} & \text{if} \quad \boldsymbol{E} \in \Re_2 \\ \boldsymbol{C}_{tan} & \text{if} \quad \boldsymbol{E} \in \Re_3 \end{cases} \tag{40}$$

where ρ is a coefficient that progressively decrease to zero when the convergence is attained. The properties of the elasticity tensor $\boldsymbol{C}$, when $\boldsymbol{E}$ belongs to $\Re_3$, were not discussed.

4. The numerical implementation

In this section we propose the numerical implementation of the problem described in the Section 3 by means of the finite element method. In particular, we use the four nodes finite element discussed by Simo *et al.* (1990) that is based on the incompatibility modes method. Of course, for the applicability of this procedure, the assumption of $\delta > 0$ is crucial.

Here we recall only the main features of the method remanding the reader to the work of Simo (1990) for further details.

According to the common usage in the finite element method, we use the matrix and vector notation. In the two-dimensional case we set:

$$\begin{aligned} \boldsymbol{\varepsilon} &= [E_{11} \;\; E_{22} \;\; 2E_{12}]^T \\ \nabla^s\boldsymbol{u} &= [u_{1,1} \;\; u_{2,2} \;\; u_{1,2} + u_{2,1}]^T \\ \boldsymbol{\sigma} &= [T_{11} \;\; T_{22} \;\; T_{12}]^T \end{aligned} \tag{41}$$

and the components of the elasticity matrix $\boldsymbol{D}$ are collected in Eq. (20).

In the method of incompatible modes it is customary to consider the strain field of the form:

$$\boldsymbol{\varepsilon} = \nabla^s\boldsymbol{u} + \bar{\boldsymbol{\varepsilon}} \tag{42}$$

where $\nabla^s\boldsymbol{u}$ is the symmetric gradient of the displacement field and $\bar{\boldsymbol{\varepsilon}}$ is the enhanced part of the strain field.

We will denote by $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{\xi})$ the isoparametric map from the two-unitary domain onto the space of the quadrilateral finite element. Let $\boldsymbol{J}(\boldsymbol{\xi}) = \partial\boldsymbol{f}/\partial\boldsymbol{\xi}$ be the gradient of the map and

$J(\boldsymbol{\xi}) = det[\boldsymbol{J}(\boldsymbol{\xi})]$ be the Jacobian determinant. Next we denote by $\boldsymbol{J}_0 = \boldsymbol{J}(\mathbf{0})$ and $J_0 = J(\mathbf{0})$ the map gradient and the Jacobian of the map evaluated at $\boldsymbol{\xi} = \mathbf{0}$, respectively.

The main idea of Simo and Rifai (1990) is to interpolate the enhanced part of the strain field in the isoparametric space and push it forward in the physical space according to standard rules of tensor calculus. Specifically, let $\tilde{\boldsymbol{E}}$ be the strain matrix in the isoparametric space, the enhanced field $\bar{\boldsymbol{\varepsilon}}$ in the physical space is obtained by the formula:

$$\bar{\boldsymbol{\varepsilon}} = \boldsymbol{G}\boldsymbol{\alpha} = \frac{J_0}{J}\boldsymbol{F}_0^{-1}\tilde{\boldsymbol{E}}\boldsymbol{\alpha} \tag{43}$$

The components of the Jacobian are listed in the matrix $\boldsymbol{F}_0$ and $\boldsymbol{\alpha}$ is the vector of the internal parameters, (see Simo and Rifai (1990) for details).

In the aim of Simo's work, we assume the following interpolation for the enhanced isoparametric field $\tilde{\boldsymbol{E}}$ and we write explicitly the components of $\boldsymbol{F}_0$:

$$\tilde{\boldsymbol{E}} = \begin{bmatrix} \xi & 0 & 0 & 0 & \xi\eta \\ 0 & \eta & 0 & 0 & -\xi\eta \\ 0 & 0 & \xi & \eta & \xi^2 - \eta^2 \end{bmatrix}$$

$$\boldsymbol{F}_0 = \begin{bmatrix} J_{11}^2 & J_{21}^2 & J_{11}J_{21} \\ J_{12}^2 & J_{22}^2 & J_{12}J_{22} \\ 2J_{11}J_{12} & 2J_{21}J_{22} & J_{11}J_{22} + J_{12}J_{21} \end{bmatrix}_{\boldsymbol{\zeta} = \mathbf{0}} \tag{44}$$

Finally, we define by $\boldsymbol{B}$ the standard strain displacement matrix in order to obtain the relation between the symmetric gradient of the displacement field and the vector $\boldsymbol{d}$ of the nodal displacements: $\nabla^s \boldsymbol{u} = \boldsymbol{Bd}$.

