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Numerical solving of initial-value problems by R_{bf} basis functions

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Abstract. This paper presents a numerical procedure for solving initial-value problems using the special functions which belong to a class of Rvachev's basis functions R_{bf} based on algebraic and trigonometric polynomials. Because of infinite derivability of these functions, derivatives of all orders, required by differential equation of the problem and initial conditions, are used directly in the numerical procedure. The accuracy and stability of the proposed numerical procedure are proved on an example of a single degree of freedom system. Critical time step was also determined. An algorithm for solving multiple degree of freedom systems by the collocation method was developed. Numerical results obtained by R_{bf} functions are compared with exact solutions and results obtained by the most commonly used numerical procedures for solving initial-value problems.

Key words: vibrations; numerical solution; Rvachev's basis functions; collocation method.

1. Introduction

The selection of the basis functions is of special importance for the numerical procedure and quality of the approximate solution. Spline functions have an important place in the development of structural numerical analyses (Prenter 1989). Although splines are a fine approximating tool, it is clear that they are not universal basis functions for all problems of numerical approximations. In this paper, a numerical procedure will be presented in which new basis functions, not well known to engineers, are implemented.

The numerical solving of an initial-value problem is here performed by the procedure of a continuous approximation in time with smooth finite functions named after the authors Rvachev's basis functions or, in short, R_{bf} (Rvachev and Rvachev 1971), (Gotovac 1986). R_{bf} functions are classified between classic polynomials and spline functions. However, in practice, their application as basis functions is still closer to splines. Therefore, the class of R_{bf} functions can be regarded as splines of an infinitely high degree. In the study by Gotovac (1986), the existing knowledge on functions of R_{bf} class is systematized and basis functions are transformed into a numerically applicable form. Procedures for calculation of R_{bf} functions are given by Gotovac and Kozulić (1999) together with their distribution for forming numerical solutions and an illustration of basic

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possibilities for their application in practice.

For general solutions of initial-value problems belonging to a class of trigonometric functions, it is appropriate to select the functions $y_{\omega,h}(t)$ as basis functions of an approximate solution. Functions $y_{\omega,h}(t)$ are infinitely derivable functions, the linear combination of which can be used for an exact description of trigonometric functions (Gotovac and Kozulić 1999). Using the basis functions $y_{\omega,h}(t)$, exact solutions of free undamped vibrations are obtained (Gotovac 1986). In case of forced vibrations, besides functions $y_{\omega,h}(t)$, other functions of R_{bf} class can be selected as an approximate solution base depending on the character of a disturbing force. Basis functions $Fup_n(t)$ are applied here, the linear combination of which can be used for an exact description of algebraic polynomials (Rvachev and Rvachev 1979), (Gotovac 1986), (Kozulić 1999). When a disturbing force function is an algebraic polynomial, time function of load can be described exactly by basis functions $Fup_n(t)$.

A concise description of functions $y_{\omega,h}(t)$, which belong to a class of trigonometric polynomials, and functions $Fup_1(t)$ and $Fup_2(t)$, which belong to a space containing algebraic polynomials, is given in the following Sections. The $up(t)$ function, which is essential in the definition of $Fup_n(t)$ functions, is specially described. It is the simplest function and is studied in the most detail among Rvachev's basis functions. The basic properties of $up(t)$ function refer to all other functions of R_{bf} class. The procedure of solving of initial-value problem by these basis functions is illustrated in Section 5 on numerical examples.

2. Function *up*(*t*)

Rvachev's basis functions R_{bf} are defined as finite solutions of differential-functional equations of the following type:

$$L\,y(t) = \lambda \sum_{k=1}^{M} C_k y(at - b_k) \tag{1}$$

where L is a common linear differential operator with constant coefficients, λ is a scalar different from zero, C_k are solution coefficients, $a > 1$ is a parameter of the length of finite function support, b_k are coefficients which determine displacements of finite basis functions.

The type of finite function of R_{bf} class is determined by the selection of operator L in Eq. (1). The $up(t)$ function is a solution of differential-functional equation in which the differential operator of the first order, according to Eq. (1), has the following form:

$$y'(t) = \lambda[C_1 y(at - b_1) + C_2 y(at - b_2)] \tag{2}$$

Support of the $up(t)$ function is the interval $[-1, 1]$.

Parameter of "compression" i.e., "extension" of the support of function $up(t)$ is $a = 2$, the characteristic displacements of the function on the abscissa are $b_1 = -1$ and $b_2 = 1$, and value $\lambda = -2$ and coefficients $C_1 = -1$, $C_2 = 1$. Therefore, the basic equation for the function $up(t)$, according to Eq. (2), is:

$$up'(t) = 2up(2t + 1) - 2up(2t - 1) \tag{3}$$

If the length of the $up(t)$ function support is described as an union of lengths 2^{-k}, $k = 0, 1, ..., \infty$, the Fourier transform of the $up(t)$ function, using the procedure given in Fig. 1, is obtained as a product of the Fourier transforms of zero degree splines condensed to a support length 2^{-k} with the ordinates 2^k:

$$\hat{up}(\xi) = \prod_{j=1}^{\infty} \frac{\sin(\xi 2^{-j})}{\xi 2^{-j}} \tag{4}$$

The finite solution of Eq. (3), with the fulfillment of condition $\int_{-\infty}^{\infty} up(t)dt = \int_{-1}^{1} up(t)dt = 1$, has the following form:

$$up(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t}\hat{up}(\xi)d\xi \tag{5}$$

Based on Eq. (5), i.e., the fact that the function *up*(*t*) is expressed by its Fourier transform (4), function *up*(*t*) can be generated using the convolution theorem.

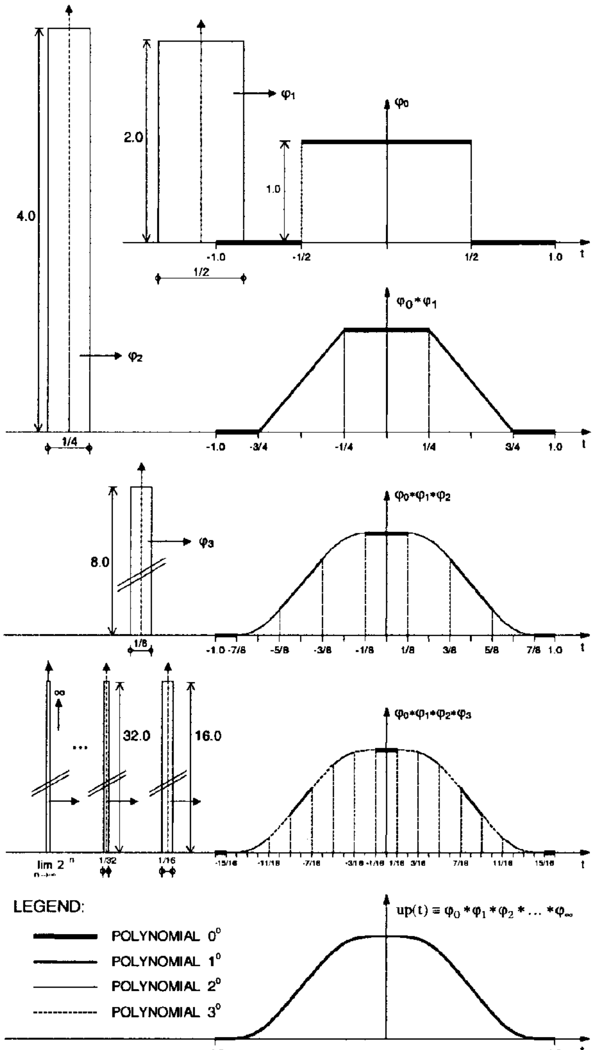


Fig. 1 Generating of function *up*(*t*)

