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Investigation of natural frequencies of multi-bay and multi-storey frames using a single variable shear deformation theory

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Abstract. This study concerns about calculating exact natural frequencies of frames using a single variable shear deformation theory (SVSDT) which considers the parabolic shear stress distribution across the cross section. Free vibration analyses are performed for multi-bay, multi-storey and multi-bay multi-storey type frame structures. Dynamic stiffness formulations are derived and used to obtain first five natural frequencies of frames. Different beam and column cross sections are considered to reveal their effects on free vibration analysis. The calculated natural frequencies are tabulated with the results obtained using Euler-Bernoulli Beam Theory (EBT) and Timoshenko Beam Theory (TBT). Moreover, the effects of inner and outer columns on natural frequencies are compared for multi-bay frames. Several mode shapes are plotted.

Keywords: dynamic stiffness; multi-bay frame; multi-storey frame; natural frequency; single variable shear deformation

1. Introduction

Natural frequency values of frames are of high importance in civil and mechanical engineering structures exposed to dynamic loading. Therefore, exact results of free vibration analysis of frame structures become an important research area. In open literatures, free vibration analysis of frame structures using finite element method (FEM) that provides approximate solutions can be found (Clough and Penzien 2003, Paz and Leigh 2004, Chopra 2012, Rao 1995, Wu 2008, Özyiğit 2009, Minghini *et al.* 2010, Ranjbaran 2014, Mehmood 2015, Ozturk *et al.* 2016).

Distributed parameter model is used in very limited number of studies about vibrations of frames as mathematical formulations and analysis procedure are complicated. In the study of Albarracin and Grossi (2005), a frame that have one column and one beam is considered and eigenfrequencies are calculated according to distributed parameter model. Caddemi and Calio (2013), developed dynamic stiffness matrix for cracked Euler-Bernoulli beams and natural frequencies of damaged frames are calculated. Labib *et al.* (2014) performed free vibration analysis of multiple cracked frames using dynamic stiffness formulation using EBT. Mei and Sha (2015) investigated vibrations of a simple spatial frame using wave propagation approach and EBT. In the study of Grossi and Albarracin (2013), free vibrations of frames that consist of inclined members are investigated using variational approach. Mei (2012) obtained natural frequencies of one storey frames via wave vibration approach according to EBT. Caddemi *et al.* (2017) investigated free vibrations of multiple cracked axially loaded frames using Dynamic Stiffness Method (DSM) according to EBT.

DSM is an effective technique for vibration analysis of beams and beam assembly structures such as frames. The DSM provides exact results as uses the exact mode shapes (Banerjee 1997). There are many studies about application of DSM for calculation of natural frequencies of different type of beams and plates (Jun *et al.* 2008, Bao-Hui *et al.* 2011, Banerjee 2012, Banerjee and Jackson 2013, Su and Banerjee 2015, Jun *et al.* 2016, Bozyigit and Yesilce 2016).

It is a known issue that EBT overestimates natural frequencies. Thus, Timoshenko Beam Theory (TBT) which considers shear deformation and rotational inertia is used in many studies that focus on more accurate results. Even TBT provides more realistic results according to EBT, there is an important parameter called shear coefficient or area reduction factor in TBT. This parameter is used to decrease the error arised from unconstant distribution of shear stress on the cross section (Han *et al.* 1999).

The high order shear deformation theories that based on realistic shear stress distribution with the assumption of cross section does not remain plane after bending have been studied by many of the researchers (Levinson 1981, Bickford 1982, Reddy 1984, Heyliger and Reddy 1988). Ghugal and Shimpi (2001) reviewed EBT, TBT and high order beam theories for isotropic and anisotropic laminated beams. Shimpi (2002) researched on refined plate theory that has shear and bending components of lateral and axial displacements. Shimpi *et al.* (2007) introduced two new displacement based first-order shear deformation theories involving only two unknown functions for plate bending. Even high-order theories provide more accurate results when compared to EBT and TBT, the solutions of these theories are significantly complicated and time-consuming. Shimpi *et al.* (2016) presented a new SVSDT that considers

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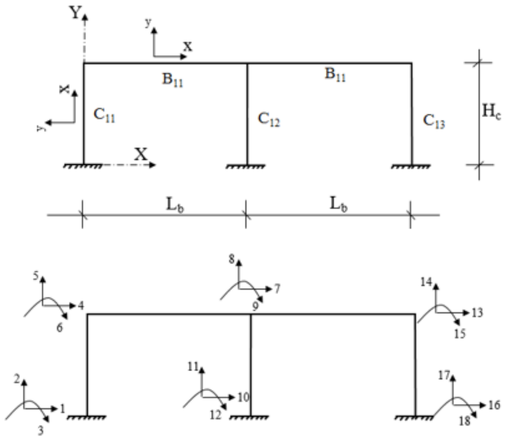

Fig. 1 One-storey two-bay frame model and degrees of freedom

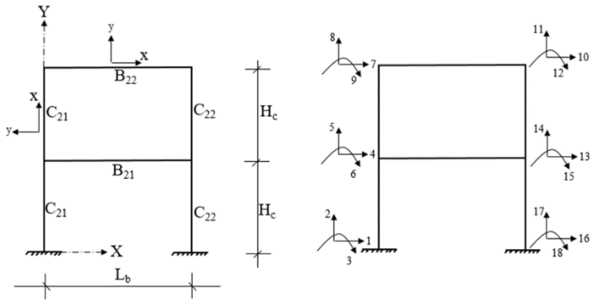

Fig. 2 Two-storey one-bay frame model and degrees of freedom

the varying shear stress distribution along the cross section. The governing equation of motion of SVSDT is a fourth order partial differential equation and the requirement of shear correction factor vanishes.

In this study, the SVSDT is applied to vibration analysis of frames for the first time. Three different frame structures are considered and first five exact natural frequencies of frames are calculated according to SVSDT using DSM. As SVSDT is used for vibrations of frames for the first time, the results are compared with the solutions of EBT and TBT. The effect of cross-sections of inner and outer columns are compared. In addition to this, different beam cross sections are used in the free vibration analysis of frames to reflect the effects on natural frequencies. The mode shapes are plotted.

