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A spatial displacement model for horizontally curved beams

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Abstract. A new approach to the analysis of horizontally curved beams is presented in this paper. The proposed method simplifies a two-dimensional structure into a one-dimensional structure just like a normal beam for structural analysis and, therefore, reduces the computational effort significantly.

Key words: curved beam; finite element; force method; stiffness method.

1. Introduction

Over the last two decades, there has been a steady increase in the use of curved bridges (Hall 1996). Although horizontally curved steel bridges constitute roughly one-third of all steel bridges being erected today, their structural behavior is not well understood (Hall 1996, Luo and Li 2000). Basically, there are four main methods used for curved bridge analysis: (1) plane grid and space frame methods that treat curved members as straight members (Hall 1996, *Li et al*. 1996); (2) numerical and analytical methods, such as finite strip, finite difference, slope-deflection, finite element, and closed form solutions to differential equations (Li, Yang, Ou, Li and Liu 2001; Li, Yang and Li 2001; Li 2001); (3) experimental methods (Zureick *et al*. 2000, Pi *et al.* 2000, Shanmugam *et al*. 1995, Rajasekaran and Padmanabhan 1989); (4) the force method (Weaver and

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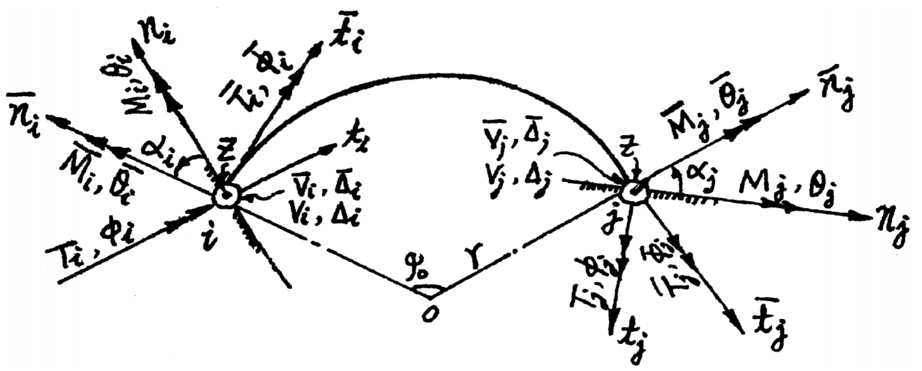

Fig. 1 Coordinate system and end forces, displacements notations

Gere 1980). Among these methods, the finite element method is probably the most involved and time consuming. However, it is still the most general and comprehensive technique. Although the force method is convenient in the analysis of continuous curved beams, it is difficult to be programmed in the analysis of curved frame structures. Therefore, it lacks generality and is no longer used as often as the displacement method (Weaver and Gere 1980). The plane grid and space frame methods are approximate methods that are only used for primary design. The experiment methods are not cost effective and are often used for checking of other methods.

In this paper, using a curved beam element in a local coordinate system, a spatial displacement field for circular beam structures was constructed. Based on structural mechanics, the equilibrium equation for a circular beam structure is derived.

2. Coordinate system and element stiffness matrix

Fig. 1 shows the circular beam element $\overset{\frown}{ij}$, with its circular angle φ_0; radius r; cross section flexure stiffness, EI; torsion stiffness, GJ; rotation angles α_i and α_j. Selecting the local coordinate system $(\bar{t}\,\bar{n}z)$ and nodal coordinate system (tnz), the element end forces and displacements are shown in Fig. 1.

2.1 Displacement equation for perpendicularly supported circular beam

As shown in Fig. 1, when $\alpha_i = \alpha_j = 0$, from the force method, the bending-torsion displacement equation for the two-fixed-ends perpendicularly supported circular beam is,

$$\{\bar{F}\} = [\bar{k}]\{\bar{\delta}\} \tag{1}$$

where $\{\bar{F}\} = [\bar{T}_i, \bar{M}_i, \bar{V}_i, \bar{T}_j, \bar{M}_j, \bar{V}_j]^T$ is the end force vector; $\{\bar{\delta}\} = [\bar{\phi}_i, \bar{\theta}_i, \bar{\Delta}_i, \bar{\phi}_j, \bar{\theta}_j, \bar{\Delta}_j]^T$ is the end displacement vector; and $[\bar{k}]$ is the element stiffness matrix. Using the force method,

$$[\bar{k}] = \begin{bmatrix} [\overline{k_{ii}}] & [\overline{k_{ij}}] \\ [\overline{k_{ji}}] & [\overline{k_{jj}}] \end{bmatrix} \tag{2}$$

where the sub matrices are,

$$[\overline{k_{ii}}] = \frac{2EI}{rC_5}\begin{bmatrix} C_8 & & symmetric \\ -C_7 & C_1 & \\ -\frac{C_{10}}{r} & -\frac{(C_1 + C_4)}{r\varphi_0} & C_{11} \end{bmatrix}$$