The solution of the structural problem requires to solve the discrete non-linear system of equations:

$$\bigcup_{e=1}^{nel} [\boldsymbol{r}_e(\boldsymbol{d}_e, \boldsymbol{\alpha}_e) - \boldsymbol{f}_e] = \mathbf{0}$$

$$\boldsymbol{h}_e(\boldsymbol{d}_e, \boldsymbol{\alpha}_e) = \mathbf{0}, (e = 1, 2, \ldots, nelem) \tag{45}$$

where $\bigcup_{e=1}^{nel}$ denotes the standard assembly operator, $\boldsymbol{f}_e$ is the vector of the external forces including the equivalent thermal forces and:

$$\boldsymbol{r}_e = \int_B \boldsymbol{B}^T \boldsymbol{\sigma} dV$$

$$\boldsymbol{h}_e = \int_B \boldsymbol{G}^T \boldsymbol{\sigma} dV \tag{46}$$

The solution is accomplished by the Newton Raphson procedure that incorporates static condensation of the parameters $\boldsymbol{\alpha}_e$ at the element level.

The method leads to solve the following incremental system:

$$\bigcup_{e=1}^{nel} \left\{ \begin{pmatrix} \boldsymbol{r}-\boldsymbol{f} \\ \boldsymbol{h} \end{pmatrix} + \begin{bmatrix} \boldsymbol{k} & \boldsymbol{\Gamma}^T \\ \boldsymbol{\Gamma} & \boldsymbol{H} \end{bmatrix} \begin{pmatrix} \Delta \boldsymbol{d} \\ \Delta \boldsymbol{\alpha} \end{pmatrix} \right\} = \boldsymbol{0} \tag{47}$$

and starting with the elastic elasticity matrix, the iteration, over a single loop, proceeds as follow.

At the step k we suppose to know the following quantities: $\boldsymbol{d}^k$, $\boldsymbol{\alpha}_e^k$, $\boldsymbol{h}_e^k$, $\boldsymbol{H}_e^k$ and $\boldsymbol{\Gamma}_e^k$, then:

(a) Given an increment in displacement, recover the total value:

$$\boldsymbol{d}^{k+1} = \boldsymbol{d}^k + \Delta \boldsymbol{d}^k \tag{48}$$

(b) Update at the element level $\boldsymbol{\alpha}_e^k$ by setting:

$$\boldsymbol{\alpha}_e^{k+1} = \boldsymbol{\alpha}_e^k - [\boldsymbol{H}_e^k]^{-1}[\boldsymbol{\Gamma}_e^k \Delta \boldsymbol{d}_e^k + \boldsymbol{h}_e^k] \tag{49}$$

(c) For each of the four Gauss point, compute the total strain by Eq. (42):

$$\boldsymbol{\varepsilon}^{k+1} = \boldsymbol{B}\boldsymbol{d}_e^{k+1} + \boldsymbol{G}\boldsymbol{\alpha}_e^{k+1} \tag{50}$$

in order to find, by Eqs. (23), $(25)_1$ and (28), the regions $\Re_i$, $i = 1, 2, 3$.

(d) Then, for each Gauss point, compute the value of β_0, β_1, γ_0 and γ_1 in order to find, by Eq. (15) the stress vector $\boldsymbol{\sigma}^{k+1}$

(e) Compute, by Eqs. (19) and (20), the components of the tangent elasticity matrix $\boldsymbol{D}_t^{k+1}$