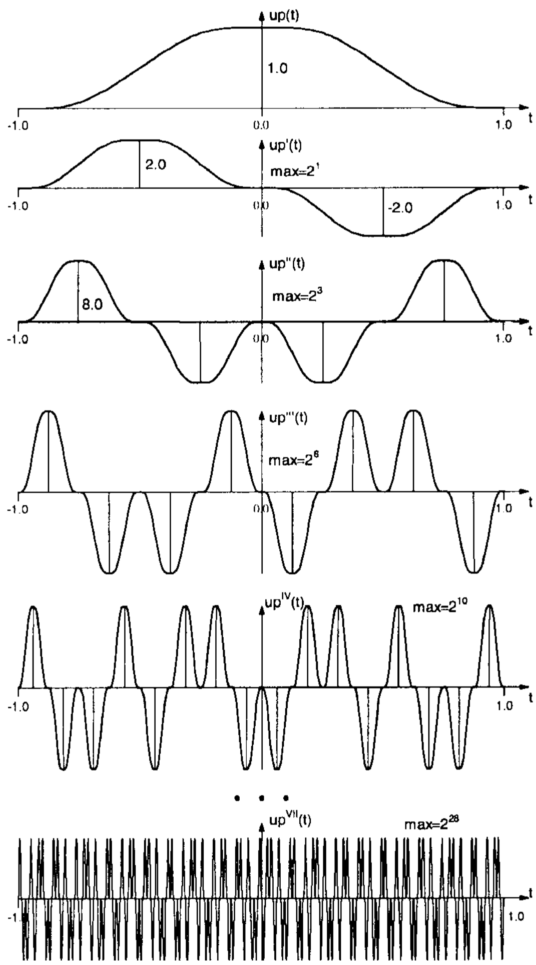


Fig. 2 Function *up*(*t*), its first four and the seventh derivative

2.1 Derivatives of the up(t) function

As it can be observed in Eq. (3), the first derivative can be expressed as a linear combination of the displaced and compressed *up*(*t*) function. By differentiating the basic Eq. (3) and by replacing the first derivative of *up*(*t*) function with the right side of the initial Eq. (3), the second derivative can also be expressed as a linear combination of the compressed and displaced function *up*(*t*).

If the procedure of differentiating and replacement of the first derivative from the basic Eq. (3) continues, a general expression for the derivative of the *m*-th degree is obtained:

$$up^{(m)}(t) = 2^{C^2_{m+1}} \sum_{k=1}^{2^m} \delta_k up(2^m t + 2^m + 1 - 2k), \quad m \in N \tag{6}$$

where $C^2_{m+1} = m(m+1)/2$ are the binomial coefficients and δ_k are the coefficients of value ± 1 which determine the sign of each term. They change according to the following recursive formulas:

$$\delta_{2k-1} = \delta_k, \ \ \delta_{2k} = -\delta_k, \ \ k \in N, \ \ \delta_1 = 1 \tag{7}$$

Fig. 2 shows the *up*(*t*) function and its derivatives. It can be observed that the derivatives consist of the function *up*(*t*) “compressed” to the interval of length $2^{-m+1}$and with ordinates “extended” with the factor $2^{C^2_{m+1}}$. A high degree derivative of the *up*(*t*) function when $m \to \infty$ becomes a series in which every single member corresponds to Dirac’s function.

2.2 Moments of function up(t)

Expression (5) is numerically inadequate for the calculation of the *up*(*t*) function values. Reference Gotovac (1986) shows that the *up*(*t*) function values can be calculated using the function *up*(*t*) moments.

Function *up*(*t*) moments with an even index (odd ones are equal to zero because *up*(*t*) is an even function):

$$a_{2k} = \int_{-1}^{1} t^{2k} up(t) dt \tag{8}$$

can be calculated according to formula:

$$a_{2k} = \frac{(2k)!}{2^{2k}-1} \sum_{l=1}^{k} \frac{a_{2k-2l}}{(2k-2l)!(2l+1)!}, \quad k \in N; \ \ a_0 = 1 \tag{9}$$

The scalar product of a polynomial and function *up*(*t*) on an even half of the support is:

$$b_n = \int_0^1 t^n up(t) dt, \ \ n = -1, 0, 1, \ldots \tag{10}$$

Since the *up*(*t*) function is even, the comparison of expressions (8) and (10) gives:

$$b_{2k} = \frac{1}{2} a_{2k}, \quad k = 0, 1, \ldots \tag{11}$$

According to Eq. (10) and using Eq. (9), the following is obtained for odd indices:

$$b_{2k+1} = \frac{1}{(k+1)2^{2k+3}}\sum_{l=0}^{k+1} a_{2l}C_{2(k+1)}^{2l}; \quad k = 0, 1, 2, 3, \dots$$

$$b_{-1} = 1 \quad \text{(by definition)} \tag{12}$$

Scalar products of the $up(t)$ function and algebraic polynomials can be easily calculated using Eqs. (11) and (12).

2.3 Function up(t) value in a characteristic point

Characteristic points $t_m^{(n)}$ are the points in which the function $up(t)$ values and values of the first n derivatives are calculated exactly in the form of a rational number. In the other points of the support, the values are calculated with a computer precision i.e., the accuracy depends on the possibility of describing a selected point coordinate in the base used by the computer.

A set of characteristic points of the given density on the $up(t)$ function support can be described in a simpler manner as:

$$t_k = -1 + k2^{-n}, \quad n \in N, \quad 1 \le k \le 2^{n+1} \tag{13}$$

where n determines the distance between the characteristic points on the $up(t)$ function support:

$$\Delta t_n = 2^{-n} \tag{14}$$

The function $up(t)$ value in a characteristic point $t_k = -1 + k2^{-n}, n \in N, 1 \le k \le 2^{n+1}$ can be expressed in the following form:

$$up(t_k) = \frac{2^{-n(n+1)/2}}{n!}\sum_{j=1}^{k}\delta_j\sum_{l=0}^{[n/2]} C_n^{2l}(2(k-j)+1)^{n-2l} \cdot a_{2l} \tag{15}$$

where δ_j are the coefficients in the role of sign according to Eq. (7), C_n^{2l} are binomial coefficients, a_{2l} are even moments of the $up(t)$ function while square brackets in expression [n/2] denote the maximum integer of the fraction within the brackets.

In a characteristic point $t_1 = -1 + 2^{-n}$, Eq. (15) can be written as:

$$up(-1+2^{-n}) = \frac{b_{n-1}}{2^{n(n-1)/2}(n-1)!}, \quad n = 0, 1, \dots \tag{16}$$

Introducing Eq. (16) into a general expression of the $up(t)$ function derivative (6), the following value of function $up(t)$ derivative in a characteristic point $t_1 = -1 + 2^{-n}$ is obtained:

$$up^{(l)}(-1+2^{-n}) = \frac{2^{-n(n-2l-1)/2}}{(n-l-1)!}b_{n-l-1} \tag{17}$$

2.4 Polynomial as a linear combination of displaced up(t) functions

The polynomial of n-th degree can be expressed as a linear combination of displaced $up(t)$ functions, for example:

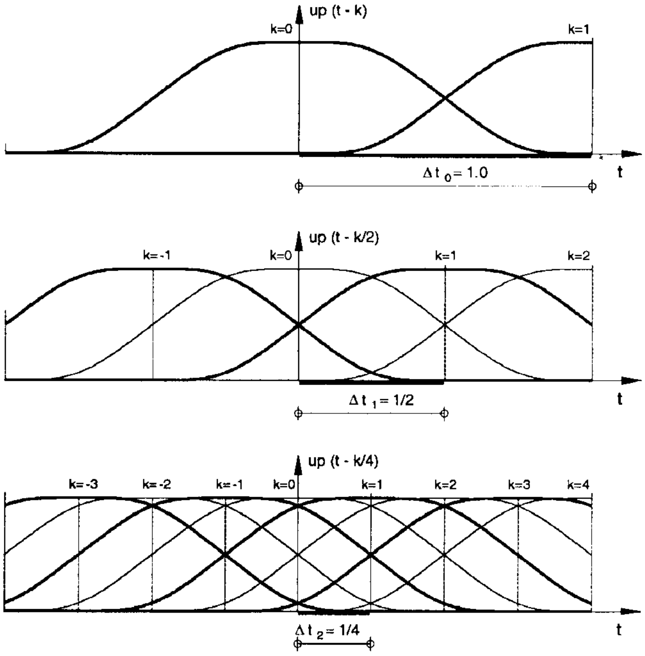


Fig. 3 Distribution of basis functions for an exact description of 0,1 and 2 degree polynomials

$$n = 0 \rightarrow 1 = \sum_{k=-\infty}^{\infty} up(t-k) \ ; \ n = 1 \rightarrow t = 2^{-2} \sum_{k=-\infty}^{\infty} k \ up(t-k/2)$$

$$n = 2 \rightarrow t^2 = \frac{2^{-6}}{9} \sum_{k=-\infty}^{\infty} (9k^2 - 16) up(t-k/4) \tag{18}$$

Fig. 3 shows the distribution of the basis functions obtained by displacement of function *up*(*t*) by $k \cdot 2^{-n}$, $k \in Z$. According to Eq. (18), polynomials of $n = 0$, 1 and 2 degrees can be expressed exactly as a linear combination of those basis functions on the interval $\Delta t_n = 2^{-n}$. Coefficient *k* measures the displacement of the function *up*(*t*) with reference to the origin of a global coordinate system with a step 2^{-n}, which gives a basis function. Therefore, *k* is a global index of the basis function.