$$[\overline{k_{ij}}] = \frac{2EI}{rC_5}\begin{bmatrix} C_9 & C_6 & \frac{C_{10}}{r} \\ -C_6 & C_4 & \frac{(C_1 + C_4)}{r\varphi_0} \\ \frac{C_{10}}{r} & -\frac{(C_1 + C_4)}{r\varphi_0} & -C_{11} \end{bmatrix}$$

$$[\overline{k_{ji}}] = \frac{2EI}{rC_5}\begin{bmatrix} C_9 & -C_6 & \frac{C_{10}}{r} \\ C_6 & C_4 & -\frac{(C_1 + C_4)}{r\varphi_0} \\ \frac{C_{10}}{r} & \frac{(C_1 + C_4)}{r\varphi_0} & -C_{11} \end{bmatrix}$$

$$[\overline{k_{ii}}] = \frac{2EI}{rC_5}\begin{bmatrix} C_8 & & symmetric \\ C_7 & C_1 & \\ -\frac{C_{10}}{r} & \frac{(C_1 + C_4)}{r\varphi_0} & C_{11} \end{bmatrix}$$

where, $$C_1 = \varphi_0^2(k+1) + \varphi_0(k-1)\sin\varphi_0\cos\varphi_0 - 2k\sin^2\varphi_0;$$

$$C_2 = \frac{(k+1)(\varphi_0 - \sin\varphi_0\cos\varphi_0)}{2k};$$

$$C_3 = \frac{[\varphi_0(k+1) + (k-1)\sin\varphi_0\cos\varphi_0]}{2k};$$

$$C_4 = 2k\sin^2\varphi_0 - \varphi_0(k-1)\sin\varphi_0 - \varphi_0^2(k+1)\cos\varphi_0;$$

$$C_5 = \frac{(C_1^2 - C_4^2)}{\varphi_0\sin^2\varphi_0};$$

$$C_6 = (k+1)[(1-\cos\varphi_0)(\varphi_0+\sin\varphi_0)-\varphi_0^2\sin\varphi];$$

$$C_7 = (1-\cos\varphi_0)\{(3k-1)\sin\varphi_0-\varphi_0[2k+(k-1)\cos\varphi_0]\};$$

$$C_8 = \frac{[C_7^2+C_5(\varphi_0+C_3-2\sin\varphi_0)]}{C_1};$$

$$C_9 = \frac{[C_6C_7+C_5(\sin\varphi_0-\varphi_0\cos\varphi_0-C_2)]}{C_1};$$

$$C_{10} = \frac{C_6(C_1+C_4)}{\varphi_0C_1}+\frac{C_5C_2}{C_1};$$

$$C_{11} = \frac{[(C_1+C_4)^2+\varphi_0^2C_3C_5]}{r^2\varphi_0^2C_1}$$

$$k = \frac{EI}{GJ}.$$

2.2 Stiffness matrix of the rotated supported circular beam element

As shown in Fig. 1, in the nodal coordinate system (*tnz*), the vector $\{\delta\} = [\phi_i, \theta_i, \Delta_i, \phi_j, \theta_j, \Delta_j]^T$ and the vector $\{F\} = [T_i, M_i, V_i, T_j, M_j, V_j]^T$ are, respectively, the displacement and internal force vectors of the rotated cross section at the support line. The rotated circular element equilibrium equations can be expressed as $\{\bar{F}\} = [\bar{k}]\{\bar{\delta}\}$, where [*k*] is the element stiffness matrix in the nodal coordinate system. The transformation relationship between the local coordinate system and the nodal coordinate system is

$$\{\bar{\delta}\} = [T]\{\delta\}, \quad \{\bar{F}\} = [T]\{F\} \tag{3}$$

The coordinate transformation matrix is

$$[T] = \begin{bmatrix} [T_i] & [0] \\ [0] & [T_j] \end{bmatrix}, \qquad [T_s] = \begin{bmatrix} \cos\alpha_s & \sin\alpha_s & 0 \\ -\sin\alpha_s & \cos\alpha_s & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (s = i, j) \tag{4}$$