(f) Integrate element matrices and residuals:

$$\begin{aligned} \boldsymbol{H}_e^{k+1} &= \int_B \boldsymbol{G}^T \boldsymbol{D}_t^{k+1} \boldsymbol{G}\, dV \\ \boldsymbol{\Gamma}_e^{k+1} &= \int_B \boldsymbol{G}^T \boldsymbol{D}_t^{k+1} \boldsymbol{B}\, dV \\ \boldsymbol{k}_e^{k+1} &= \int_B \boldsymbol{B}^T \boldsymbol{D}_t^{k+1} \boldsymbol{B}\, dV \\ \boldsymbol{h}_e^{k+1} &= \int_B \boldsymbol{G}^T \boldsymbol{\sigma}^{k+1}\, dV \\ \boldsymbol{r}_e^{k+1} &= \int_B \boldsymbol{B}^T \boldsymbol{\sigma}^{k+1}\, dV \end{aligned} \tag{51}$$

(g) Perform static condensation:

$$\begin{aligned} \hat{\boldsymbol{r}}_e^{k+1} &= \boldsymbol{r}_e^{k+1} - \boldsymbol{\Gamma}_e^{(k+1)^T} \boldsymbol{H}_e^{(k+1)^{-1}} \boldsymbol{h}_e^{k+1} \\ \boldsymbol{K}_e^{k+1} &= \boldsymbol{k}_e^{k+1} - \boldsymbol{\Gamma}_e^{(k+1)^T} \boldsymbol{H}_e^{(k+1)^{-1}} \boldsymbol{\Gamma}_e^{k+1} \end{aligned} \tag{52}$$

(h) Assemble and solve for the new increment of displacements.

$$\left[\bigcup_{e=1}^{nel} \boldsymbol{K}_e^{k+1}\right]\Delta\boldsymbol{d}^{k+1} = \bigcup_{e=1}^{nel}(\boldsymbol{f}_e - \hat{\boldsymbol{r}}_e^{k+1}) \tag{53}$$

(i) Compute the relative error and check for the convergence:

$$\varepsilon = \frac{\|\Delta\boldsymbol{d}^{k+1}\|}{\|\boldsymbol{d}^k\|} \tag{54}$$

(j) If $\varepsilon \leq Tol$ then finish and test for the global balance forces, else go to (a).

5. Numerical examples

The proposed numerical method has been implemented on an existing FEM code named *Solver* that is distributed and commercialized in Italy by a software house.

In order to illustrate the effectiveness of the numerical method developed in the preceding section, we perform some numerical simulations. First, we consider a rectangular block subjected to a trapezoidal load and to a thermal load. Next we consider a cantilever beam subjected to a constant curvature and to a uniform thermal strain in the cases of no-tensile strength and bounded tensile strength, respectively.

Finally, we analyze in detail a concrete example inherent a masonry panel in order to show the convergence properties of the proposed numerical method.

5.1 The rectangular block

This is a rectangular block in plane stress state supported by a rigid plane. A trapezoidal load, as shown in Fig. 2, loads the block that is subject also to a thermal strain. The material is assumed with no-tensile strength.

This example has been studied by Lucchesi *et al.* (1990) which shown the analytical solution in

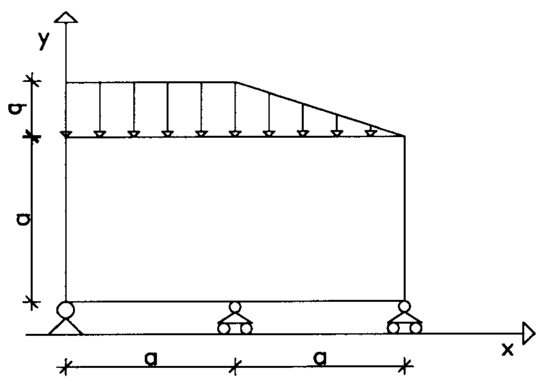

Fig. 2 The rectangular block

the case of the plane strain. In the case of the plane stress and assuming a unitary thickness, the solution is:

$$v(x,y) = \begin{Bmatrix} -\dfrac{qy}{E} + \alpha\Delta\theta y, & x \in [0,a] \\ -\dfrac{1}{E}\dfrac{q}{a}(2a-x)y + \alpha\Delta\theta y, & x \in [a,2a] \end{Bmatrix}$$

$$\sigma_y(x,y) = \begin{Bmatrix} -q, & x \in [0,a] \\ -\dfrac{q}{a}(2a-x), & x \in [a,2a] \end{Bmatrix}$$

The block is discretized first into fifty finite elements for a total of sixty-six nodes and next into two hundred elements for a total of two hundred and thirty-one nodes.