2^{n+1} basis functions, which form the vector space UP_n, are required for an exact description of the monomial t^n on the interval of length 2^{-n}. For an exact description of the monomial t^{n+1}, 2^{n+2} basis functions, which form the vector space UP_{n+1}, are required. As it can be observed in Fig. 3, linear vector space of functions UP_{n+1} contains the space UP_n. Therefore, the space of *up*(*t*) basis functions is **universal**, i.e., $UP_0 \subset UP_1 \subset \ldots \subset UP_n \subset UP_{n+1} \subset \ldots \subset UP_\infty$.

2.5 Function up(t) value in an arbitrary point

Based on the fact that the development of the *up*(*t*) function in a Taylor series, in characteristic points t_k, is a polynomial of *n*-th degree, a special series for the calculation of function *up*(*t*) values in an arbitrary point $t \in [0, 1]$ is proposed by Rvachev & Rvachev (1979), and Gotovac & Kozulić (1999):

$$up(t) = 1 - \sum_{k=1}^{\infty} (-1)^{1 + p_1 + \ldots + p_k} p_k \sum_{j=0}^{k} C_{jk}(t - 0, p_1 \ldots p_k)^j \tag{19}$$

where coefficients C_{jk} are rational numbers determined according to the following expression:

$$C_{jk} = \frac{1}{j!} 2^{j(j+1)/2} up(-1 + 2^{-(k-j)}) \ ; \quad j = 0, 1, \ldots, k \ ; \quad k = 1, 2, \ldots, \infty \tag{20}$$

Expression $(t - 0,\ p_1 \ldots p_k)$ in Eq. (19) is the difference between the real value of coordinate t and its binary form with k bits, where $p_1 \ldots p_k$ are the digits 0 or 1 of the binary development of the coordinate t value. Therefore, the accuracy of coordinate t computation, and, thus, the accuracy of the $up(t)$ function in an arbitrary point, depend upon the accuracy of a computer. For n, an error of the calculated function $up(t)$ value in an arbitrary point t, i.e., the residue of a series given in Eq. (19)

FORTRAN Code for the function *up(t)* values:

```
C.........................................................................................................
C   Functional subprogram for the calculation of function up(t) values in an arbitrary point t ∈ (-∞,∞)
C.........................................................................................................
      REAL*8 FUNCTION UPX(X)
      IMPLICIT REAL*8 (A-H, O-Z)
      DIMENSION UN(0:10),XK(10),FAK(0:10),DIV(0:10),UNN(0:10)
      INTEGER*4 PK(10),SPK(10)
      DVA(M) = 2.0D0**M
C
      DATA DIV/ 1.0D0, 1.0D0, 5.0D0, 1.0D0, 143.0D0, 19.0D0, 1153.0D0, 583.0D0, 1616353.0D0, 132809.0D0, 134926369.0D0/
      DATA UNN/ 1.0D0, 2.0D0, 72.0D0, 288.0D0, 2073600.0D0, 33177600.0D0, 561842749440.0D0, 179789679820800.0D0,
     &          70420021792210944000O.0D0, 18027525578806001664000O.0D0, 12463948513585393872383508480000.0D0/
      DATA ZERO/0.0D0/,   ONE/1.0D0/
      DATA FAK/ 1.0D0, 1.0D0, 2.0D0, 6.0D0, 24.0D0, 120.0D0, 720.0D0, 5040.0D0, 40320.0D0, 362880.0D0, 3628800.0D0/
C
       XX = DABS(X)
       IF(XX .GE. ONE )  THEN
          UPX = ZERO
       ELSE
              DO K = 1,10
                 PK(K) = 0
                 SPK(K) = 0
                 XK(K) = ZERO
              END DO
                 DO I = 0,10
                    UN(I) = DIV(I)/UNN(I)
                 END DO
                   XK(1) = XX
                   IF(XX .GE. 0.5D0) XK(1) = XX-0.5D0
                   IF(XX .GE. 0.5D0) PK(1) = 1
                   SPK(1) = 1+PK(1)
                        DO K = 2,10
                             DVAMK = ONE/DVA(K)
                             IF(XK(K-1) .GE. DVAMK) THEN
                                    XK(K) = XK(K-1)-DVAMK
                                    PK(K) = 1
                                    SPK(K)= 1+SPK(K-1)
                             ELSE
                                    XK(K) = XK(K-1)
                                    SPK(K)= SPK(K-1)
                             END IF
                        END DO
                 SUMAK = ZERO
                 DO K = 1,10
                 PRED = (-ONE)**SPK(K)
                             SUMA = ZERO
                             IF(PK(K) .EQ. 1) THEN
                                DO J = 0,K
                                    PR = DVA(J*(J+1)/2)/FAK(J)*UN(K-J) * XK(K)**J
                                    SUMA = SUMA + PR
                                END DO
                             ELSE
                                CYCLE
                             END IF
                 SUMAK = SUMAK + PRED * SUMA
                 END DO
           UPX = ONE - SUMAK
       END IF
      END
```

when $k = 1, \ldots, n$, does not exceed the function $up(-1 + 2^{-n})$ value obtained from Eq. (16). Using the given functional subprogram, the function $up(t)$ value for $t \in [0, 1]$ is calculated with an error smaller than 10^{-21} using only ten terms in the series.

3. Basis functions $Fup_n(t)$

A family of $Fup_n(t)$ functions was developed according to the $up(t)$ function. $Fup_n(t)$ functions and their derivations retain the properties of $up(t)$ function, but they are more suitable for numerical analyses. Index n denotes the greatest degree of a polynomial which can be expressed accurately in the form of a linear combination of basis functions obtained by the displacement of function $Fup_n(t)$ by a characteristic interval 2^{-n}. When $n = 0$:

$$Fup_0(t) = up(t) \tag{21}$$

Function $Fup_n(t)$ values are calculated using a linear combination of displaced $up(t)$ functions:

$$Fup_n(t) = \sum_{k=0}^{\infty} C_k(n) up\left(t - 1 - \frac{k}{2^n} + \frac{n+2}{2^{n+1}}\right) \tag{22}$$

where coefficient $C_0(n)$ is:

$$C_0(n) = 2^{C_{n+1}^2} = 2^{n(n+1)/2} \tag{23}$$

and other coefficients of the linear combination are determined as $C_k(n) = C_0(n) \cdot C_k'(n)$, where a recursive formula is used for the calculation of auxiliary coefficients $C_k'(n)$:

$$C_0'(n) = 1, \text{when } k = 0; \text{ when } k > 0:$$

$$C_k'(n) = (-1)^k C_{n+1}^k - \sum_{j=1}^{\min\{k;2^{n+1}-1\}} C_{k-j}'(n) \cdot \delta_{j+1} \tag{24}$$

Function $Fup_n(t)$ support is determined according to:

$$sup\, p\, Fup_n(t) = [-(n+2)2^{-n-1}; (n+2)2^{-n-1}] \tag{25}$$

Practically, it is enough to include only $(n + 2)$ functions $up(t)$ in the linear combination according to Eq. (22) to determine function $Fup_n(t)$ values in the points of support defined by Eq. (25).

Finite functions $Fup_n(t)$ are not analytical in any point of their support, similarly as the $up(t)$ function (see Fig. 2).

Derivatives of the function $Fup_n(t)$ are also obtained by a linear combination of derivatives of displaced $up(t)$ functions according to Eq. (22).