2. Models and formulation

The first model of the study is a one-storey two-bay frame that consists of three columns and two beams (Fig. 1). The second model is a one-bay two-storey frame structure (Fig. 2). The third model is a two-bay two-storey frame that can be experienced for architectural purposes (Fig. 3). The columns of all three models are fixed supported. In Figs. 1-3, X and Y are global axes of frames, x and y are local axes of frame members, H_c is height of the columns, L_b is length of the beams. Columns and beams are denominated as C_{mn} and B_{mn} (m: 1,2,3; n: 1,2,3) where m and n represent model

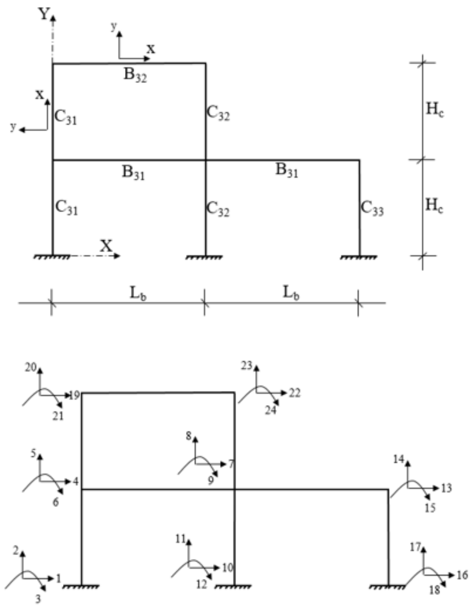

Fig 3. Two-storey two-bay frame model and degrees of freedom

number and member number, respectively. Degrees of freedom of models can be seen in Figs. 1-3. The assumptions listed below are considered for all models:

1) The cross-sections of members are uniform.
2) The material of members is isotropic.
3) The frame members behave linear and elastic.
4) The damping is neglected.

The displacement and force functions are derived to begin the DSM solution. Firstly, equation of flexural motion of an Euler-Bernoulli beam in free vibration (Mei 2012) is written in Eq. (1)

$$EI\frac{\partial^4 y^E(x,t)}{\partial x^4}+\bar{m}\frac{\partial^2 y^E(x,t)}{\partial t^2}=0 \tag{1}$$

where x is the position along the beam, t is time, $y^E(x,t)$ is transverse displacement function according to EBT, E is elastic modulus, I is the area moment of inertia of cross section, $\bar{m}$ is mass per unit length. Separation of variables method is applied to Eq. (1) with the assumption of harmonic motion using $y^E(x,t) = y^E(x)e^{i\omega t}$ to obtain Eq. (2).

$$\frac{d^4 y^E(z)}{dz^4}-\frac{\bar{m}\omega^2 L^4}{EI}y^E(z)=0 \tag{2}$$

where ω is natural frequency, $z = x/L$. It should be noted that L represents the member length. The solution is assumed as:

$$y^E(z)=\{C\}e^{isz} \tag{3}$$

where $i=\sqrt{-1}$.

Substituting Eq. (3) into Eq. (2), the transverse displacement $y^E(z)$ and slope $\theta^E(z)$ functions are presented in Eqs. (4)-(5), respectively.

$$y^E(z)=(C_1e^{is_1z}+C_2e^{is_2z}+C_3e^{is_3z}+C_4e^{is_4z}) \tag{4}$$

$$\theta^E(z)=(is_1C_1e^{is_1z}+is_2C_2e^{is_2z}+is_3C_3e^{is_3z}+is_4C_4e^{is_4z}) \tag{5}$$

Bending moment function and shear force function are defined in Eqs. (6)-(7), respectively.

$$\begin{aligned} M^E(z) = (&Hs_1^2C_1e^{is_1z} + Hs_2^2C_2e^{is_2z} \\ &+Hs_3^2C_3e^{is_3z} + Hs_4^2C_4e^{is_4z}) \end{aligned} \tag{6}$$

$$\begin{aligned} Q^E(z) = (&Jis_1^3C_1e^{is_1z} + Jis_2^3C_2e^{is_2z} \\ &+Jis_3^3C_3e^{is_3z} + Jis_4^3C_4e^{is_4z}) \end{aligned} \tag{7}$$

where $H=EI/L^2$ and $J=EI/L^3$.

Shimpi *et al.* (2016) defined the transverse displacement function of SVSDT as

$$y^S = y_b + y_s \tag{8}$$

where y^s is total transverse displacement, y_b is displacement component of bending and y_s is displacement component of shearing. The governing equations of bending motion of a beam in free vibration according to SVSDT is written as follows (Shimpi *et al.* 2016)

$$\begin{aligned} &EI\frac{\partial^4 y_b}{\partial x^4} - \frac{\bar{m}I}{A}\left(1+\frac{12(1+\mu)}{5}\right)\frac{\partial^4 y_b}{\partial x^2\partial t^2} \\ &+\bar{m}\frac{\partial^4 y_b}{\partial t^2} + \frac{\bar{m}^2 I}{A^2E}\frac{12(1+\mu)}{5}\frac{\partial^4 y_b}{\partial t^4} = 0 \end{aligned} \tag{9}$$

where A and μ represent cross-sectional area and Poisson's ratio, respectively. y_b is obtained from the solution of Eq. (9). Using separation of variables method with the assumption of $y_b(x,t)=y_b(x)e^{i\omega t}$, ordinary differential equation below is obtained

$$\begin{aligned} &A_0\frac{d^4y_b(z)}{dz^4} + B_0\omega^2\frac{d^2y_b(z)}{dz^2} \\ &-C_0\omega^2y_b(z) + D_0\omega^4y_b(z) = 0 \end{aligned} \tag{10}$$

where

$$A_0 = \frac{EI}{L^4};\, B_0 = -\frac{\bar{m}I}{AL^2}\left(1+\frac{12(1+\mu)}{5}\right);\, C_0 = \bar{m};$$

$$D_0 = \frac{\bar{m}^2I\left(1+\frac{12(1+\mu)}{5}\right)}{A^2E}\frac{12(1+\mu)}{5}$$

The solution is assumed as

$$y_b(z) = \{D\}e^{ikz} \tag{11}$$

where $i=\sqrt{-1}$.