The following data are assumed:

$a = 5.0$ m, $h = 5.0$ m, $th = 1.0$ m
$q = 0.001$ GPa, $\nu = 0.1$, $E = 5.0$ GPa
$\sigma = 0$, $\alpha_t = 1.0\ \mathrm{E}^{-5}(^{\circ}\mathrm{C})^{-1}$, $\Delta\theta = -20.0\ ^{\circ}\mathrm{C}$
$\delta = 0.002$, $Tol = 1.0\ \mathrm{E}^{-5}$

and the analysis results are summarized in Table 1.

The inspection of the Table 1 show that the results are in agreement to the analytical values and that the refinement of the mesh has not a meaningful effect on the solution. The use of the coarse mesh is sufficient to fully describe the behavior of the loaded block. Furthermore, the *principal extra stresses* assume very small values.

In order to assess the effectiveness of the proposed numerical method, i.e., the consequences of

Table 1 The rectangular block

Displacements v(m) × 10^{-4} and stress $\sigma\sigma$, Δt (GPa) × 10^{-3}			
	50 elements	200 elements	Theoretical
$v(x = 0, y = a)$	–20.005	–19.999	–20.0
$v(x = a, y = a)$	–19.661	–19.631	–20.0
$v(x = 2a, y = a)$	–10.004	–9.9987	–10.0
$\sigma_Y(x = 0, y = a)$	–1.0013	–0.9995	–1.00
$\sigma_Y(x = a, y = a)$	–0.9996	–0.9999	–1.00
$\sigma_Y(x = 2a, y = a)$	$-1.2809\mathrm{E}^{-3}$	$2.2817\mathrm{E}^{-6}$	0
$\sigma_X(x = 0, y = a)$	–0.0103	0.0000221	0
$\sigma_X(x = a, y = a)$	–0.108	–0.095	0
$\sigma_X(x = 2a, y = a)$	–0.010	0.000021	0
$\Delta t_1(x = 0, y = a)$	0	$2.12\mathrm{E}^{-6}$	0
Δt_1 (x = a, $y = a$)	0	0	0
Δt_1 ($x = 2a, y = a$)	0	$2.28\mathrm{E}^{-6}$	0
n. of. iterations	11	12	=

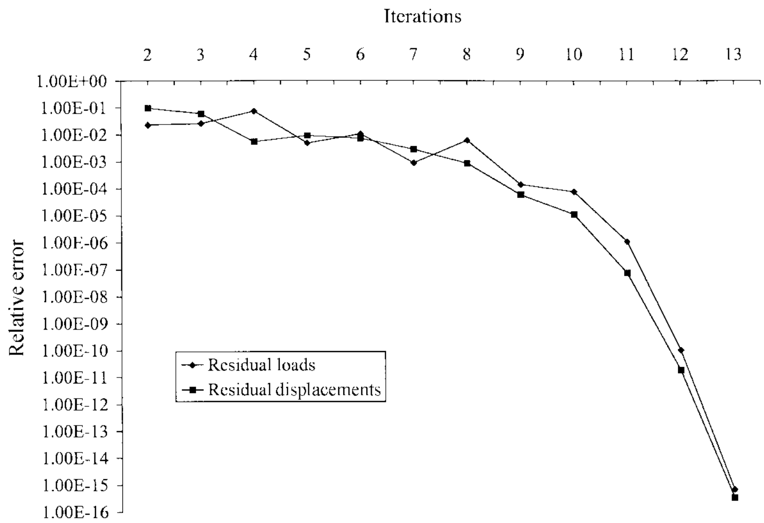

Fig. 3 The rectangular block loaded by a trapezoidal load: relative error vs. number of iterations

the assumption on the modified constitutive equations (see Eqs. (8) and (9)) and the choice of the finite element type (the model of Simo and Rifai), we reconsider the example of the block comparing the results in terms of convergence with that shown in the paper of Lucchesi *et al.* (1994).

In the cited work, the authors used an eight-node element and discretized the block into one hundred and twenty-eight elements for a total number of four hundred and thirty-tree joints. They defined the relative convergence error ξ as the ratio of the norm of the residual forces vector on the norm of the applied forces vector. Setting $\xi = 10^{-3}$, the convergence was reached in fifteen iteration and the norm of the residual forces was $0.2 \times 10^{-3} \, |\boldsymbol{f}_e|$.

