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Free vibration analysis of Reissner plates by mixed finite element

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Abstract. In this study, free vibration analysis of Reissner plates on Pasternak foundation is carried out by mixed finite element method based on the Gâteaux differential. New boundary conditions are established for plates on Pasternak foundation. This method is developed and applied to numerous problems by Aköz and his co-workers. In dynamic analysis, the problem reduces to the solution of a standard eigenvalue problem and the mixed element is based upon a consistent mass matrix formulation. The element has four nodes and bending and torsional moments, transverse shear forces, rotations and displacements are the basic unknowns. The element performance is assessed by comparison with numerical examples known from literature. Validity limits of Kirchhoff plate theory is tested by dynamic analysis. Shear locking effects are tested as far as $h/2a = 10^{-6}$ and it is observed that REC32 is free from shear locking.

Key words: Reissner plate, free vibration, Pasternak, mixed-finite element.

1. Introduction

The Reissner-Mindlin theory provides more reliable representation of structural behavior (Reissner 1946, Mindlin 1951). Those theories that include the effects of transverse shear deformations eliminate the inaccuracies of the classical Kirchhoff-Love theory. A considerable number of publications are concerned with the problem of the free vibration of plates. Theoretical studies are based on various methods, such as, the finite difference method, the Rayleigh-Ritz method, the finite element method, the Lagrangian multiplier method, the finite strip method, the superposition method and others.

Because of its versatility, finite element method founds a great application in engineering field. The great majority of results for flexural vibration of plates are based on Kirchhoff plates theory that ignore the transverse shear deformation. Its requirement of C^1 continuity causes substantial difficulties and also it introduces errors since the effects of transverse shear are ignored and inertia terms are also neglected. This theory overestimates the plate frequencies. For simply supported rectangular plates comparison studies show that classical theory solutions are significantly in error for all modes of the plates. The errors increase with the increasing thickness (Srinivas *et al.* 1970, Lee and Reissmann 1969, Hughes *et al.* 1977). An improved thick plate theory, which includes the effects of transverse shear and rotary inertia, were presented by numerous authors, for example,

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Reissner (1946) and Mindlin (1951). Hughes *et al*. (1977) based on Mindlin theory, which require C^0 continuity has developed a very efficient form of the bilinear four-node element (S1). Hinton and Bicanic (1979) have performed free vibration analysis with this element. C^0 continuity causes a problem called shear locking when the plate thickness approaches zero. Various modifications of formulation as well as numerical tricks have been introduced in order to overcome this problem such as reduced/selective integration (Cook 1972, Zienkiewicz *et al.* 1971, Pugh *et al.* 1978, Paswey and Clough 1971) and Discrete Kirchhoff-Mindlin element (Katılı 1993a, b, Batoz and Lardeur 1989). Belytschko suggests a method to remove zero-energy (kinematics) mode by perturbing the stiffness by stabilization matrix (Belytschko *et al.* 1981). The theory's requirement for free parameters may be a disadvantage. Recently Eratlı and Aköz 1997 have obtained an element for thick plates using Gâteaux approach, which eliminates shear locking.

In this study, assuming Reissner plate theory and Pasternak foundation, a new element is developed. Having the field equations, one needs a method to reach a functional. Hu-Washizu and Hellinger-Reissner principles or weak formulation are very popular approaches, which provide functionals that are essential for finite element formulation Reddy (1993). Aköz and his co-workers (Aköz 1985, Aköz *et al*. 1991 used Gâteaux differential approach first time 1991, Omurtag and Aköz 1992, Aköz and Uzcan (Eratlı) 1992, Omurtag and Aköz 1993, Omurtag and Aköz 1994, Eratlı (Uzcan) 1995, Omurtag *et al.* 1997, Özçelikörs and Aköz 1993, Aköz and Kadioglu 1996, Aköz and Eratlı 2000, Aköz and Özütok 2000, Eratlı 2000) to obtain a functional. Although, Hellinger-Reissner and Gâteaux method can produce the same functional, it is believed that Gâteaux approach has some advantages over Hellinger-Reissner or Hu-Washizu approaches, which has the following nice properties:

1. All field equations are enforced to the functional by systematic way.
2. Boundary conditions can be constructed.
3. Potential test provides accuracy checking of field equations.