The bending component of transverse displacement function is written by substituting Eq. (11) into Eq. (10)

$$y_b(z) = (D_1e^{ik_1z} + D_2e^{ik_2z} + D_3e^{ik_3z} + D_4e^{ik_4z}) \tag{12}$$

The slope function due to bending can be derived as follows

$$\frac{dy_b(z)}{dz} = (ik_1D_1e^{ik_1z} + ik_2D_2e^{ik_2z} + ik_3D_3e^{ik_3z} + ik_4D_4e^{ik_4z}) \tag{13}$$

The bending moment function and shear force function are defined in Eq. (14) and Eq. (15), respectively (Shimpi *et al.* 2016).

$$M^S(z) = -\frac{EI}{L^2}\frac{d^2y_b(z)}{dz^2} \tag{14}$$

$$Q^S(z) = -\frac{EI}{L^3}\frac{d^3y_b(z)}{dz^3} - \frac{\bar{m}I\omega^2}{AL}\frac{dy_b(z)}{dz} \tag{15}$$

Eqs. (14)-(15) can be rewritten by using Eq. (12) as follows

$$\begin{aligned} M^S(z) = (&Hk_1^2D_1e^{ik_1z} + Hk_2^2D_2e^{ik_2z} \\ &+Hk_3^2D_3e^{ik_3z} + Hk_4^2D_4e^{ik_4z}) \end{aligned} \tag{16}$$

$$\begin{aligned} Q^S(z) = &(Jik_1^3 - Kik_1)D_1e^{ik_1z} + (Jik_2^3 - Kik_2)D_2e^{ik_2z} \\ &+(Jik_3^3 - Kik_3)D_3e^{ik_3z} + (Jik_4^3 - Kik_4)D_4e^{ik_4z} \end{aligned} \tag{17}$$

where $K = (\bar{m}I\omega^2)/(AL)$

The displacement function due to shearing y_s and total displacement y^S function are defined in Eqs. (18) and (19), respectively.

$$y_s = T\left(-H\frac{d^2y_b}{dz^2} - Py_b(z)\right) \tag{18}$$

$$\begin{aligned} y^S = &\left(THk_1^2 - TP + 1\right)D_1e^{ik_1z} + \left(THk_2^2 - TP + 1\right)D_2e^{ik_2z} \\ &+\left(THk_3^2 - TP + 1\right)D_3e^{ik_3z} + \left(NHk_4^2 - TP + 1\right)D_4e^{ik_4z} \end{aligned} \tag{19}$$

where $T = \frac{12(1+\mu)}{5AE};\, P = \left(\bar{m}I\omega^2/A\right)$

Finally, the total slope function is obtained by assembly of $\frac{dy_b(z)}{dz}$ and $\frac{dy_s(z)}{dz}$ as follows

$$\begin{aligned} \frac{dy^S(z)}{dz} = &\left(ik_1 + TJik_1^3 - TRik_1\right)D_1e^{ik_1z} \\ &+\left(ik_2 + TJik_2^3 - TRik_2\right)D_2e^{ik_2z} + \left(ik_3 + TJik_3^3 - TRik_3\right)D_3e^{ik_3z} \\ &+\left(ik_4 + TJik_4^3 - TRik_4\right)D_4e^{ik_4z} \end{aligned} \tag{20}$$

where $R=P/L$

It should be importantly noted that the further details of derivations and definitions of displacement, slope, bending moment and shear force functions of SVSDT are clearly presented by Shimpi *et al.* (2016).

The axial vibrations of frame members are also considered in this study. Thus, axial displacement function and axial force function are obtained by solving the equation of motion of a beam in free axial vibration (Rao 1995) given below

$$AE\frac{\partial^2u(x,t)}{\partial x^2} - \bar{m}\frac{\partial^2u(x,t)}{\partial t^2} = 0 \tag{21}$$

where $u(x,t)$ is axial displacement function. Separation of variables method is applied to Eq. (21) using $u(x,t)=u(x)e^{i\omega t}$ with the assumption of harmonic motion, the following equation is obtained

$$\frac{d^2u(z)}{dz^2} + \frac{\bar{m}\omega^2L^2}{AE}u(z) = 0 \tag{22}$$

where $z = x/L$.

Substituting Eq. (23) into Eq. (22), the axial displacement function $u(z)$ and axial force function $N(z)$ are given in Eqs.(24) - (25), respectively.

$$u(z) = \{D\}e^{ikz} \tag{23}$$

$$u(z) = (D_5e^{ik_5z} + D_6e^{ik_6z}) \tag{24}$$

$$N(z) = V(is_5D_5e^{ik_5z} + is_6D_6e^{ik_6z}) \tag{25}$$

where $V=AE/L$.

The equations of motion of a Timoshenko beam can be written as

$$\frac{AG}{\bar{k}}\left(\frac{\partial^2 y^T(x,t)}{\partial x^2}-\frac{\partial\varphi(x,t)}{\partial x}\right)-\bar{m}\frac{\partial^2 y^T(x,t)}{\partial t^2}=0 \tag{26}$$

$$EI\frac{\partial \varphi^2(x,t)}{\partial x^2}-\frac{\bar{m}I}{A}\frac{\partial \varphi^2(x,t)}{\partial t^2}+\frac{AG}{\bar{k}}\left(\frac{\partial y^T(x,t)}{\partial x}-\varphi(x,t)\right)=0 \tag{27}$$

where $\bar{k}$ is shear correction factor, G is shear modulus.

The bending moment function $M^T(x,t)$ and the shear force function $Q^T(x,t)$ of the Timoshenko beam are written as

$$M^T(x,t)=EI\frac{\partial\varphi(x,t)}{\partial x} \tag{28}$$

$$Q^T(x,t)=\frac{AG}{\bar{k}}\gamma(x,t)=\frac{AG}{\bar{k}}\left(\frac{\partial y^T(x,t)}{\partial x}-\varphi(x,t)\right) \tag{29}$$

where $\gamma(x,t)$ is the associated shearing deformation.

Assuming that the motion is harmonic we substitute for $y^T(x,t)$ and $\varphi(x,t)$ the following

$$y^T(z,t)=y^T(z)e^{i\omega t} \tag{30}$$

$$\varphi(z,t)=\varphi(z)e^{i\omega t} \tag{31}$$

where $y^T(z)$ and $\varphi(z)$ are the amplitudes of the total transverse deflection and the angle of rotation due to bending, respectively; $i=\sqrt{-1}$. Eqs. (26) and (27) can be written as ordinary differential equations by using Eqs. (30) and (31) as

$$\left(\frac{AG}{\bar{k}L^2}\right)\frac{d^2y^T(z)}{dz^2}-\left(\frac{AG}{L\bar{k}}\right)\frac{d\varphi(z)}{dz}+(\bar{m}\omega^2y^T(z)=0 \tag{32}$$

$$\frac{EI}{L^2}\frac{d^2\varphi(z)}{dz^2}+\left(\frac{AG}{L\bar{k}}\right)\frac{dy(z)}{dz}+\left(\frac{\bar{m}\omega^2I}{A}-\frac{AG}{\bar{k}}\right)\varphi(z)=0 \tag{33}$$

where $z=x/L$.

