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# Effect of shear deformation on the critical buckling of multi-step bars

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**Abstract.** The governing differential equation for buckling of a one-step bar with the effect of shear deformation is established and its exact solution is obtained. Then, the exact solution is used to derive the eigenvalue equation of a multi-step bar. The new exact approach combining the transfer matrix method and the closed form solution of one step bar is presented. The proposed methods is convenient for solving the entire and partial buckling of one-step and multi-step bars with various end conditions, with or without shear deformation effect, subjected to concentrated axial loads. A numerical example is given explaining the proposed procedure and investigating the effect of shear deformation on the critical buckling force of a multi-step bar.

**Key words:** stability; buckling; shear deformation; bar.

## 1. Introduction

It is well known that the analytical model for buckling analysis of various civil, aeronautical and mechanical structures can be treated as a uniform or non-uniform bar subjected to concentrated and/or variable distributed axial loads. However, it is difficult to determine the exact solution for the buckling of a non-uniform bar subjected to complicated loads, especially for a multi-step bar. Simple problems, such as a one-step non-uniform bar subjected to a concentrated axial load, a uniform bar subjected to uniformly distributed axial loads or a uniform bar under its own weight, were studied by Timoshenko (1936), Karman and Biot (1940), Dinnik (1950). More complicated problems, such as buckling of columns under variably distributed axial loads, were investigated numerically by Vaziri and Xie (1992). The exact buckling solutions of a one-step bar and a multi-step bar with varying cross-section under concentrated and variably distributed axial loads were investigated by Li *et al*. (1995) and Li (2000, 2001, 2002). It is noted that all the studies mentioned above did not consider the effect of shear deformation on the critical buckling of bars or columns.

In deriving the governing differential equation for buckling of a bar, it is usually assumed that the cross-sectional dimensions of the bar are small compared to its length, and thus the effect of shear deformation on the buckling load can be neglected. However, Iyengar (1988), among others, reported that as the depth of the bar increases, this effect has to be taken into account for the correct

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estimation of the critical load since shear deformation reduces the critical load. Ari-Gur and Elishkoff (1990) discussed the influence of shear deformation on buckling of uniform columns with overhang. Banerjee and Williams (1994) studied the effect of shear deformation on the critical buckling of a one-step uniform column in detail. They indicated that fiber-reinforced composite beams are generally more shear sensitive than metallic ones, because of their low shear modulus ($G$) to Young's modulus ($E$) ratio, so that the effect of shear deformation on the critical buckling can be significant.

If the effect of shear deformation is taken into account, it is difficult to obtain the exact buckling solution of a bar with varying cross-section subjected to concentrated and distributed axial loads, especially, for a multi-step bar. As reported by Li *et al*. (1994), in buckling analysis, a one-step bar with varying cross-section subjected to distributed axial load may be simplified as a multi-step uniform bar under concentrated loads. Thus, this paper addresses the effect of shear deformation on the critical buckling load of a multi-step bar, for which each step is assumed to have constant parameters. In the paper published by Banerjee and Williams (1994), the eigenvalue equation of a one-step uniform bar was established by using two differential equations. In this paper, however, using one differential equation and the transfer matrix method the eigenvalue equation of a multi-step bar is obtained, and the stability problem of a one-step bar is shown to be a special case of the general problem.

## 2. Theory

A multi-step bar is shown in Fig. 1. The axial load in the $i$-th step bar, $N_i$, is given by

$$N_i = \sum_{j=1}^{i} a_j P \tag{1}$$

where the load $a_jP$ is directly acting on the $j$-th step bar.