So, the element stiffness matrix in the nodal coordinate system is

$$[k] = [T]^T[\bar{k}][T] \tag{5}$$

3. Equivalent nodal forces

The equivalent nodal force vector for the circular beam element under loads between the end points is

$$\{P_F\} = -[T]^T\{\bar{F}_0\} \tag{6}$$

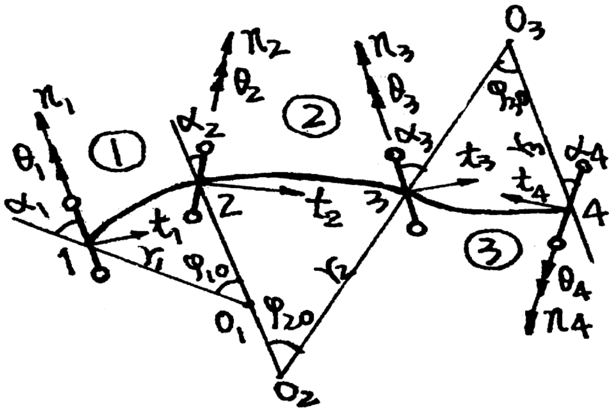

Fig. 2 Three span continuous circular beams

Table 1 Reaction and torsion of the three span circular beam at supports

Support Condition	Support Reaction (MN)				Support Torsion (MNm)			
	V_1	V_2	V_3	V_4	T_1	T_2	T_3	T_4
Perpendicular	5.63	38.71	40.63	8.05	-2.29	40.07	21.81	16.72
Rotated	5.82	38.50	40.72	8.07	-29.16	26.64	42.48	0.88

where $\{\bar{F}_0\}$ is the end force vector of the element in the local coordinate system.

The direct stiffness method can be used to establish the global stiffness matrix of the structure in the global coordinate system (nodal coordinate system),

$$[K_0]\{\Delta_0\} = \{P_0\} \tag{7}$$

where the structural stiffness matrix, $[K_0]$, can be constructed using each element in the nodal coordinate system; $\{\Delta_0\}$ is the nodal displacement vector; $\{P_0\}$ is the nodal force vector, and it is composed of the equivalent nodal forces $\{P_F\}$ and the structural nodal loads. Introducing support conditions into Eq. (7), the global stiffness matrix can be modified and solved for the nodal displacements and support reactions (Fang *et al.* 1998).

4. Examples

4.1 Example 1

As shown in Fig. 2, a three span constant cross section circular beam, under uniformly distributed load $q = 220$ kN/m; $EI = 3 \times 10^9$ kNm^2; $k = EI/GJ = 1.5$; $r_1 = 150$ m; $\varphi_{10} = 40^o$; $r_2 = 250$ m, $\varphi_{20} = 45^o$; $r_3 = 200$ m, $\varphi_{30} = 35^o$.

Table 1 lists the results of the support forces and torsions under the perpendicular ($\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$) and the rotated ($\alpha_1 = 30^o$, $\alpha_2 = -\alpha_3 = 20^o$, $\alpha_4 = 30^o$) support conditions. The results are identical with those obtained by the force method (not shown here).

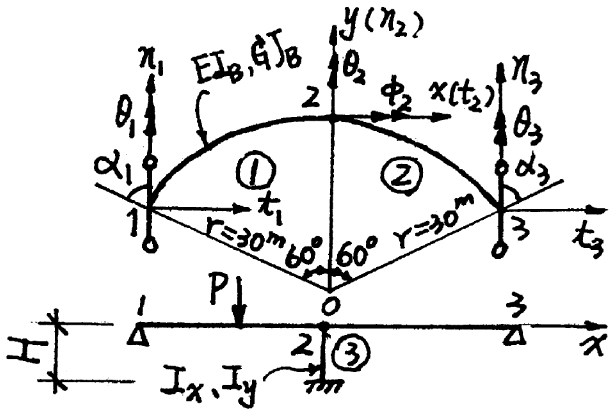

Fig. 3 Two span circular beam frame

Table 2 Torsion and bending moment of the two span circular beam frame at ends

Support Condition	Internal Torsion at ends (MNm)				Internal Moment at ends (MNm)			
	$T_{12}(1)$	$T_{21}(1)$	$T_{23}(2)$	$T_{32}(2)$	$M_{12}(1)$	$M_{21}(1)$	$M_{23}(2)$	$M_{32}(2)$
Perpendicular	5.63	38.71	40.63	8.05	0	2.94	−1.0	0
Rotated	5.82	38.50	40.72	8.07	−1.28	2.19	−0.88	0.28

4.2 Example 2

Fig. 3 shows a two-span constant-cross-section circular beam frame system, under concentrated load $P = 0.568$ kN/m; $k = EI_B/GJ_B = 2$; $i = EI_B/r$; $i_x = EI_x/H$, $i_y = EI_y/H$; let $i = i_x = i_y = 1$.

The stiffness matrices for the circular elements (1) and (2) are determined using Eq. (5); the column element (3) is a two-end-fixed straight beam element. Table 2 presents the results of the support forces and torsions under the perpendicular ($\alpha_1 = \alpha_2 = \alpha_3 = 0$) and rotated ($\alpha_1 = -\alpha_3 = 60^o$, $\alpha_2 = 0$) supported conditions.

5. Conclusions

The proposed approach can be used for the analysis of curved beams under various support conditions. The approach can be easily programmed to solve large-scale problems, or used on a calculator for two or three span horizontally curved structures.

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