It is assumed that the solution is

$$y^T(z)=Be^{ibz} \tag{34}$$

$$\varphi^T(z)=Fe^{ibz} \tag{35}$$

Substituting Eqs. (34) and (35) into Eqs. (32) and (33) results in

$$\left[\bar{m}\omega^2-\left(\frac{AG}{\bar{k}L^2}\right)b^2\right]B-\left(\frac{AG}{L\bar{k}}ib\right)F=0 \tag{36}$$

$$\left(\frac{AG}{L\bar{k}}ib\right)B+\left(\frac{\bar{m}\omega^2I}{A}-\frac{AG}{\bar{k}}-\frac{EI}{L^2b^2}\right)F=0 \tag{37}$$

Eqs. (36) and (37) can be written in matrix form for the two unknowns B and F as

$$\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{Bmatrix}B\\ F\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{38}$$

where

$$A_{11}=\bar{m}\omega^2-\left(\frac{AG}{\bar{k}L^2}\right)b^2;A_{21}=-A_{12}=\frac{AG}{L\bar{k}}ib;$$

$$A_{22}=\frac{\bar{m}\omega^2I}{A}-\frac{AG}{\bar{k}}-\frac{EI}{L^2}b^2$$

The non-trivial solution is obtained when the determinant of the coefficient matrix will be zero. Thus, we have a fourth-order equation with the unknowns, resulting in four values and the general solution functions can be written as

$$y^T(z)=[B_1e^{ib_1z}+B_2e^{ib_2z}+B_3e^{ib3_1z}+B_4e^{ib_4z}] \tag{39}$$

$$\varphi(z)=[F_1e^{ib_1z}+F_2e^{ib_2z}+F_3e^{ib_3z}+F_4e^{ib_4z}] \tag{40}$$

The eight constants, B_1, ..., B_4 and F_1, ..., F_4 will be found from Eqs. (39), (40) and boundary conditions where $B_n=j_nF_n$, $j_n=\left(-AGib_n/L\bar{k}\right)/\left(\left(\bar{m}\omega^2I/A\right)-\left(AG/\bar{k}\right)-\left(EIb_n^2/L^2\right)\right)$, (n:1,2,3,4).

The bending moment and shear force functions of the Timoshenko beam can be obtained as

$$M^T(z)=\left[\frac{EI}{L}\frac{d\varphi(z)}{dz}\right] \tag{41}$$

$$Q^T(z)=\left[\frac{AG}{\bar{k}L}\frac{dy^T(z)}{dz}-\left(\frac{AG}{\bar{k}}\right)\varphi(z)\right] \tag{42}$$

3. Dynamic stiffness method for SVSDT and EBT

The end forces and end displacements of a frame member are used to construct the dynamic stiffness matrix. After obtaining global stiffness matrices of all members, the global dynamic stiffness matrix of frame can be formed to calculate natural frequencies. The vector of end displacements of a frame member and the vector of coefficients for SVSDT are written in Eqs. (43)-(44), respectively.

$$\delta^S=[u_0 \quad y_0^S \quad \theta_0^S \quad u_1 \quad y_1^S \quad \theta_1^S]^T \tag{43}$$

$$D=[D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad D_6]^T \tag{44}$$

where

$$u_0=u(z=0),y_0^S=y^S(z=0),\theta_0^S=\theta^S(z=0),$$
$$u_1=u(z=1),y_1^S=y_1(z=1),\theta_1^S=\theta^S(z=1)$$

Using Eqs. (19)-(20) and Eq. (24), the following equation is obtained

$$\delta^S=\Delta^S D \tag{45}$$

where

$$\Delta^S=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & 1\\ \lambda_1^S & \lambda_2^S & \lambda_3^S & \lambda_4^S & 0 & 0\\ \eta_1^S & \eta_2^S & \eta_3^S & \eta_4^S & 0 & 0\\ 0 & 0 & 0 & 0 & e^{ik_5} & e^{ik_6}\\ \lambda_1^Se^{ik_1} & \lambda_2^Se^{ik_2} & \lambda_3^Se^{ik_3} & \lambda_4^Se^{ik_4} & 0 & 0\\ \eta_1^Se^{ik_1} & \eta_2^Se^{ik_2} & \eta_3^Se^{ik_3} & \eta_4^Se^{ik_4} & 0 & 0\end{bmatrix}$$

$$\lambda_n^S = \left(THk_n^2 - TP + 1\right), \eta_n^S = \left(ik_n + TJik_n^3 - TRik_n\right)$$
$$(n = 1,2,3,4)$$

The vector of end forces of the frame member is given below

$$F^S = [N_0 \quad Q_0^S \quad M_0^S \quad N_1 \quad Q_1^S \quad M_1^S]^T \tag{46}$$

where

$$N_0 = N(z=0); Q_0^S = Q^S(z=0), M_0^S = M^S(z=0),$$
$$N_1 = N(z=1), Q_1^S = Q^S(z=1), M_1^S = M^S(z=1)$$

It should be noted that the following relations are valid for sign convention

$$N_0 = -N_1, \quad Q_0^S = -Q_1^S, \quad M_0^S = -M_1^S \tag{47}$$

The matrix form of end force functions given in Eq. (46) is presented using Eqs. (16)-(17) and Eq. (25) as

$$\begin{bmatrix} N_0 & Q_0^S & M_0^S & N_1 & Q_1^S & M_1^S \end{bmatrix}^T = \kappa^S \begin{bmatrix} D_1 & D_2 & D_3 & D_4 & D_5 & D_6 \end{bmatrix}^T \tag{48}$$

where

$$\kappa^S = \begin{bmatrix} 0 & 0 & 0 & 0 & Vik_5 & Vik_6 \\ \Lambda_1^S & \Lambda_2^S & \Lambda_3^S & \Lambda_4^S & 0 & 0 \\ \Psi_1^S & \Psi_2^S & \Psi_3^S & \Psi_4^S & 0 & 0 \\ 0 & 0 & 0 & 0 & -Vik_5 e^{ik_5} & -Vis_5 e^{ik_6} \\ -\Lambda_1^S e^{ik_1} & -\Lambda_2^S e^{ik_2} & -\Lambda_3^S e^{ik_3} & -\Lambda_4^S e^{ik_4} & 0 & 0 \\ -\Psi_1^S e^{ik_1} & -\Psi_2^S e^{ik_2} & -\Psi_3^S e^{ik_3} & -\Psi_4^S e^{ik_4} & 0 & 0 \end{bmatrix}$$