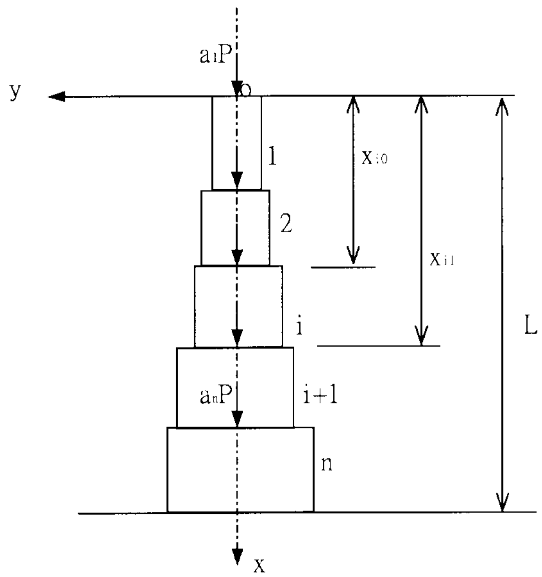


Fig. 1 A multi-step bar

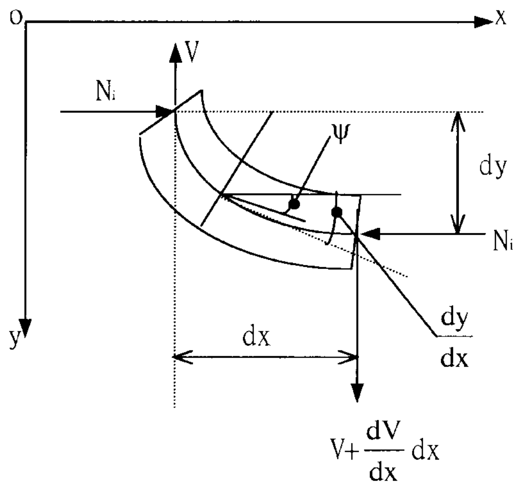


Fig. 2 Force acting on a typical element

In order to establish the differential equation for buckling of the *i*-th step bar with shear deformation, an infinitesimal element of the *i*-th step bar is considered. For convenience, the element is rotated through an angle of 90°, as shown in Fig. 2. Now considering the equilibrium of all the forces acting on the element leads to (Banerjee and Williams 1994)

$$\frac{dV}{dx} = 0 \tag{2}$$

$$V = \frac{dM}{dx} - N_i\frac{dy}{dx} \tag{3a}$$

where $V$ is the shear force due to bending, and the total shear force, $Q$, is as follows

$$Q = \frac{dM}{dx} = V + N_i\frac{dy}{dx} \tag{3b}$$

As is well known, the relationship between $M$ and $\psi$ is given by

$$M = -EI_i\frac{d\psi}{dx} \tag{4}$$

where $\psi$ is the angle of rotation of the cross-section due to bending, $EI_i$ is the flexural rigidity of the *i*-th step bar.

The total slope, $dy/dx$, equals the sum of bending slope and the slope due to shearing of the element. Thus, the following equation can be established.

$$\psi = \left(1 - \frac{N_i}{K_i}\right)\frac{dy}{dx} - \frac{V}{K_i} \tag{5}$$

in which $K_i$ is the shear rigidity of the *i*-th step bar, it is given by

$$K_i = kGA_i \tag{6}$$

where $k$, $G$, $A_i$ are the section shape factor, shear modulus and cross-sectional area, respectively.

Substituting Eq. (5) into Eq. (4) yields

$$M = -EI_i\left(1-\frac{N_i}{K_i}\right)\frac{d^2y}{dx^2} \tag{7}$$

Substituting Eq. (7) into Eq. (3a) leads to

$$V = -EI_i\left(1-\frac{N_i}{K_i}\right)\frac{d^3y}{dx^3} - N_i\frac{dy}{dx} \tag{8}$$

Substituting Eq. (8) into Eq. (5) one obtains

$$\psi = \frac{EI_i}{K_i}\left(1-\frac{N_i}{K_i}\right)\frac{d^3y}{dx^3} + \frac{dy}{dx} \tag{9}$$

Using Eqs. (4), (7) and (9) one obtains the governing differential equation for buckling of the *i*-th step bar with shear deformation as follows

$$\frac{d^4y}{dx^4} + \alpha_i^2\frac{d^2y}{dx^2} = 0 \tag{10}$$

where

$$\alpha_i^2 = \frac{K_iN_i}{EI_i(K_i-N_i)} = \frac{N_i}{EI_i\left[1-\dfrac{N_i}{\dfrac{kG}{E}\left(\dfrac{l_i^2}{r_i^2}\right)\dfrac{EI_i}{l_i^2}}\right]} \tag{11}$$

where $r_i$ is the radius of gyration of the cross-section, $l_i$ is the length of the *i*-th step bar.