Polynomial of the m-th degree t^m is developed over the functions $Fup_n(t)$ base in the following form:

$$t^m = \sum_{k=-\infty}^{\infty} D_k(m,n) Fup_n\left(\frac{t}{2^n \Delta t} - \frac{k_*}{2^n}\right) \tag{26}$$

where Δt is a characteristic interval which determines mutual displacement of basis functions (time step); k is the counter of basis functions, $k_* = k$ for even n, $k_* = (2k + 1)/2$ for odd n; $D_k(m, n)$ are the coefficients of a linear combination of basis functions $Fup_n(t)$ for an exact development of an algebraic polynomial of the m-th degree. When $n = 1$, coefficients $D_k(m, n)$ are:

$$D_k(0,1) = 2^{-1} \cdot \Delta t^0 \cdot k_*^0 = 1/2$$
$$D_k(1,1) = 2^{-1} \cdot \Delta t^1 \cdot k_* = \frac{\Delta t}{2} \cdot \frac{2k+1}{2} = \frac{2k+1}{4}\Delta t \tag{27}$$

3.1 Function $Fup_1(t)$

Basis function $Fup_1(t)$ has a support with the length $\left[-\frac{3}{2}\Delta t, \frac{3}{2}\Delta t\right]$. Calculations of function $Fup_1(t)$ values and its derivatives are given by Gotovac (1986). Fig. 4 shows the function $Fup_1(t)$ and its first two derivatives for $\Delta t = 1/2$. Their values are given in Table 1.

3.2 Function $Fup_2(t)$

According to Eq. (22), function $Fup_2(t)$ can be written as linear combination of displaced $up(t)$ functions:

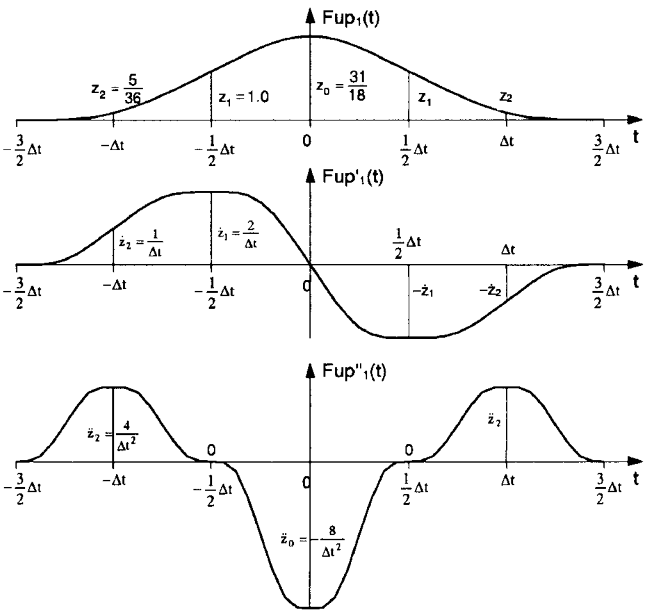


Fig. 4 Function $Fup_1(t)$ and its first two derivatives

Table 1 Function $Fup_1(t_k)$ values and its first two derivatives

$t_k = \left(-\frac{3}{4}+\frac{k}{16}\right)2\Delta t$ $k = 0, 1, \ldots, 24$	$Fup_1(t_k)$	$\frac{Fup_1'(t_k)}{2\Delta t}$	$\frac{Fup_1''(t_k)}{(2\Delta t)^2}$
–.7500	.000000000	.000000000	.000000000
–.6875	.000137924	.013888889	1.111111111
–.6250	.006944444	.277777778	8.000000000
–.5625	.045000965	1.013888889	14.888888889
–.5000	.138888889	2.000000000	16.000000000
–.4375	.295000965	2.986111111	14.888888889
–.3750	.506944444	3.722222222	8.000000000
–.3125	.750137924	3.986111111	1.111111111
–.2500	1.000000000	4.000000000	.000000000
–.1875	1.249724151	3.972222222	–2.222222222
–.1250	1.486111111	3.444444444	–16.000000000
–.0625	1.659998071	1.972222222	–29.777777778
.0000	1.722222222	.000000000	–32.000000000
.0625	1.659998071	–1.972222222	–29.777777778
.1250	1.486111111	–3.444444444	–16.000000000
.1875	1.249724151	–3.972222222	–2.222222222
.2500	1.000000000	–4.000000000	.000000000
.3125	.750137924	–3.986111111	1.111111111
.3750	.506944444	–3.722222222	8.000000000
.4375	.295000965	–2.986111111	14.888888889
.5000	.138888889	–2.000000000	16.000000000
.5625	.045000965	–1.013888889	14.888888889
.6250	.006944444	–.277777778	8.000000000
.6875	.000137924	–.013888889	1.111111111
.7500	.000000000	.000000000	.000000000

$$Fup_2(t) = \sum_{k=0}^{\infty} C_k \, up\left(t - 1 - \frac{k}{4} + \frac{1}{2}\right) \tag{28}$$

where coefficients C_k are given by Eqs. (23) and (24). The function support is $[-2\Delta t, 2\Delta t]$. By a linear combination of basis functions obtained by mutual displacement of only one $Fup_2(t)$ function, a polynomial of the 2^{nd} degree can be expressed exactly on a characteristic interval with the length Δt. Fig. 5 shows the function $Fup_2(t)$ and its first two derivatives with the normed characteristic interval $\Delta t = 1/4$. The values of the basis function and its derivatives are given in Table 2.

4. Basis functions $y_{\omega,h}(t)$

For approximate solutions belonging to a class of trigonometric functions or containing trigonometric functions, finite basis functions $y_{\omega,h}(t)$ are developed. They are determined as a

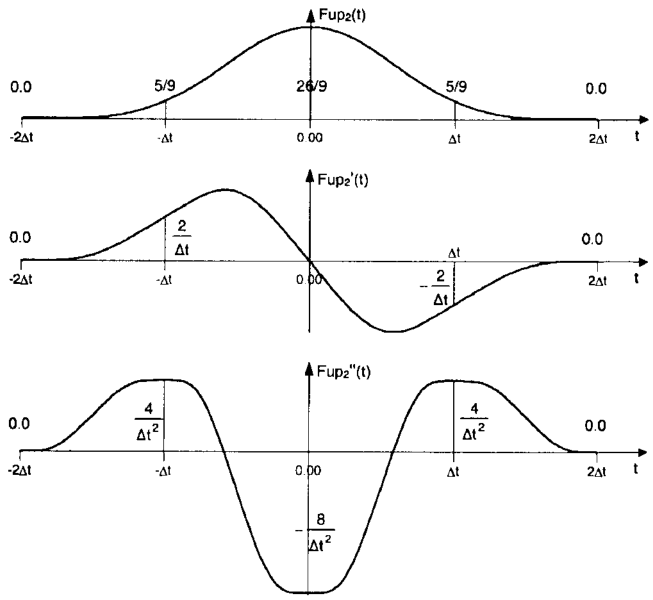


Fig. 5 Function $Fup_2(t)$ and its first two derivatives

Table 2 Function $Fup_2(t_k)$ values and its first two derivatives when $\Delta t = 1/4$

$t_k = \left(-\frac{1}{2}+\frac{k}{16}\right)4\Delta t$, $k = 0, 1, \ldots, 16$	$Fup_2(t_k)$	$\frac{Fup_2'(t_k)}{4\Delta t}$	$\frac{Fup_2''(t_k)}{(4\Delta t)^2}$
–.5000	.00000000	.00000000	.00000000
–.4375	.00055169	.05555555	4.44444444
–.3750	.02777777	1.11111111	32.00000000
–.3125	.18000385	4.05555555	59.55555555
–.2500	.55555555	8.00000000	64.00000000
–.1875	1.17890046	11.83333333	50.66666666
–.1250	1.97222222	12.66666666	–32.00000000
–.0625	2.64054398	7.83333333	–114.66666666
.0000	2.88888888	.00000000	–128.00000000
.0625	2.64054398	–7.83333333	–114.66666666
.1250	1.97222222	–12.66666666	–32.00000000
.1875	1.17890046	–11.83333333	50.66666666
.2500	.55555555	–8.00000000	64.00000000
.3125	.18000385	–4.05555555	59.55555555
.3750	.02777777	–1.11111111	32.00000000
.4375	.00055169	–.05555556	4.44444444
.5000	.00000000	.00000000	.00000000

solution of differential-functional Eq. (1) written in the following form:

$$y''_{\omega,h}(t) + \omega^2 y_{\omega,h}(t) = a\, y_{\omega,h}(3t+2h) - b\, y_{\omega,h}(3t) + a\, y_{\omega,h}(3t-2h) \tag{29}$$

where ω is the circular frequency, h is the length of the half of function $y_{\omega,h}(t)$ support, while coefficients a and b are:

$$a = \frac{3}{2} \cdot \frac{\omega^2}{1 - \cos(2\omega h/3)}, \quad b = 2a\cos(2\omega h/3) \tag{29a}$$