$$\Lambda_n^S = (Jik_n^3 - Kik_n), \Psi_n^S = Hk_n^2, (n = 1,2,3,4)$$

The closed form of Eq. (48) can be written as

$$F^S = \kappa^S D \tag{49}$$

Eqs. (45) and (49) are used to construct the dynamic stiffness matrix of a frame member as

$$F^S = \kappa^S (\Delta^S)^{-1} \delta^S \tag{50}$$

$$K^{*S} = \kappa^S (\Delta^S)^{-1} \tag{51}$$

Here, K^{*S} is the local dynamic stiffness matrix of a frame member modeled according to SVSDT. The calculation of natural frequencies of distributed parameter frames is performed after obtaining the global dynamic stiffness matrix. The global dynamic stiffness matrix is constructed using transformation of local member dynamic stiffness matrices to global stiffness matrices. The angular transformation matrix and transformed dynamic stiffness matrix of a frame member are given in Eqs. (52)-(53), respectively (Paz and Leigh 2004).

$$TM = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 & 0 & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & 0 & 0 & -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{52}$$

$$\bar{K}^{*S} = (TM)^{-1} (K^{*S}) (TM) \tag{53}$$

Here, α is the angle between local axes of the frame member and global axes of the frame, $\bar{K}^{*S}$ is the global dynamic stiffness matrix of frame member according to SVSDT.

The derivation procedure of dynamic stiffness formulation for EBT and TBT are same as SVSDT. The global dynamic stiffness matrix of frame models according to EBT can be obtained by using displacement functions in Eqs. (4)-(5), (23) and force functions in Eqs. (6)-(7), (25). Similarly, Eqs. (24)-(25) and Eqs. (39)-(42) are used to construct dynamic stiffness formulations of frames according to TBT.

In this study, three different frame models are used. Let the denomination of frames in Fig. 1 and Fig. 2 as Frame-I and Frame-II, respectively. The third model that presented in Fig. 3 is entitled as Frame-III. It is seen that Frame-I and Frame-II have 18 degrees of freedom. Thus, the dimension of the global dynamic stiffness matrices of Frame-I and Frame-II is 18×18. Frame-III has 24 degrees of freedom and its global dynamic stiffness matrix is formed as 24×24 square matrix.

The related global dynamic stiffness matrices of frames can be constructed by a standard coding technique. It should be noted that the global dynamic stiffness matrices of Frame-I, Frame-II and Frame-III are reduced by erasing rows and columns due to restriction of displacements and rotations.

The ω values that equate the determinant of reduced global dynamic stiffness matrices of the frames to zero are obtained as natural frequencies using Wittrick-Williams algorithm. Besides, an alternative procedure such as a trial and error method based on interpolation and bisection approach can be used for calculating roots. When there is a change of sign between trial values, there must be a root lying in this interval. Using some iterations, the natural frequencies can be determined (Tuma and Cheng 1983). The calculation of natural frequencies are performed using computer programs on Matlab (2014).

4. Numerical analysis and discussions

In the numerical examples, three different frame structures that can be designed for civil projects are considered. The cross-sections of frame elements that used in numerical examples are rectangular. As one of the aims of study is to reflect the effect of frame element geometries on natural frequencies, the dimensions of members are increased step by step. For all frame models, the length of the beams L_b and height of the columns H_c are selected as 5.00 m and 3.50 m, respectively. It is assumed that beams and columns are designed using same material that has following properties: E=29430000 kN/m^2, μ=0.3, weight per unit volume γ=24.5 kN/m^3.

The first five natural frequencies of Frame-I for different inner and outer column cross-sections are presented in Table 1 and Table 2, respectively. It is observed that the effect of geometry of inner columns on natural frequencies is more remarkable in comparison with outer columns for two-bay one-story frame structures. Table 1 and Table 2 also revealed that SVSDT provides more realistic results according to EBT as it is known that EBT overestimates

Table 1 First five natural frequencies of Frame-I (B_{11}=25×50, C_{11}=C_{13}=30×50)

C_{12} (cm)	Theory	Natural Frequency (Hz)				
$Z\times X$		1st	2nd	3rd	4th	5th
	EBT	16.1367	48.7472	57.3666	102.2397	111.0760
30×50	TBT	16.0807	47.0969	56.1850	97.3475	104.7325
	SVSDT	15.8508	46.9930	55.0043	96.7709	104.1406
	EBT	16.8051	50.1277	57.5484	102.3432	112.2561
30×55	TBT	16.6767	49.1970	56.3348	97.4323	105.5077
	SVSDT	16.4870	48.2301	55.1638	96.8544	104.9951
	EBT	17.4589	51.3999	57.6997	102.4269	112.8797
30×60	TBT	17.3476	50.2881	56.4595	97.5010	105.9516
	SVSDT	17.1088	49.3653	55.2965	96.9222	105.4323
	EBT	18.1095	52.5383	57.8275	102.4960	113.2784
30×65	TBT	18.0501	51.2506	56.5649	97.5576	106.2629
	SVSDT	17.7266	50.3786	55.4087	96.9783	105.7106
	EBT	18.7680	53.5355	57.9369	102.5540	113.5525
30×70	TBT	18.5431	52.1570	56.6551	67.6051	106.4949
	SVSDT	18.3510	51.2653	55.5047	97.0255	105.9017

Table 2 First five natural frequencies of Frame-I (B_{11}=25×50, C_{12}=C_{13}=30×50)

C_{11} (cm)	Theory	Natural Frequency (Hz)				
$Z\times X$		1st	2nd	3rd	4th	5th
	EBT	16.1367	48.7472	57.3666	102.2397	111.0760
30×50	TBT	16.0807	47.0969	56.1850	97.3475	104.7325
	SVSDT	15.8508	46.9930	55.0043	96.7709	104.1406
	EBT	16.5757	49.3534	58.2186	103.6869	113.8331
30×55	TBT	16.4187	48.6595	56.9241	98.5381	106.7951
	SVSDT	16.2757	47.5416	55.7580	98.0480	106.5033
	EBT	17.0443	49.7761	59.0133	104.2935	115.8333
30×60	TBT	16.9339	48.0815	56.6600	99.0558	108.3850
	SVSDT	16.7286	47.9266	56.4589	98.5745	108.1259
	EBT	17.5498	50.0731	59.7164	104.5628	116.8253
30×65	TBT	17.4430	49.3956	58.3002	99.2911	109.2395
	SVSDT	17.2160	48.1990	57.0791	98.8055	108.8366
	EBT	18.0970	50.2858	60.3189	104.6854	117.3257
30×70	TBT	18.0616	48.6305	58.8886	99.3988	109.6840
	SVSDT	17.7418	48.3953	57.6115	98.9095	109.1721

natural frequencies. For SVSDT, TBT and EBT, an augmentation on natural frequencies is observed with increasing column dimension.