Using our proposed numerical method, we have tested the block subjected only to the trapezoidal load forcing the relative error ξ (or ε) to be 10^{-14}. Thus, the convergence was attained in thirteen iterations and for nine iterations the relative error was about 10^{-4}. The plot is presented in Fig. 3.

The same examples were examined using a four nodes isoparametric finite element assuming $\delta = 0$. The elaboration was stopped for the apparition, in the phase of the stiffness matrix reduction, of a negative term on the diagonal.

5.2 The cantilever beam

This is a cantilever beam subjected to a constant curvature and to a uniform thermal strain. The geometry is shown in Fig. 4.

This example is necessary to test the performance of the proposed numerical method in bending dominated situations. We assume the following data:

$$L = 10.0\text{ m}, \quad h = 2.0\text{ m}, \quad th = 1.0\text{ m}, \quad E = 1500\text{ GPa},$$
$$\nu = 0.25, \quad \alpha_t = 0.00025(^{\circ}\mathrm{C})^{-1}, \quad \Delta\theta = 100^{\circ}\mathrm{C}, \quad \delta = 0.001$$

For $\sigma = 0$ and under a constant curvature $\chi = 0.0010$, $q = 1.50$ GPa. In the second load condition we assume $\sigma = 0.75$ GPa.

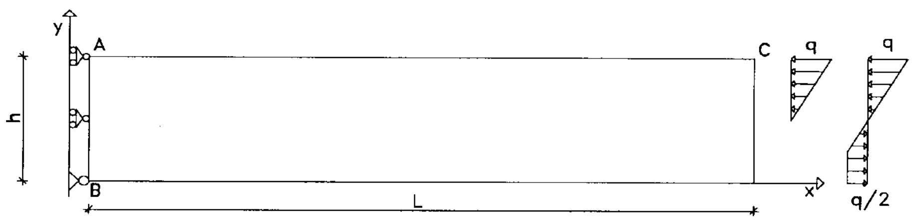

Fig. 4 The cantilever beam

Table 2 The cantilever beam

	Displacements u(m) and stress σ, Δt (GPa)			
	$\sigma = 0$		$\sigma = 0.75$	
	Computed	Theoretical	Computed	Theoretical
u_y(C)	0.09979	0.10	0.10001	0.10
u_x(C)	0.24002	0.24	0.24000	0.24
σ_x(B)	0.001581	0	0.7508	0.75
σ_x(A)	−1.4953	−1.50	−1.4996	−1.50
Δt_2(B)	0.001581	0	0.000798	0
Δt_1(A)	1.678E^{-9}	0	0	0

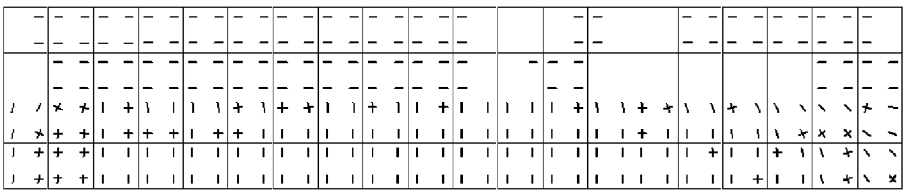

Fig. 5 The fractured cantilever beam

The theoretical vertical displacement of the cantilever tip beam is $v = \chi L^2/2 + \alpha_t \Delta\theta h = 0.10$ m. Discretizing the beam into eighty elements and setting the relative error $\varepsilon = 10^{-5}$, the convergence was attained in eight iterations and the analysis results are summarized in Table 2. Notice that the results are in perfect agreement to the theoretical one.

In the case of no-tensile strength and in absence of the thermal loads, the Fig. 5 illustrates the distribution of the fractures evaluated at the four Gauss points of the finite element belonging to the mesh. The directions of the fractures are obtained considering the eigenvectors of the anelastic strains. By the fracture distribution, we note that at the Gauss points, almost everywhere, the strain $\boldsymbol{E}$ belongs to the region $\Re_3$, that is, for $\delta = 0$, the tangent elasticity matrix is semi-definite positive.