Function $y_{\omega,h}(t)$ support is selected in dependence of the value of circular frequency ω:

$$\operatorname{supp} y_{\omega,h}(t) = [-h, h] \tag{30}$$

Finite solution of Eq. (29) must satisfy the normed condition:

$$\int_{-\infty}^{\infty} y_{\omega,h}(t)dt = \int_{-h}^{h} y_{\omega,h}(t)dt = 1 \tag{31}$$

and in that case has the following form:

$$y_{\omega,h}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t}\hat{y}_{\omega,h}(\xi)d\xi \tag{32}$$

Table 3 Function $y_{\pi,1}(t_k)$ values and its first two derivatives

$t_k = -1 + k/9$, $k = 0, 1, \ldots, 2\cdot 9$	$y_{\pi,1}(t_k)$	$y'_{\pi,1}(t_k)$	$y''_{\pi,1}(t_k)$
−1.0000000	.000000000	.000000000	.000000000
−.8888889	.000529392	.030931029	1.419777536
−.7777778	.026498778	.569906388	8.183069521
−.6666667	.144382932	1.551917694	8.444601979
−.5555556	.364408349	2.346744822	4.848035737
−.4444444	.635591651	2.346744822	−4.848035737
−.3333333	.855617068	1.551917694	−8.444601979
−.2222222	.973501222	.569906388	−8.183069521
−.1111111	.999470608	.030931029	−1.419777536
.0000000	1.000000000	.000000000	.000000000
.1111111	.999470608	−.030931029	−1.419777536
.2222222	.973501222	−.569906388	−8.183069521
.3333333	.855617068	−1.551917694	−8.444601979
.4444444	.635591651	−2.346744822	−4.848035737
.5555556	.364408349	−2.346744822	4.848035737
.6666667	.144382932	−1.551917694	8.444601979
.7777778	.026498778	−.569906388	8.183069521
.8888889	.000529392	−.030931029	1.419777536
1.0000000	.000000000	.000000000	.000000000

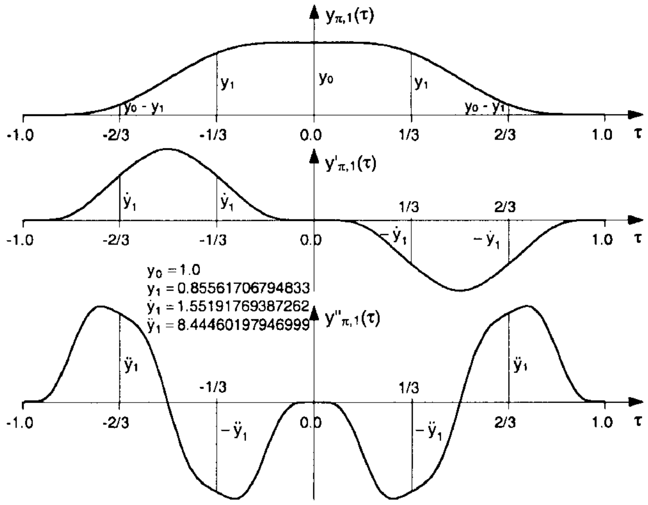


Fig. 6 Function $y_{\pi,1}(\tau)$ and its first two derivatives

where $\hat{y}_{\omega,h}(\xi)$ is the Fourier transform of function $y_{\omega,h}(t)$:

$$\hat{y}_{\omega,h}(\xi) = \prod_{j=1}^{\infty} \left\{ \frac{2}{3} a \frac{\cos(2\xi h/3^j) - \cos(2\omega h/3)}{\omega^2 - \xi^2/9^{j-1}} \right\} \tag{33}$$

Depending on the support length 2 h, frequency ω and the required accuracy (SP or DP), between 10 and 20 first factors are used in the product (33) since the remaining ones are practically equal to 1.

Similar to Eq. (15) for the computation of the $up(t)$ function values in characteristic points $t_k = -1 + k2^{-n}$, numerically more adequate expressions (Gotovac and Kozulić 1999) for the calculation of function $y_{\omega,h}(t)$ values and its derivatives in characteristic points $t_k = (-1 + k3^{-n}) \cdot h$, $n = 1, 2, 3, \ldots$; $1 \le k \le 2 \cdot 3^n$, are developed. For conciseness, those expressions are not given here; although, using a computer they are helpful for an easy calculation of the values for any density of characteristic points.

For example, values of the function $y_{\omega,h}(t)$ and its derivatives for the frequency $\omega = \pi$, $h = 1$ and $n = 2$ are given in Table 3. Basis function $y_{\pi,1}(\tau)$ and its first two derivatives are shown in Fig. 6.

The distance between characteristic points of the function $y_{\omega,h}(t)$ support is:

$$\Delta t_n = h \cdot 3^{-n},\ n = 1, 2, 3, \ldots \tag{34}$$

The distance between characteristic points determines a displacement of a basis function in order to obtain a suitable base. In such a base, an arbitrary function $\varphi(t)$ can be developed as:

$$\varphi(t) = \sum_{k=-\infty}^{\infty} C_k y_{\omega,h}\left(t - \frac{2hk}{3}\right) \tag{35}$$

Trigonometric functions of the given frequency ω, can be described exactly in a base of displaced $y_{\omega,h}(t)$ functions according to Eq. (35):

$$\sin(\omega t) = C \cdot \sum_{k=-\infty}^{\infty} \sin\left(\frac{2\omega h}{3}k\right) \cdot y_{\omega,h}\left(t - \frac{2h}{3}k\right)$$

$$\cos(\omega t) = C \cdot \sum_{k=-\infty}^{\infty} \cos\left(\frac{2\omega h}{3}k\right) \cdot y_{\omega,h}\left(t - \frac{2h}{3}k\right) \tag{36}$$

where:

$$C = \frac{y_{\omega,h}(h/3) - 2y_{\omega,h}(2h/3) \cdot \cos(\omega h/2)}{y_{\omega,h}(h/3)(y_{\omega,h}(0) - 2y_{\omega,h}(2h/3))} \tag{36a}$$

5. Application

5.1 Forced vibrations of a particle

Forced vibrations of a particle of unit mass without damping are described by the following differential equation:

$$\ddot{x}(t) + \omega^2 x(t) = f(t) \tag{37}$$

and initial conditions:

$$x(t_0) = x(0) = x_0 \;\; ; \;\; \dot{x}(t_0) = \dot{x}(0) = \dot{x}_0 \tag{38}$$

As an example, observed are the oscillations of a material point with the circular frequency of vibration $\omega = \pi$, homogeneous initial conditions $x_0 = \dot{x}_0 = 0$, and time function of a disturbing force $f(t)$ given in Fig. 7.

Numerical solution of the given problem will be determined by the collocation method with finite basis functions, described in Sections 4 and 3.1., in the following form:

$$\tilde{x}(t) = \tilde{x}_h + \tilde{x}_p = \sum_{k=-1}^{\infty} C_k \cdot y_{\pi,1}\left(t - \frac{2k}{3}\right) + \sum_{l=-1}^{\infty} D_l \cdot Fup_1\left(\frac{t}{2\Delta t} - \frac{2l+1}{4}\right) \tag{39}$$

Selected time step is $\Delta t = 2/3$. An approximate solution base is formed by a mutual displacement of functions $y_{\pi,1}(t)$ and $Fup_1(t)$ by the value which corresponds to the time step Δt, as shown in Fig. 8.