Table 3 that prepared for Frame-II shows that increasing beam height with constant beam width increases natural frequencies of one-bay two-storey frames. However, a decrease in natural frequencies is seen when beam width is increased with constant beam height. According to Table 4, an increment of column dimension increases natural frequencies of one-bay two-storey frames. SVSDT results are consistent as well for Frame-II.

Tables 5-7 are prepared to reflect the effects of inner and outer columns on natural frequencies for two-bay two-storey frames like Frame-III. For EBT, TBT and SVSDT, it is seen that the most effective column on natural frequencies

Table 3 First five natural frequencies of Frame-II (C_{21}=C_{22}=30×50)

$B_{21,22}$(cm)	Theory	Natural Frequency (Hz)				
$Z\times Y$		1st	2nd	3rd	4th	5th
	EBT	7.4992	27.5364	46.8720	55.4259	92.0980
25×50	TBT	7.4452	27.4617	46.6522	54.9296	89.4017
	SVSDT	7.3946	26.9501	45.4370	53.2991	87.5133
	EBT	7.8174	27.5723	48.2064	57.4002	95.1024
25×55	TBT	7.7622	27.2403	47.2085	56.4489	92.4773
	SVSDT	7.6928	26.9647	46.6439	55.0568	90.1069
	EBT	8.0894	27.5525	49.4277	58.8589	98.1559
25×60	TBT	8.0200	27.4444	48.4305	57.6156	95.5791
	SVSDT	7.4241	26.9269	47.7452	56.3527	92.8234
	EBT	7.4470	26.6038	45.3808	53.7150	92.4678
30×50	TBT	7.3991	26.3985	44.9199	52.3112	89.9377
	SVSDT	7.3394	26.0387	44.0466	51.7378	87.9118
	EBT	7.7338	26.6702	46.6629	55.4248	95.5102
30×55	TBT	7.6543	26.5939	45.8819	54.6522	93.2251
	SVSDT	7.6070	25.9871	45.2128	53.2657	90.8099

Table 4 First five natural frequencies of Frame-II (B_{21}=B_{22}=25×50, C_{22}=30×50)

C_{21} (cm)	Theory	Natural Frequency (Hz)				
$Z\times X$		1st	2nd	3rd	4th	5th
	EBT	7.4992	27.5364	46.8720	55.4259	92.0980
30×50	TBT	7.4452	27.4617	46.6522	54.9296	89.4017
	SVSDT	7.3946	26.9501	45.4370	53.2991	87.5133
	EBT	7.6980	28.9091	48.2478	56.4691	94.4248
30×55	TBT	7.6033	28.6084	47.1551	55.9416	91.2492
	SVSDT	7.5883	28.2415	46.6952	54.2381	89.4218
	EBT	7.8976	30.3631	49.4414	57.2813	95.4157
30×60	TBT	7.8237	29.9955	48.2573	55.7552	92.1242
	SVSDT	7.7826	29.5977	47.7836	54.9692	90.1117
	EBT	8.1013	31.9067	50.4517	57.9163	95.8007
30×65	TBT	8.0517	31.7842	49.2363	56.4059	92.5019
	SVSDT	7.9809	31.0238	48.7032	55.5412	90.3489
	EBT	8.3114	33.5419	51.2946	58.4168	95.9539
30×70	TBT	8.2503	33.2098	49.8181	56.9280	92.6656
	SVSDT	8.1854	32.5192	49.4700	55.9926	90.4276

Table 5 First five natural frequencies of Frame-III (B_{31}=B_{32}=25×50, C_{31}=C_{33}=30×50)

C_{32} (cm)	Theory	Natural Frequency (Hz)				
$Z\times X$		1st	2nd	3rd	4th	5th
	EBT	8.6782	24.2036	46.3221	53.7029	57.8976
30×50	TBT	8.4475	24.1555	46.1949	53.2827	57.0339
	SVSDT	8.2428	23.5211	45.2590	53.0086	57.0084
	EBT	8.8576	25.4108	47.6861	54.8616	58.1537
30×55	TBT	8.5200	25.2045	46.6548	54.5643	57.2455
	SVSDT	8.3838	24.6420	46.5071	54.0577	57.0094
	EBT	9.0241	26.6942	48.8979	55.7851	58.3836
30×60	TBT	8.6515	26.5874	47.7384	54.8987	57.4338
	SVSDT	8.5253	25.8283	47.6012	54.8715	57.1467

Table 5 Continued

	EBT	9.1856	28.0613	49.9442	56.5054	58.6007
30×65	TBT	8.8865	28.0141	48.7158	55.5226	57.6074
	SVSDT	8.6706	27.0822	48.5357	55.4969	57.4763
	EBT	9.3480	29.5149	50.8314	57.0544	58.8145
30×70	TBT	9.0555	29.2884	49.3666	56.1159	57.7721
	SVSDT	8.8221	28.4019	49.3210	55.9790	57.4883

Table 6 First five natural frequencies of Frame-III (B_{31}=B_{32}=25×50, C_{32}=C_{33}=30×50)

C_{31} (cm) Z×X	Theory	Natural Frequency (Hz)				
		1st	2nd	3rd	4th	5th
	EBT	8.6782	24.2036	46.3221	53.7029	57.8976
30×50	TBT	8.4475	24.1555	46.1949	53.2827	57.0339
	SVSDT	8.2428	23.5211	45.2590	53.0086	57.0084
	EBT	8.8013	25.3594	47.4771	54.2057	58.7593
30×55	TBT	8.5927	25.2572	46.8798	53.9613	58.5516
	SVSDT	8.3780	24.6293	46.4201	53.7800	58.3719
	EBT	8.9292	26.5815	48.4163	54.6137	59.4997
30×60	TBT	8.7176	26.4854	47.5948	54.5479	59.1495
	SVSDT	8.5180	25.7898	47.4036	54.3482	58.7192
	EBT	9.0655	27.8752	49.1529	54.9767	60.1116
30×65	TBT	8.8561	27.8474	48.7985	54.7986	59.6695
	SVSDT	8.6657	27.0064	48.2165	54.7752	59.0342
	EBT	9.2122	29.2407	49.7179	55.3104	60.6097
30×70	TBT	9.0605	29.0463	47.6897	55.2364	59.6492
	SVSDT	8.8219	28.2777	48.8786	55.1055	59.3096