2. The field equations for Reissner plates on elastic foundations

2.1 Reissner plate

Reissner plate theory includes the transverse shear effects and field equations are given in Eq. (1). For more information one can refer literature (Panc 1975). The positive directions of internal forces are illustrated in Fig. 1 and the geometric parameters such as w, Ω_x, $\partial w/\partial x$ are shown in Fig. 2. The governing equations of Reissner plate are

$$\begin{aligned}
&\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x = 0\\
&\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y = 0\\
&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q = 0 \qquad \text{(1a)}\\
&\frac{\partial \Omega_x}{\partial x}-\frac{12}{Eh^3}\left(M_x-\mu M_y-\frac{h^2}{10}\mu q\right) = 0
\end{aligned}$$

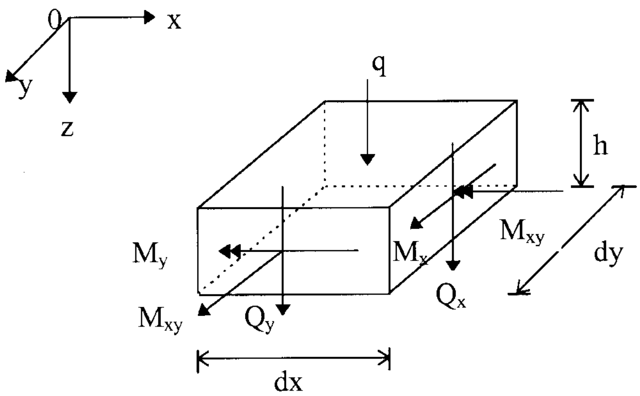

Fig. 1 Internal forces

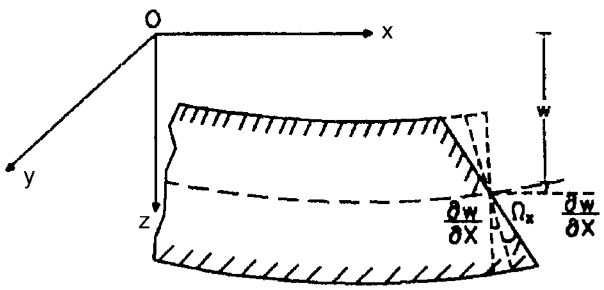

Fig. 2 Definition of deformation state

$$\frac{\partial \Omega_y}{\partial y} - \frac{12}{Eh^3}\left(M_y - \mu M_x - \frac{h^2}{10}\mu q\right) = 0 \tag{1b}$$

$$\frac{\partial \Omega_x}{\partial y} + \frac{\partial \Omega_y}{\partial x} - \frac{12}{Gh^3}M_{xy} = 0$$

$$\Omega_x + \frac{\partial w}{\partial x} - \frac{6}{5Gh}Q_x = 0$$

$$\Omega_y + \frac{\partial w}{\partial y} - \frac{6}{5Gh}Q_y = 0 \tag{1c}$$

2.2 Pasternak foundation

In the Pasternak model, a shear interaction between the spring elements exists. The reaction-deflection relation of Pasternak model is given as;

$$p = kw - G_f(w_{,xx} + w_{,yy}) \tag{2}$$

where k is spring coefficient and G_f is shear coefficient of foundation (see Fig. 3 and Pasternak

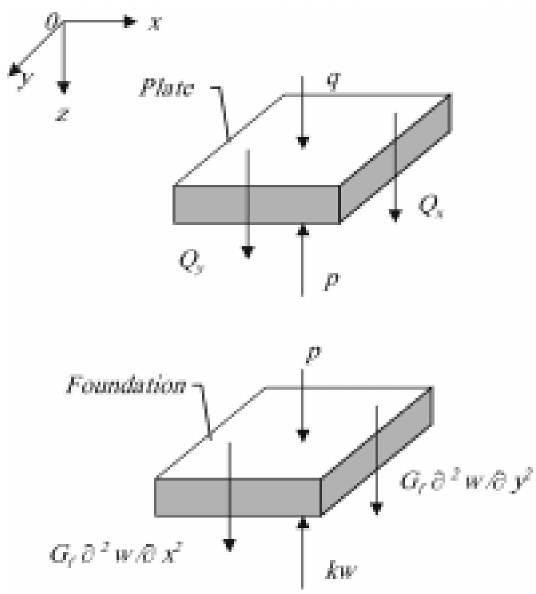

Fig. 3 Interaction of plate-elastic foundation

1954 as well). If G_f is neglected, Pasternak model reduces to Winkler model. If we substitute expression (3)

$$q - kw + G_f(w_{,xx} + w_{,yy}) \tag{3}$$

for lateral load q in Eq. (1), we obtain

$$\begin{aligned} &\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \\ &\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \\ &\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q - kw + G_f(w_{,xx} + w_{,yy}) = 0 \end{aligned} \tag{4a}$$