The general solution of Eq. (10) is given by

$$y(x) = C_1 + C_2x + C_3\sin\alpha_ix + C_4\cos\alpha_ix \tag{12}$$

Using Eqs. (7), (8), (9) and (12) one obtains

$$\begin{bmatrix} y(x) \\ \varphi(x) \\ M(x) \\ V(x) \end{bmatrix} = [A(x)]\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{13}$$

where

$$[A(x)] = \begin{bmatrix} 1 & x & \sin\alpha_ix & \cos\alpha_ix \\ 0 & 1 & \alpha_i\left(1-\dfrac{N_i}{K_i}\right)\cos\alpha_ix & \alpha_i\left(1+\dfrac{N_i}{K_i}\right)\sin\alpha_ix \\ 0 & 0 & N_i\sin\alpha_ix & N_i\cos\alpha_ix \\ 0 & -N_i & 0 & 0 \end{bmatrix} \tag{14}$$

It can be seen from Eq. (14) that the shear force due to bending of the *i*-th step is a constant. This observation is consistent with Eq. (2).

The relationship between the parameters introduced above at the two ends of the *i*-th step bar can be expressed as

$$\begin{bmatrix} y(x_{i1}) \\ \varphi(x_{i1}) \\ M(x_{i1}) \\ V(x_{i1}) \end{bmatrix} = [T] \begin{bmatrix} y(x_{i0}) \\ \varphi(x_{i0}) \\ M(x_{i0}) \\ V(x_{i0}) \end{bmatrix} \quad (15)$$

where

$$[T_i] = [A(x_{i1})][A(x_{i0})]^{-1} \quad (16)$$

$[T_i]$ is called the transfer matrix because it transfers the parameter at the end $x_{i0}$ to those at the end $x_{i1}$ of the *i*-th step bar.

Using Eq. (15) for the $(i + 1)$-th step bar one obtains

$$\begin{bmatrix} y(x_{i+1,1}) \\ \varphi(x_{i+1,1}) \\ M(x_{i+1,1}) \\ V(x_{i+1,1}) \end{bmatrix} = [T_{i+1}] \begin{bmatrix} y(x_{i+1,0}) \\ \varphi(x_{i+1,0}) \\ M(x_{i+1,0}) \\ V(x_{i+1,0}) \end{bmatrix} \quad (17)$$

Considering the following relationship

$$\begin{bmatrix} y(x_{i+1,0}) \\ \psi(x_{i+1,0}) \\ M(x_{i+1,0}) \\ V(x_{i+1,0}) \end{bmatrix} = \begin{bmatrix} y(x_{i1}) \\ \psi(x_{i1}) \\ M(x_{i1}) \\ V(x_{i1}) \end{bmatrix} \quad (18)$$

and using Eq. (15) lead to

$$\begin{bmatrix} y(x_{i+1,1}) \\ \varphi(x_{i+1,1}) \\ M(x_{i+1,1}) \\ V(x_{i+1,1}) \end{bmatrix} = [T_{i+1}][T_i] \begin{bmatrix} y(x_{i0}) \\ \varphi(x_{i0}) \\ M(x_{i0}) \\ V(x_{i0}) \end{bmatrix} \quad (19)$$

The equation for the top step bar can be established by using Eqs. (15), (18) and (19) repeatedly as follows

$$\begin{bmatrix} y(x_{n1}) \\ \varphi(x_{n1}) \\ M(x_{n1}) \\ V(x_{n1}) \end{bmatrix} = [T] \begin{bmatrix} y(x_{10}) \\ \varphi(x_{10}) \\ M(x_{10}) \\ V(x_{10}) \end{bmatrix} \tag{20}$$

where

$$[T] = [T_n][T_{n-1}]\ldots[T_1] \tag{21}$$

and $[T]$ has the following form

$$[T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \tag{22}$$

The element $T_{ij}$ of $[T]$ can be found from Eq. (21).