Table 7 First five natural frequencies of Frame-III (B_{31}=B_{32}=25×50, C_{31}=C_{32}=30×50)

C_{33} (cm) Z×X	Theory	Natural Frequency (Hz)				
		1st	2nd	3rd	4th	5th
	EBT	8.6782	24.2036	46.3221	53.7029	57.8976
30×50	TBT	8.4475	24.1555	46.1949	53.2827	57.0339
	SVSDT	8.2428	23.5211	45.2590	53.0086	57.0084
	EBT	8.8633	24.3264	46.4610	54.2991	58.6237
30×55	TBT	8.7054	24.1756	45.6301	53.3587	57.6292
	SVSDT	8.4517	23.6470	45.2695	53.0329	57.2064
	EBT	9.0531	24.4738	46.5522	54.6604	59.3815
30×60	TBT	8.9032	24.2059	45.7222	53.7684	59.2616
	SVSDT	8.6863	23.8077	45.2779	53.0504	58.0425
	EBT	9.2494	24.6486	46.6147	54.8805	60.0850
30×65	TBT	9.1029	24.2640	45.7871	53.5537	59.3782
	SVSDT	8.9429	24.0058	45.2852	53.0640	58.4463
	EBT	9.4526	24.8538	46.6593	55.0208	60.6978
30×70	TBT	9.3058	24.5471	45.8340	52.2527	58.4448
	SVSDT	9.2169	24.2440	45.2920	53.0751	57.4300

of Frame-III is C_{33} which does not create a joint with an upper floor column. The inner column C_{32} and the outer column C_{31} effects natural frequencies of frame similarly. However, the natural frequencies are affected a little more from C_{32} when compared to C_{31}.

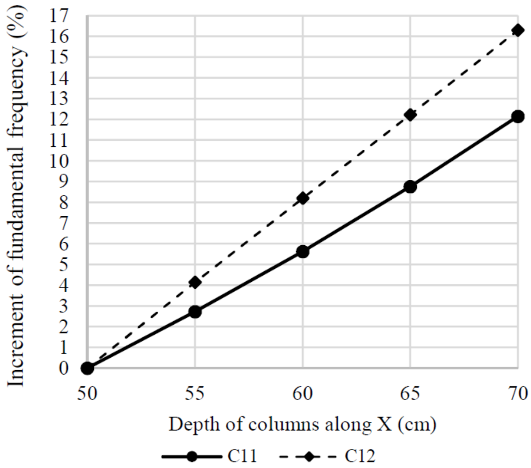

Fig. 4 Increment of fundamental frequency of Frame-I for different column dimensions using EBT

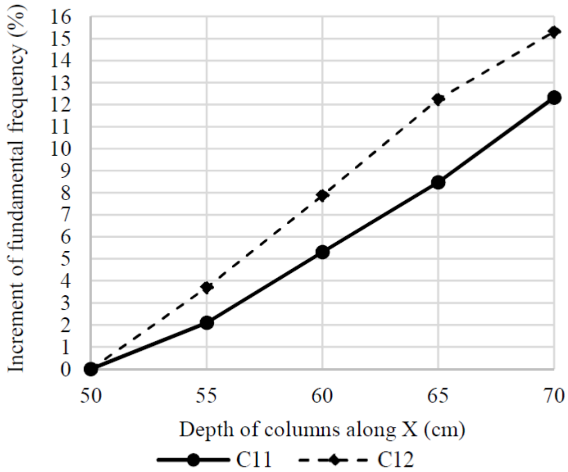

Fig. 5 Increment of fundamental frequency of Frame-I for different column dimensions using TBT

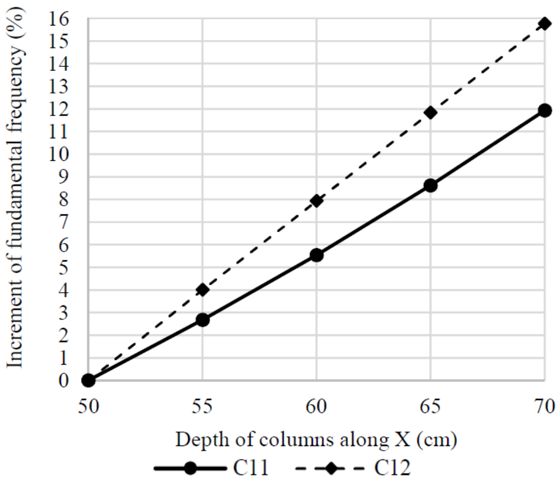

Fig. 6 Increment of fundamental frequency of Frame-I for different column dimensions using SVSDT

The augmentation of fundamental frequency of Frame-I by increasing column size are presented in Figs. 4-6. The effects of column size on fundamental frequency of Frame-

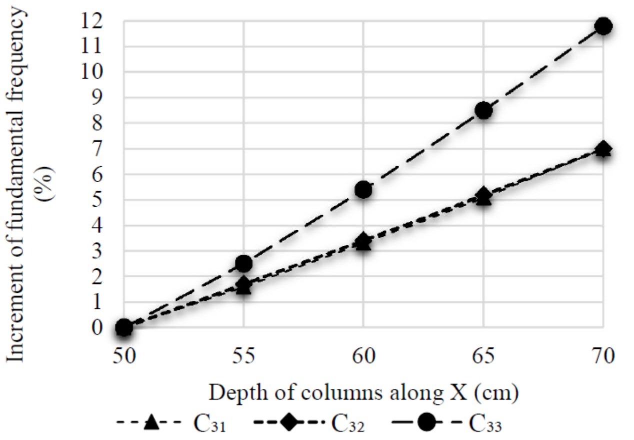

Fig. 7 Increment of fundamental frequency of Frame-III for different column dimensions using SVSDT