5.3 The masonry panel

This example was exhaustively studied by Alfano *et al.* (2000) in the case of no tensile strength and it is inherent to a masonry wall endowed by apertures that is progressively loaded by a lateral

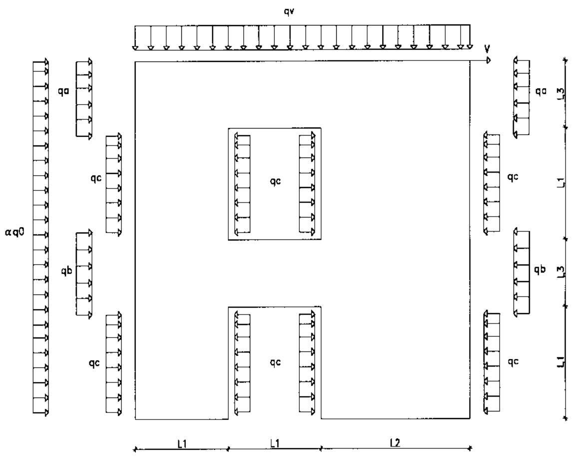

Fig. 6 The masonry panel with apertures

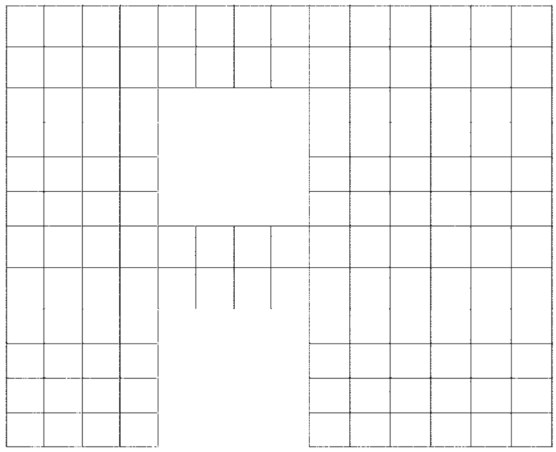

Fig. 7 The discretized masonry panel

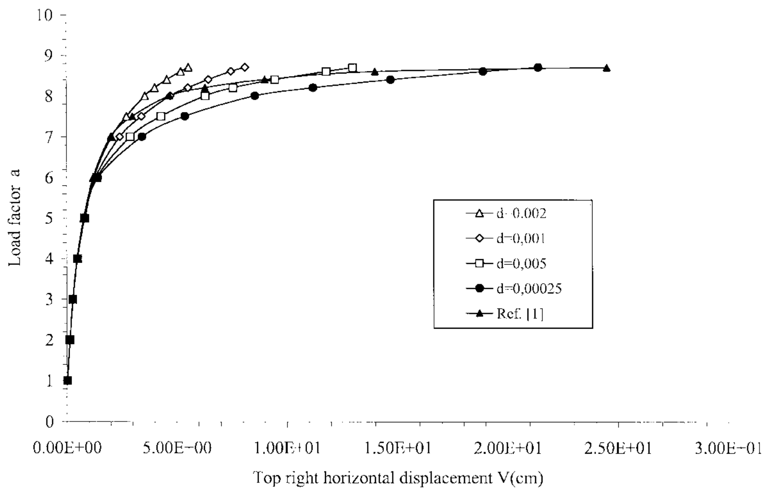

Fig. 8 The masonry panel: load-displacement curve

load which intensity is governed by a multiplier α. The geometrical scheme is shown in Fig. 6 and the following data are assumed:

$$L_1 = 3.0 \text{ m}, \quad L_2 = 4.80 \text{ m}, \quad L_3 = 1.8 \text{ m}, \quad th = 1.0 \text{ m}, \quad E_m = 1.0 \text{ GPa}, \quad \nu = 0.2$$
$$q_a = 1.0E^5 Pa, \quad q_b = 7.0E^4 Pa, \quad q_c = 1.0E^3 Pa, \quad q_0 = 2.0E^4 Pa, \quad q_v = 1.428E^5 Pa$$

For this example, Alfano *et al.* (2000) used an eight nodes finite element, a nine Gauss points scheme for the integration and a tolerance $\varepsilon = 10^{-16}$.

In our numerical simulation, we assume a discretization into one hundred and thirty-six elements for a total of one hundred and seventy-four nodes in contrast to a fifty hundred and ninety-four nodes necessary for the discretization with the eight nodes element. The mesh is shown in Fig. 7.