The eigenvalue equation can be established by using Eq. (20) and the boundary conditions as follows

*(1) A multi-step bar with clamped-free (C - F) end conditions*

For this case Eq. (20) becomes

$$\begin{bmatrix} 0 \\ 0 \\ M(x_{n1}) \\ V(x_{n1}) \end{bmatrix} = [T] \begin{bmatrix} y(x_{10}) \\ \varphi(x_{10}) \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

From the above equation, we have

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} y(x_{10}) \\ \psi(x_{10}) \end{bmatrix} = 0 \tag{24}$$

Because $y(x_{10}) \neq 0$, $\psi(x_{10}) \neq 0$, the eigenvalue equation is

$$T_{11}T_{22} - T_{12}T_{21} = 0 \tag{25}$$

*(2) A multi-step bar with hinged-hinged (H - H) end conditions*

Eq. (20) for this case becomes

$$\begin{bmatrix} 0 \\ \varphi(x_{n1}) \\ 0 \\ V(x_{n1}) \end{bmatrix} = [T] \begin{bmatrix} 0 \\ \varphi(x_{10}) \\ 0 \\ V(x_{10}) \end{bmatrix} \tag{26}$$

The eigenvalue equation can be established from the above equation as

$$T_{12}T_{34} - T_{14}T_{32} = 0 \tag{27}$$

*(3) A multi-step bar with clamped-hinged (C - H) end conditions*

Eq. (20) for this case becomes

$$\begin{bmatrix} 0 \\ 0 \\ M(x_{n1}) \\ V(x_{n1}) \end{bmatrix} = [T]\begin{bmatrix} 0 \\ \varphi(x_{10}) \\ 0 \\ V(x_{10}) \end{bmatrix} \tag{28}$$

The eigenvalue equation is

$$T_{12}T_{24} - T_{14}T_{22} = 0 \tag{29}$$

*(4) A multi-step bar with clamped-clamped (C - C) end conditions*

Eq. (20) for this case becomes

$$\begin{bmatrix} 0 \\ 0 \\ M(x_{n1}) \\ V(x_{n1}) \end{bmatrix} = [T]\begin{bmatrix} 0 \\ 0 \\ M(x_{10}) \\ V(x_{10}) \end{bmatrix} \tag{30}$$

The eigenvalue equation is

$$T_{13}T_{24} - T_{14}T_{23} = 0 \tag{31}$$

All the boundary conditions used above are listed in Table 1

It is evident that the method presented above can also be used to solve the problem of partial buckling. For example, it is assumed that partial buckling occurs on steps from the 1st step to the *i*-th step of a cantilever bar (Fig. 1), then the low end of the *i*-th step should be fixed. The eigenvalue equation is also given by Eq. (24), but all $T_{jk}$ in Eq. (24) must be determined from

$$[T] = [T_i][T_{i-1}]\dots[T_1] \tag{32}$$

Letting $i = 1, 2, ..., n-1, n$, a set of critical forces can be determined from Eqs. (32) and (25). The minimum one among the $n$ critical forces is the critical force for partial buckling of the cantilever

Table 1 Four common boundary conditions

| Case | End conditions at $x = x_{10}$ and | $x = x_{n1}$ |
|---|---|---|
| C - F | $M(x_{10}) = 0,\ V(x_{10}) = 0$ | $y(x_{n1}) = 0,\ \psi(x_{n1}) = 0$ |
| H - H | $y(x_{10}) = 0,\ M(x_{10}) = 0$ | $y(x_{n1}) = 0,\ M(x_{n1}) = 0$ |
| C - H | $y(x_{10}) = 0,\ M(x_{10}) = 0$ | $y(x_{n1}) = 0,\ \psi(x_{n1}) = 0$ |
| C - C | $y(x_{10}) = 0,\ \psi(x_{10}) = 0$ | $y(x_{n1}) = 0,\ \psi(x_{n1}) = 0$ |

bar considered.

Similarly, the problem of partial buckling of a bar with other end conditions can also be solved by use of the same procedure presented above.