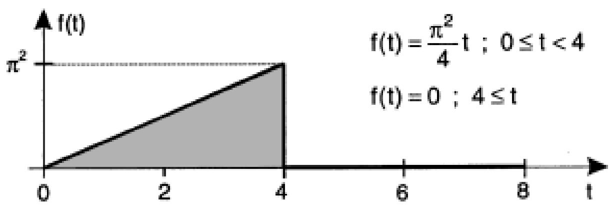


Fig. 7 Disturbing force $f(t)$

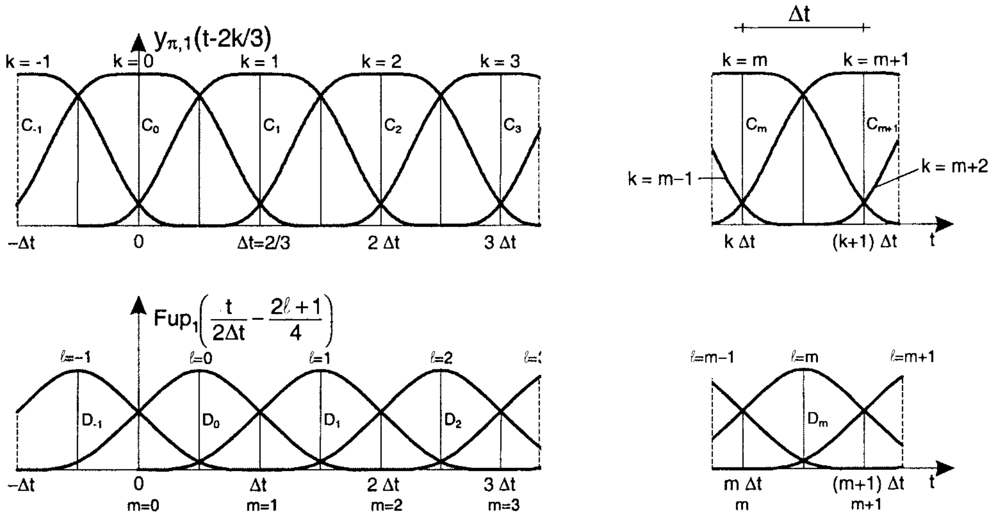


Fig. 8 Distribution of basis functions

The particular part $\tilde{x}_p(t)$ of the solution is determined from the condition that the second sum from Eq. (39) satisfies the equation of the problem (37) from the moment $t = m\Delta t$ to the moment $t = (m + 1)\Delta t$. Using the collocation method, for $t_m = m\Delta t$, $t_{m+1/2} = (m + 1/2)\Delta t$, $t_{m+1} = (m + 1)\Delta t$, appurtenant coefficients D_l, $l = m - 1$, m, $m + 1$, can be calculated:

$$D_{m-1} = \frac{1}{4\pi^2}[3f(m\Delta t) - f((m+1)\Delta t)]$$

$$D_m = \frac{1}{4\pi^2}[f(m\Delta t) + f((m+1)\Delta t)]$$

$$D_{m+1} = \frac{1}{4\pi^2}[-f(m\Delta t) + 3f((m+1)\Delta t)] \tag{40}$$

It is obvious that at any time t, only three terms of the sum participate in the second sum of Eq. (39).

The homogeneous part of an approximate solution $\tilde{x}_h(t)$ must satisfy the homogeneous form of the Eq. (37):

$$\ddot{x}_h(t) + \pi^2 x_h(t) = 0 \tag{41}$$

According to Eq. (36), there are coefficients C_k in the first sum of Eq. (39) that satisfy the equation of free oscillations in an accurate manner. In the time interval Δt, only four coefficients C_k are not equal to zero. They are determined from the condition that a homogenous Eq. (41) is satisfied at time $m\Delta t$ and $(m+1)\Delta t$ and that conditions at the beginning of the interval $m\Delta t$ are satisfied i.e., position x_m and velocity $\dot{x}_m$.

Using the values of $y_{\pi,1}(\tau)$ basis function and its derivatives given in Fig. 6, the previously calculated coefficients D_l and conditions at the beginning of each interval Δt, yields:

$$C_{m-1} = \frac{1}{2y_1 \cdot \dot{y}_1} \cdot [-x_m\dot{y}_1 - \dot{x}_m y_1 - D_{m-1} \cdot (3y_1 - \dot{y}_1) + D_m \cdot (3y_1 + \dot{y}_1)]$$

$$C_m = \frac{1}{y_1} \cdot [x_m - D_{m-1} - D_m]$$

$$C_{m+1} = \frac{1}{2y_1 \cdot \dot{y}_1} \cdot [-x_m\dot{y}_1 + \dot{x}_m y_1 + D_{m-1} \cdot (3y_1 + \dot{y}_1) - D_m \cdot (3y_1 - \dot{y}_1)]$$

$$C_{m+2} = \frac{1}{2y_1 \cdot \dot{y}_1} \cdot [x_m\dot{y}_1 + \dot{x}_m y_1 + D_{m-1} \cdot (3y_1 - \dot{y}_1) - D_m \cdot (3y_1 + \dot{y}_1)] \tag{42}$$

For an initial time interval $m = 0$; $0 \le t \le \Delta t$, coefficients D_l have the following values:

$$D_{-1} = -1/24, \quad D_0 = 1/24, \quad D_1 = 3/24$$

$$C_{-1} = 1/(8\dot{y}_1) = 0.080545508626864,$$

$$C_0 = 0,\ C_1 = -1/(8\dot{y}_1) = -C_{-1},\ C_2 = -1/(8\dot{y}_1) = -C_{-1} \tag{43}$$

The position and velocity of a particle at the time $t = \Delta t = 2/3$ according to Eq. (39), are:

$$\tilde{x}_1(2/3) = C_0(1 - y_1) + C_1 + C_2(1 - y_1) + D_0 + D_1$$

$$\dot{\tilde{x}}_1(2/3) = -C_0\dot{y}_1 + C_2\dot{y}_1 - 3D_0 + 3D_1 \tag{44}$$

Substituting the coefficients given in (43) into expressions (44), the values are obtained:

$$\tilde{x}_1 = 0.097750554738942, \quad \dot{\tilde{x}}_1 = 0.37500000000000$$

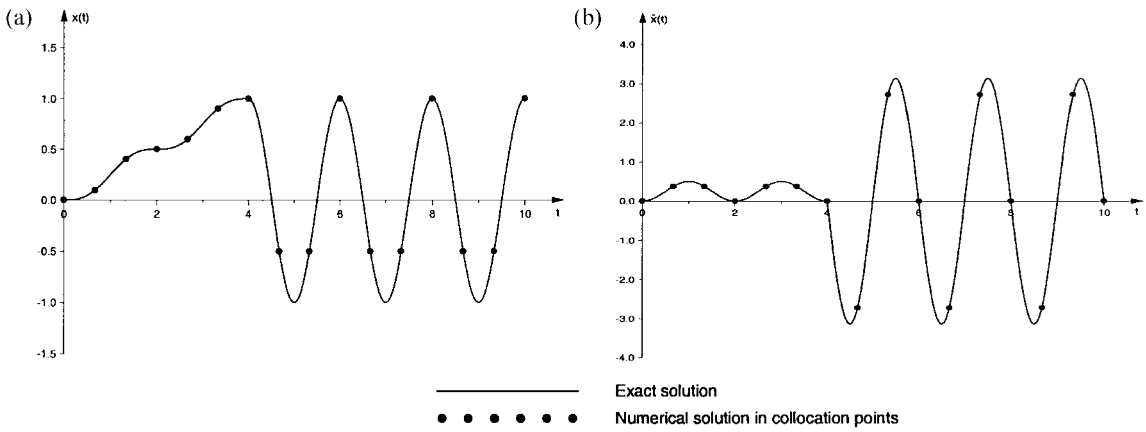


Fig. 9 Forced vibrations of the one degree of freedom system: a) Displacements, b) Velocities

that coincide with the values of the known exact solution:

$$x(t) = \frac{t}{4} - \frac{\sin(\pi t)}{4\pi} \rightarrow x_1 = 0.097750554738942$$

$$\dot{x}(t) = \frac{1}{4} \cdot [1 - \cos(\pi t)] \rightarrow \dot{x}_1 = 0.37500000000000$$

Introducing the coefficients from Eqs. (40) and (42) into general numerical solution, Eq. (39), an approximate solution is obtained, which corresponds to an exact solution in every moment. Fig. 9 shows the response of a system.

Efficiency of the proposed procedure consists in the following:

- The procedure is adaptive, which means that Δt can change from step by step;
- Given load is approximated with the chosen accuracy (basis functions and length of the time step are selected);
- For the system with frequency ω, basis functions $y_{\omega,h}(\tau)$ are calculated and used for obtaining an exact dynamic response for accurately approximated load;
- Accuracy of the procedure does not depend on the time increment Δt (only an approximation of the given load can depend on Δt);
- The homogeneous part of the solution for system frequency ω is obtained in an accurate manner because $\sin(\omega t)$ and $\cos(\omega t)$ can be accurately developed using $y_{\omega,h}(\tau)$ finite functions. Functions $y_{\omega,h}(\tau)$ are the only ones to have that property.
- The number of calculations of the proposed procedure is significantly lower than in e.g., Runge-Kutta method for a continuous approximation of high accuracy.