In order to force the tolerance to a value $\varepsilon = 10^{-14}$ and the parameter δ to small values, we assume a nine Gauss points integration scheme, that is, it is necessary to evaluate accurately the

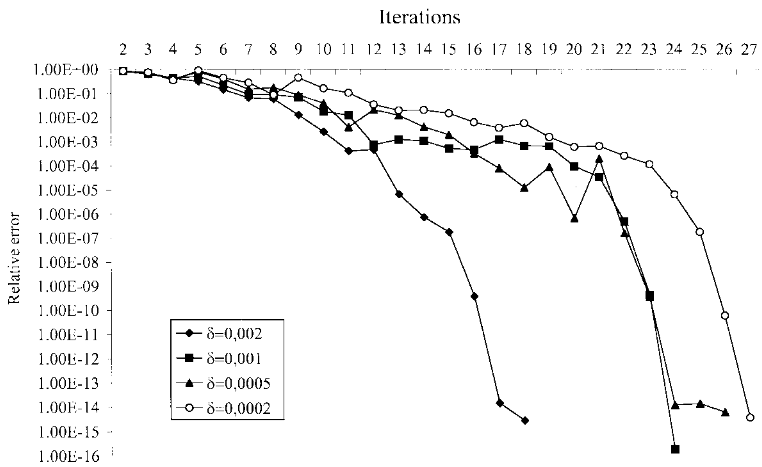

Fig. 9 The masonry panel: relative error vs. number of iterations

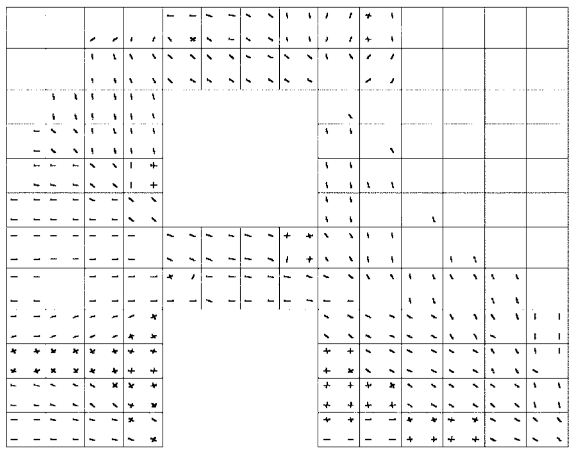

Fig. 10 The fractured masonry panel

stiffness matrix and the residual loads.

In Fig. 8 we present the curves load-displacement of the top right corner of the panel obtained increasing the load factor α and decreasing the value of δ. We see that the collapse load was attained for a value of $\alpha \cong 8.7$ that is in agreement to the value found by Alfano *et al.* (2000). We remark that the value of δ was stopped at 0.00025 because, for its smaller values and for some load factor α, the proposed numerical method became unstable. This is not amazing because the numerical instability is implicitly contained in the problem when $\delta = 0$.

In the Fig. 9 for $\alpha = 8.7$, we present the curves relating the relative error to the iterations number. Notice that for $\delta = 0.00025$, the convergence was obtained in twenty-seven iteration whereas the best performance obtained by Alfano *et al.* (2000) was fifty-five iterations for an error on the energy norm as $1.0E^{-9}$.

Finally, only for an illustrative picture, in Fig. 10 it is presented the distribution the fracture evaluated at the four Gauss points.

6. Conclusions

We have presented an approximated numerical method to study structural problems regarding bodies made of masonry-like materials with bounded tensile strength in presence of thermal strains. By setting $\delta > 0$, we have shown that the tangent elasticity tensor is positive definite and thus the proposed numerical method is stable and convergent.

The numerical examples show that the proposed numerical method is effective too. The use of the four nodes finite element based on the model by Simo and Rifai (1990) contributes to reduce the equations system number and to obtain the same accuracy that may be reached by means of the employment of the eight nodes element.

Furthermore, the convergence rate is fast, especially when δ is not too much small. Moreover, the illustrative examples show that the analysis results obtained by the proposed numerical method, are in agreement with the theoretical solution.

We conclude remarking that, for values of $\delta = 0.002 \div 0.005$, four Gauss points are sufficient to integrate the stiffness matrix and to compute the residual load vector whereas, for smaller values of δ, a nine Gauss points integration scheme is recommended.

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