Clearly, two important special cases can be obtained from the procedure proposed above as follows:

(1) Setting $n = 1$ obtaining the solutions of one-step bars

(2) Setting $K_i \to \infty$ obtains the solutions of multi-step bars without the effect of shear deformation.

## 3. An illustrative example

The example will show how to determine the critical axial force of a two-step cantilever bar with thin-walled rectangular section subjected to concentrated loads as shown in Fig. 3. The parameters of the bar are selected as follows

$$\left.\begin{array}{l} b = 3d,\ k = 0.153,\ F_1 = F,\ F_2 = 1.2F_1,\ r_1 = r,\ r_2 = 1.3r,\ E = 2.6G \\ l_1 = l,\ l_2 = l,\ \dfrac{l_1}{r_1} = 25,\ \dfrac{l_2}{r_2} = 19.2308 \end{array}\right\} \quad (33)$$

The definition of the parameters presented above can be found in the last section and Fig. 3. The procedure for determining the critical force is as follows

### 3.1 Determination of the flexural and shear rigidities

It can be found from Eq. (33) that

$$K_1 = kGF,\ K_2 = 1.2kGF,\ EI_1 = EI,\ EI_2 = 2.028EI \quad (34)$$

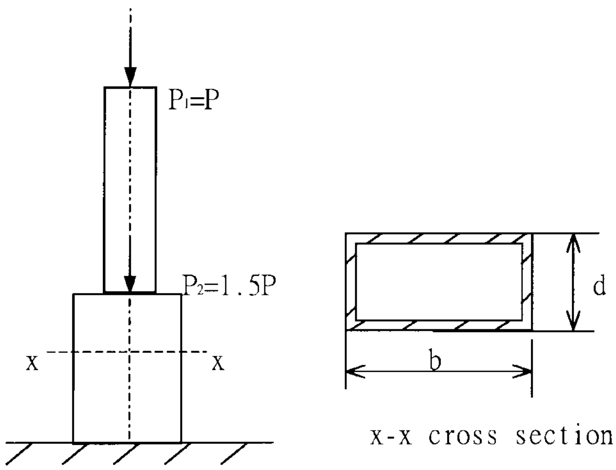


Fig. 3 A two-step cantilever bar

### 3.2 Determination of the transfer matrix

The transfer matrix for this example is given by

$$[T] = [T_2][T_1] \tag{35}$$

where

$$[T] = [A(x_{i1})][A(x_{i0})]^{-1} \qquad i = 1, 2$$

The expression of $A(x)$ is the same as Eq. (14), the parameters in that equation are as follows

$$N_1 = P_1 = P, \quad N_2 = P_1 + P_2 = 2.5P$$

$$\alpha_1^2 = \frac{P}{EI\left(1 - \frac{P}{36.7788/l^2}\right)}, \quad \alpha_2^2 = \frac{2.5P}{2.028EI\left(1 - \frac{P}{44.1346EI/l^2}\right)}$$

$$x_{10} = 0, \quad x_{11} = l, \quad x_{20} = l, \quad x_{21} = 2l$$

### 3.3 To establish the eigenvalue equation

The eigenvalue equation is the same as Eq. (25) while $T_{11}$, $T_{12}$, $T_{21}$ and $T_{22}$ are determined by Eq. (35). It is evident that the unknown variable is $P$ only.

Solving the eigenvalue equation obtains the critical axial force as

$$P_{cr} = 0.6937EI/l^2$$

If the effect of shear deformation on the critical buckling of the bar is not taken into account, then the eigenvalue equation of the bar can be established by using the static method as

$$\tan(r_1 l_1)\tan(r_2 l_2) = r_1/r_2 \tag{36}$$

where

$$r_2 = \sqrt{\frac{P_1 + P_2}{EI_2}} = 1.1103\sqrt{\frac{P}{EI}}, \qquad r_1 = \sqrt{\frac{P_1}{EI_1}} = \sqrt{\frac{P}{EI}} \tag{37}$$

Substituting Eq. (37) into Eq. (36) one obtains

$$\tan\left(l\sqrt{\frac{P}{EI}}\right)\tan\left(1.1103l\sqrt{\frac{P}{EI}}\right) = 0.9007 \tag{38}$$