5.2 Free vibrations of a particle

Free vibrations of a particle are described by the following differential equation:

$$\ddot{x}(t) + \omega^2 x(t) = 0 \tag{45}$$

and initial conditions:

$$x(0) = x_0 \ ; \ \dot{x}(0) = \dot{x}_0 \tag{46}$$

From the given initial conditions (46) and Eq. (45), an initial acceleration can be calculated as:

$$\ddot{x}_0 = -\omega^2 x_0 \tag{47}$$

An approximate solution base is formed by a mutual displacement of function $Fup_2(t)$ by the value which corresponds to the time increment Δt. Distribution of basis functions is shown in Fig. 10.

Applying the collocation method in a point, numerical solution of the problem (45)-(46) at time $t = k\Delta t$ is sought in the form of a linear combination:

$$\tilde{x}(t) = \sum_{j=k-1}^{k+1} C_j \cdot F_j(t) \tag{48}$$

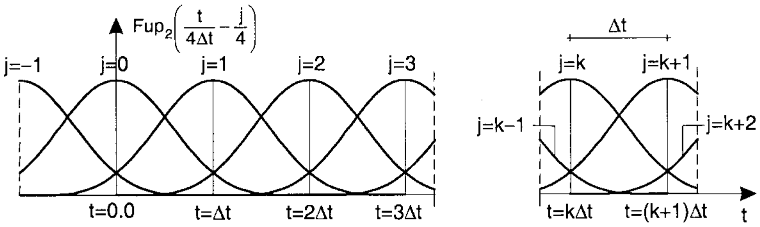


Fig. 10 Distribution of basis functions

where $F_j(t)$ is the basis function $Fup_2(t)$ with the vertex in collocation point of index j. Satisfying the initial conditions (46) and differential equation of a problem at time $t = 0$ according to Eq. (47), the following system of collocation equations is obtained:

$$(5C_{-1} + 26C_0 + 5C_1)/9 = x_0$$

$$2(-C_{-1} + C_1)/\Delta t = \dot{x}_0$$

$$4(C_{-1} - 2C_0 + C_1)/\Delta t^2 = \ddot{x}_0 \tag{49}$$

Solving a system of Eq. (49), unknown coefficients of linear combination C_j, $j = -1, 0, 1$ are obtained expressed by known values in an initial moment $t = 0$:

$$C_{-1} = \frac{36 - 13\omega^2\Delta t^2}{144}x_0 - \frac{\Delta t}{4}\dot{x}_0$$

$$C_0 = \frac{36 + 5\omega^2\Delta t^2}{144}x_0$$

$$C_1 = \frac{36 - 13\omega^2\Delta t^2}{144}x_0 + \frac{\Delta t}{4}\dot{x}_0 \tag{50}$$

From a collocation equation which satisfies the differential equation of a problem (45) at time $t = \Delta t$, a coefficient of the basis function for $j = 2$ is obtained:

$$C_2 = -C_0 + 2\frac{36 - 13\Delta t^2\omega^2}{36 + 5\Delta t^2\omega^2}C_1 \tag{51}$$

The response of the one degree of freedom system for $t \in [0, \Delta t]$ is defined by the coefficients in Eqs. (50) and (51). By analogy, the values of coefficients of any two arbitrary moments, which are mutually displaced by Δt on a time axis, can be written as:

$$C_{k-1} = x_k/4 - \Delta t\dot{x}_k/4 - 13\omega^2\Delta t^2 x_k/144$$

$$C_k = x_k/4 + 5\omega^2\Delta t^2 x_k/144 = x_{k+1}/4 - \Delta t\dot{x}_{k+1}/4 - 13\omega^2\Delta t^2 x_{k+1}/144$$

$$C_{k+1} = x_k/4 + \Delta t\dot{x}_k/4 - 13\omega^2\Delta t^2 x_k/144 = x_{k+1}/4 + 5\omega^2\Delta t^2 x_{k+1}/144$$

$$C_{k+2} = x_{k+1}/4 + \Delta t\dot{x}_{k+1}/4 - 13\omega^2\Delta t^2 x_{k+1}/144 \tag{52}$$

From the second and third equations in expressions (52), the displacement and velocity in the moment $t = (k+1)\Delta t$ are expressed by the displacement and velocity in the moment $t = k\Delta t$ in the following form:

$$\begin{bmatrix} x_{k+1} \\ \dot{x}_{k+1} \end{bmatrix} = \frac{1}{36 + 5\Delta t^2\omega^2}\begin{bmatrix} 36 - 13\Delta t^2\omega^2 & 36\Delta t \\ -4(9 - \Delta t^2\omega^2)\Delta t\omega^2 & 36 - 13\Delta t^2\omega^2 \end{bmatrix}\begin{bmatrix} x_k \\ \dot{x}_k \end{bmatrix} \tag{53}$$

The eigenvalues of the matrix in Eq. (53) are:

$$\lambda_{1,2} = \frac{(36 - 13\omega^2\Delta t^2) \pm 12\omega\Delta t\, i\sqrt{9 - \omega^2\Delta t^2}}{36 + 5\omega^2\Delta t^2} \tag{54}$$

In case when $\omega^2\Delta t^2 < 9$ a spectral radius has the following value:

$$\rho = \frac{\sqrt{36^2 + 10 \cdot 36\omega^2\Delta t^2 + 25\omega^4\Delta t^4}}{36 + 5\omega^2\Delta t^2} = 1.0 \tag{55}$$

Therefore, the time increment must be $\Delta t_{cr} \le 3T/2\pi$, where T is the period of observed oscillations of a one degree of freedom system or a period of the mode with the highest frequency of a multiple degree of freedom system. Thus, in the proposed numerical procedure, the length of the time increment is 50% greater than in the central difference method (Bathe 1982) when $\Delta t_{cr} = T/\pi$. In Eq. (55) it can be observed that the accuracy of the procedure is very good when $\Delta t < \Delta t_{cr}$ because when a spectral radius $\rho = 1$ there is no numerical damping in an analysis of dynamic system behavior.

The proposed numerical procedure with basis functions $Fup_2(t)$ is conditionally stable. Its different variants are possible either with regard to an increase in accuracy or providing an unconditional stability.

5.3 Dynamic system with multiple degrees of freedom

The linear dynamic response of a multiple degree of freedom system is described by the governing equation:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F}(t) \tag{56}$$

and initial conditions:

$$\boldsymbol{x}_{t=0} = \boldsymbol{x}_{\boldsymbol{0}} \;;\; \dot{\boldsymbol{x}}_{t=0} = \dot{\boldsymbol{x}}_{\boldsymbol{0}} \tag{57}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$, and $\boldsymbol{K}$ are the mass, damping, and stiffness matrices; $\boldsymbol{F}$ is the external load vector; $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ are the displacement, velocity, and acceleration vectors.

Table 4 The time integration scheme using continuous approximation in time with basis functions $Fup_2(t)$

A. INITIAL CALCULATIONS 1. Form stiffness matrix $\boldsymbol{K}$, mass matrix $\boldsymbol{M}$, and damping matrix $\boldsymbol{C}$ 2. Initialize $\boldsymbol{x}, \dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ 3. Select time step Δt and calculate coefficients: $\boldsymbol{C}_{-1} = \frac{1}{4}\boldsymbol{x}_0 - \frac{\Delta t}{4}\dot{\boldsymbol{x}}_0 + \frac{13\Delta t^2}{144}\ddot{\boldsymbol{x}}_0$ $\boldsymbol{C}_0 = \frac{1}{4}\boldsymbol{x}_0 - \frac{5\Delta t^2}{144}\ddot{\boldsymbol{x}}_0$ $\boldsymbol{C}_1 = \frac{1}{4}\boldsymbol{x}_0 + \frac{\Delta t}{4}\ddot{\boldsymbol{x}}_0 + \frac{13\Delta t^2}{144}\ddot{\boldsymbol{x}}_0$ 4. Calculate effective stiffness matrix $\hat{\boldsymbol{K}}$: $\hat{\boldsymbol{K}} = \frac{5}{9}\boldsymbol{K} + \frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{C}$ 5. Triangularize $\hat{\boldsymbol{K}}$: $\hat{\boldsymbol{K}} = \boldsymbol{LDL}^T$
B. FOR EACH TIME STEP 1. Calculate effective loads at time $t = k\Delta t, k \in N$ $\hat{\boldsymbol{F}}_{k\Delta t} = \boldsymbol{F}_{k\Delta t} - \frac{4}{\Delta t^2}\boldsymbol{M} \cdot (\boldsymbol{C}_{k-1} - 2\boldsymbol{C}_k) - \frac{5}{9}\boldsymbol{K}(\boldsymbol{C}_{k-1} + 5.2\boldsymbol{C}_k) + \frac{2}{\Delta t}\boldsymbol{C} \cdot \boldsymbol{C}_{k-1}$ 2. Calculate coefficients of the solution: $\hat{\boldsymbol{K}} \cdot \boldsymbol{C}_{k+1} = \hat{\boldsymbol{F}}_{k\Delta t}$ 3. Evaluate displacements, velocities and accelerations at time $t = k\Delta t$: $\boldsymbol{x}_{k\Delta t} = \frac{5}{9}(\boldsymbol{C}_{k-1} + 5.2\boldsymbol{C}_k + \boldsymbol{C}_{k+1})$ $\dot{\boldsymbol{x}}_{k\Delta t} = \frac{2}{\Delta t}(-\boldsymbol{C}_{k-1} + \boldsymbol{C}_{k+1})$ $\ddot{\boldsymbol{x}}_{k\Delta t} = \frac{4}{\Delta t^2}(\boldsymbol{C}_{k-1} - 2\boldsymbol{C}_k + \boldsymbol{C}_{k+1})$