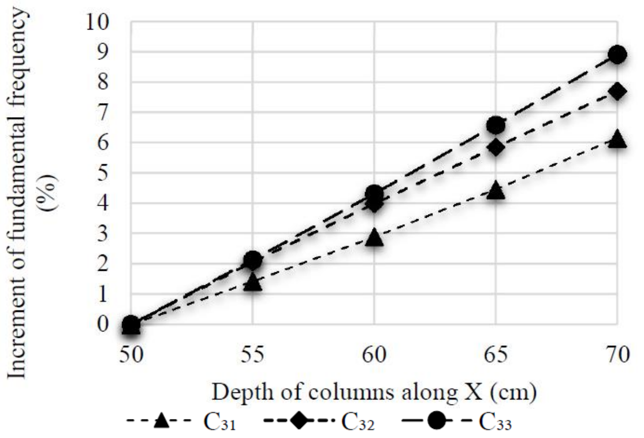

Fig. 8 Increment of fundamental frequency of Frame-III for different column dimensions using EBT

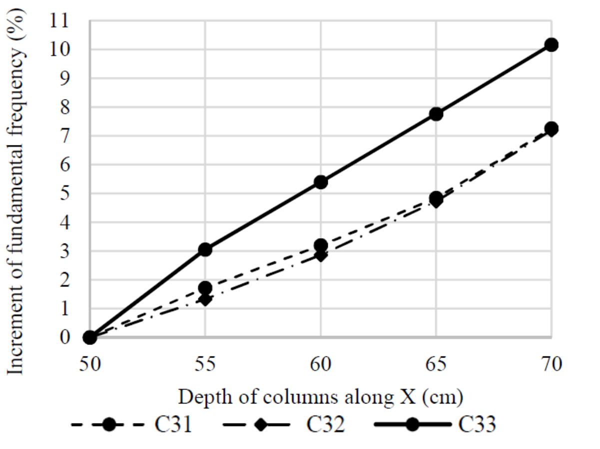

Fig. 9 Increment of fundamental frequency of Frame-III for different column dimensions using TBT

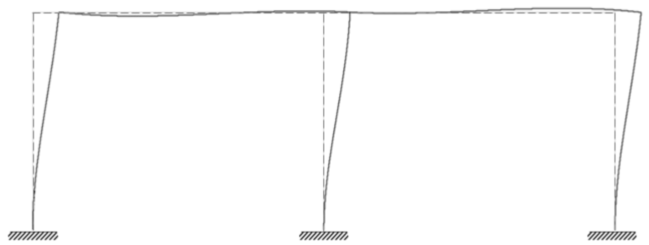

Fig. 10 First mode shape of Frame-I (B_{11}=25×50, C_{11}=C_{12}=C_{13}=30×50)

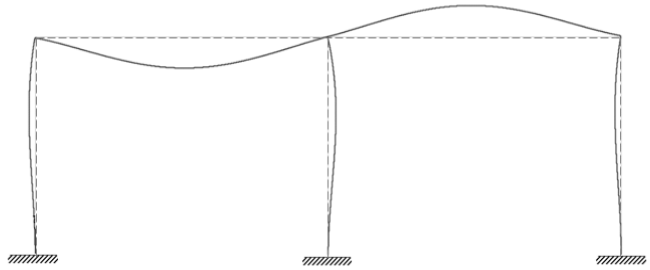

Fig. 11 Second mode shape of Frame-I (B_{11}=25×50, C_{11}=C_{12}=C_{13}=30×50)

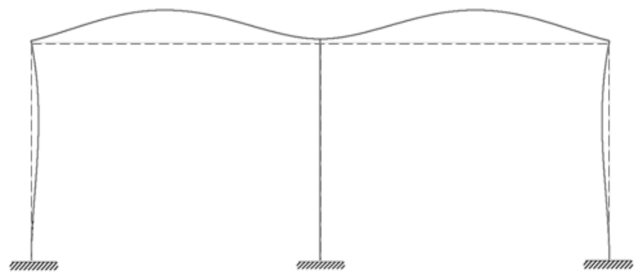

Fig. 12 Third mode shape of Frame-I (B_{11}=25×50, C_{11}=C_{12}=C_{13}=30×50)

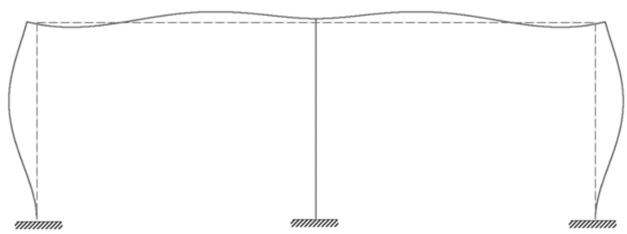

Fig. 13 Fourth mode shape of Frame-I (B_{11}=25×50, C_{11}=C_{12}=C_{13}=30×50)

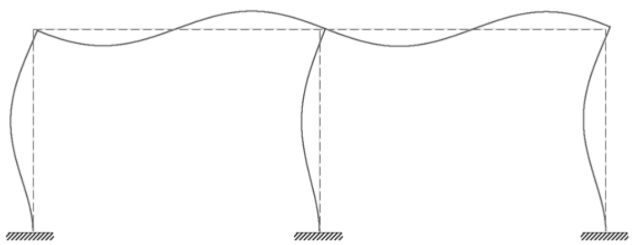

Fig. 14 Fifth mode shape of Frame-I (B_{11}=25×50, C_{11}=C_{12}=C_{13}=30×50

Frame-III can be observed from Figs. 7-9. It is clearly seen from Figs. 7-9 that column C_{33} effects significantly the vibrations of Frame-III that modeled using SVSDT and TBT in reference to EBT. It is observed that the importance of SVSDT becomes evident for large columns.

The mode shapes of frames can be plotted by equating one of the nonzero nodal displacements to an arbitrary value after calculation of natural frequencies. For instance, the first five mode shapes of Frame-I according to SVSDT are presented in Figs. 10-14.

6. Conclusions

This study introduced an application of a single variable shear deformation theory for free vibration analysis of frame structures. SVSDT which considers the varying shear stress distribution along the cross section without shear correction factor, provides an effective solution procedure as there is not more than one variable contrary to TBT. Dynamic stiffness formulation is working consistently with SVSDT. The effects of dimensions of columns and beams are indicated. Moreover, the importance of inner and outer columns on natural frequencies of frames is revealed. The computer programs that prepared for calculation of natural frequencies are working fast. The results show that SVSDT can be used for calculation of exact natural frequencies of various type of frames with any support conditions.

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