Solving Eq. (38) obtains the critical axial force as

$$P_c = 0.7196EI/l^2$$

If setting $K_i \to \infty$, $\alpha_i^2 = N_i/EI_i$ the present method gives the same value of $P_c$. It can be seen that the error due to neglecting the effect of shear deformation is 3.6% for this case. When $l_1/r_1$ and $l_2/r_2$ are variables, but $r_2/r_1 = 1.3$, the effect of shear deformation on the critical buckling load of the two-step cantilever bar is shown in Fig. 4.

If $F_1 = F$, $F_2 = 10F$, $r_2 = 5r_1$, $l_1/r_1 = 25$, $l_2/r_2 = 5$, $l_1 = l_2 = l$, $EI_1 = EI$, $EI_2 = 250EI_1$

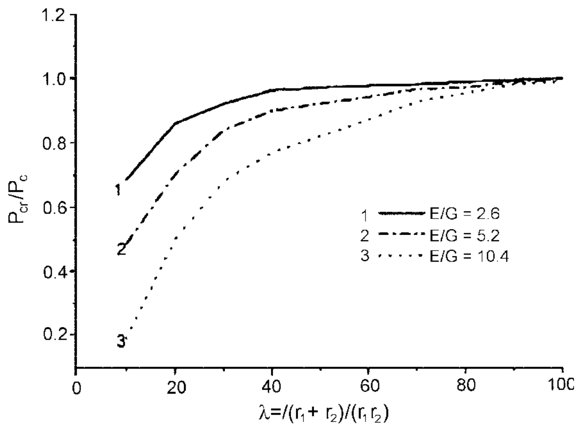


Fig. 4 Effect of shear deformation and *E*/*G* ratio on the critical buckling load of the two-step cantilever bar

Then, it is found that the critical axial loads with and without shear deformation are

$$P_{cr} = 2.2577EI/l^2, \ \ P_c = 2.4641EI/l^2$$

The effect of shear deformation causes the critical axial force, $P_{cr}$, to decrease by 8.4%. Clearly, this case belong to partial buckling because the rigidity of the lower step bar is very large and buckling of the two-step bar only occurs in the upper step bar. For this case, the critical force without the effect of shear deformation can also be determined by

$$P_c = \pi^2 EI_1/(4l_1^2) = 2.4674EI/l^2$$

If the ratio of Young's modulus to shear modulus (*E*/*G*) increases, then the effect of shear deformation on the critical axial load becomes larger. The effect of *E*/*G* ratio on the critical buckling load is also shown in Fig. 4. It can be seen from this figure that the effect of shear deformation and *E*/*G* ratio on the critical load is less than 13% when $\lambda$, ($\lambda = l(r_1 + r_2)/r_1 r_2$), is greater than 60 and *E*/*G* is less than 10.4. But, when $\lambda$ decreases, the effect is becoming more significant. For example, if $\lambda = 20$, $E/G = 2.6$ and 10.4, the effect of shear deformation causes the critical buckling load to decrease by 13.9% and 49.9%, respectively. This suggests that, for purposes of composite column design, the effect of shear deformation on the buckling load should be taken into account. For a steel column, the effect of shear deformation is not significant on the critical load.

## 4. Conclusions

This paper presents the derivation of eigenvalue equation of a multi-step bar considering the effect of shear deformation by combining the transfer matrix method and closed form solutions of one step bar. The main advantage of the proposed method is that the eigenvalue equation of a multi-step bar considering the effect of shear deformation can be conveniently determined from a second order determinant. Therefore, the proposed method is convenient for solving the entire and partial buckling of one-step and multi-step bars with various end conditions, with or without shear deformation effect, subjected to concentrated axial loads. A numerical example describing the proposed procedure is given. The calculated results show that the effect of shear deformation on the critical buckling force is significant for the cases of large Young's modulus to shear modulus (*E*/*G*) ratio, suggesting that, for purposes of composite column design, the effect of shear deformation on the buckling load may have to be taken into account.

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