Applying the procedure of a continuous approximation in time with basis functions $Fup_2(t)$, a numerical solution of the problem (56)-(57) is sought in the following form:

$$\tilde{\boldsymbol{x}}(t) = \sum_{k=-1}^{\infty} \boldsymbol{C}_k \cdot Fup_2\left(\frac{t}{4\Delta t} - \frac{k}{4}\right) \tag{58}$$

Fig. 10 shows the arrangement of basis functions for a single component of a solution $x_i(t)$. An algorithm for time integration, formed according to the procedure described in Section 5.2, is given in Table 4.

Table 5 Comparison of the solution obtained by the $Fup_2(t)$ basis functions with the exact solution

t		Δt	$2\Delta t$	$3\Delta t$	$4\Delta t$	$5\Delta t$	$6\Delta t$	$7\Delta t$	$8\Delta t$	$9\Delta t$	$10\Delta t$	$11\Delta t$	$12\Delta t$
Numerical solution	$x_1(t)$	0.004	0.042	0.180	0.485	0.984	1.631	2.302	2.828	3.037	2.824	2.185	1.239
	$x_2(t)$	0.376	1.391	2.748	4.061	4.982	5.308	5.036	4.344	3.517	2.831	2.462	2.426
Exact solution	$x_1(t)$	0.003	0.038	0.176	0.486	0.996	1.657	2.338	2.861	3.052	2.806	2.131	1.157
	$x_2(t)$	0.382	1.412	2.781	4.094	4.996	5.291	4.986	4.277	3.457	2.806	2.484	2.489

Example

The two degree of freedom system without damping for which the governing equations of motion are:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \cdot \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} \tag{59}$$

is considered. Homogeneous initial conditions $\boldsymbol{x}_0 = \mathbf{0}, \dot{\boldsymbol{x}}_0 = \mathbf{0}$ are selected, thus the vector of initial acceleration can be calculated from Eq. (59):

$$\ddot{\boldsymbol{x}}_{\mathbf{0}} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} \tag{60}$$

Using the numerical procedure described in Table 4, the response of a dynamic system is obtained for 12 time steps $\Delta t = 0.28$(s). Results of analyses are given in Table 5 complete with the exact solution.

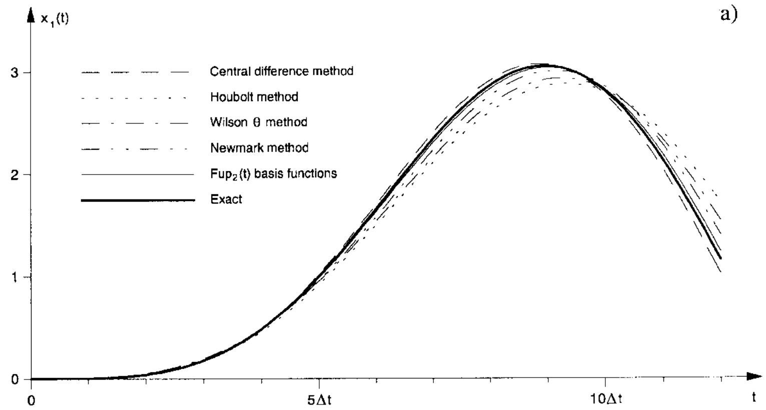
Fig. 11 shows the response of a system obtained by basis functions $Fup_2(t)$, and responses of the system obtained analytically and by the most commonly used numerical procedures of time integration (Bathe 1982).

6. Conclusions

The procedure in which a continuous approximation in time is performed by the collocation method with Rvachev's basis functions R_{bf} is presented in this paper. From the class of R_{bf}, finite functions $y_{\omega,h}(t)$ and $Fup_n(t)$ are implemented. Based on the presented numerical studies, the following concluding remarks can be made.

First, an approximate solution of free undamped vibrations of a particle obtained by the $y_{\omega,h}(t)$ functions corresponds to an exact solution. Such a result is a consequence of the fact that the $y_{\omega,h}(t)$ functions belong to a vector space which contains trigonometric functions of the given frequency ω. Simultaneously, an approximate solution base is formed by a mutual displacement of functions $y_{\omega,h}(t)$ by the value Δt (characteristic interval).

Second, an exact dynamic response can be obtained by the proposed numerical procedure for the case of forced vibrations. For the base of a particular part of the solution, the functions of R_{bf} class



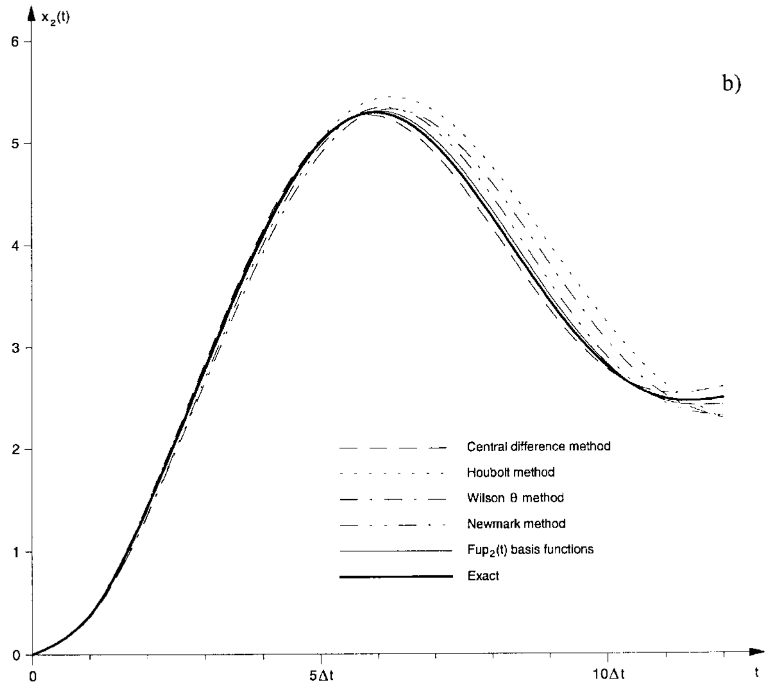


Fig. 11 a) The first component of the response of a two degree of freedom system, b) The second component of the response of a two degree of freedom system

are selected by which a disturbing force can be described exactly. Basis functions $Fup_n(t)$ are applied, the linear combination of which can be used for the exact description of algebraic polynomials. Therefore, when a disturbing force function is an algebraic polynomial, the load function is described exactly by basis functions $Fup_n(t)$.

Third, it is shown that a high quality response of a dynamic system can be attained when only basis functions $Fup_n(t)$ are applied. The numerical stability and accuracy of a proposed procedure are tested on an example of a one degree of freedom model of free vibrations of a material point by

$Fup_2(t)$ basis functions. It is shown that it is possible to use 50% longer time step than the central difference method. This knowledge can also be applied to multiple degree of freedom dynamic systems. For an illustration of the numerical procedure, a two degree of freedom dynamic system has been analyzed. Based on a comparison with the results obtained by other the most commonly used numerical methods (Fig. 11), it can be concluded that the proposed procedure gives excellent dynamic response.

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