$$\begin{aligned} &\frac{\partial \Omega_x}{\partial x} - \frac{12}{Eh^3}(M_x - \mu M_y) = 0 \\ &\frac{\partial \Omega_y}{\partial y} - \frac{12}{Eh^3}(M_y - \mu M_x) = 0 \\ &\frac{\partial \Omega_x}{\partial y} + \frac{\partial \Omega_y}{\partial x} - \frac{12}{Gh^3} M_{xy} = 0 \end{aligned} \tag{4b}$$

$$\begin{aligned} &\Omega_x + \frac{\partial w}{\partial x} - \frac{6}{5Gh} Q_x = 0 \\ &\Omega_y + \frac{\partial w}{\partial y} - \frac{6}{5Gh} Q_y = 0\,. \end{aligned} \tag{4c}$$

as the governing equations of a Reissner plate on a Pasternak foundation.

In obtaining Eq. (4b), the effect of σ_z on the bending moments were ignored because of the terms relating this effect is small comparing with remaining terms. Otherwise these field equations would not pass the Gâteaux potential test as we will see later. The necessity of the neglection of these terms can be detected only by Gâteaux differential approach and it is an evidence of the power of the method as stated in the introduction. If we had used Hellinger-Reissner, Hu-Washizu or weak formulation theories we could have not recognized incompatibility of this term.

3. The functional

Having field equations one needs a method to obtain the functional. Although Hu-Washizu or Hellinger-Reissner principles are popular methods to establish a functional, we believe that Gâteaux differential method is more suitable for this aim. Since this method was extensively used and explained in other studies, for the sake of simplicity, the basic steps and definitions will be summarizes briefly.

First, the field Eq. (4) is put in operator form $\boldsymbol{Q}$, which is given in Appendix II. Gâteaux derivative of an operator is defined as

$$d\boldsymbol{Q}(\boldsymbol{u}, \bar{\boldsymbol{u}}) = \left.\frac{\partial Q(\boldsymbol{u} + \tau\bar{\boldsymbol{u}})}{\partial \tau}\right|_{\tau=0}, \tag{5}$$

where τ is a scalar. To obtain the boundary conditions, all boundary conditions are written in symbolic form as follows:

Dynamic boundary conditions;

$$\begin{aligned} \boldsymbol{M}-\hat{\boldsymbol{M}} &= 0, \\ \boldsymbol{Q}-\hat{\boldsymbol{Q}} &= 0 \end{aligned} \tag{6}$$

and geometric boundary conditions;

$$\begin{aligned} -\boldsymbol{w}-\hat{\boldsymbol{w}} &= 0, \\ -\boldsymbol{\Omega}-\hat{\boldsymbol{\Omega}} &= 0 \end{aligned} \tag{7}$$

where $\boldsymbol{M}$, $\boldsymbol{Q}$, $\boldsymbol{\Omega}$, $\boldsymbol{w}$ are the moment, force, rotation and deflection vectors, respectively. Quantities

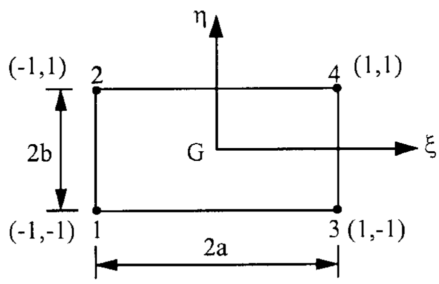

Fig. 4 Rectangular element (REC32)

with hat are known values on the boundaries. Explicit form of the boundary conditions will be obtained after the following variational manipulations. A necessary and sufficient condition that $\boldsymbol{Q}$ be potential is (Oden and Reddy 1976)

$$\langle d\boldsymbol{Q}(\boldsymbol{y},\bar{\boldsymbol{y}}),\boldsymbol{y}^*\rangle = \langle d\boldsymbol{Q}(\boldsymbol{y},\boldsymbol{y}^*),\bar{\boldsymbol{y}}\rangle, \tag{8}$$

where parantheses indicate the inner products. If the operator $\boldsymbol{Q}$ is potential, then the functional corresponding the field equations is given by (Oden and Reddy 1976)

$$\boldsymbol{I}(\boldsymbol{y}) = \int_0^1 [\boldsymbol{Q}(s\boldsymbol{y}),\boldsymbol{y}]ds, \tag{9}$$

where s is a scalar quantity. Explicit form of the functional corresponding to the field equations (Eq. 4) is

$$\begin{aligned} I(\boldsymbol{y}) = {} & [Q_x,(\Omega_x + w_{,x})] + [Q_y,(\Omega_y + w_{,y})] + [M_x,\Omega_{x,x}] + [M_y,\Omega_{y,y}] + [M_{xy},\Omega_{x,y}] \\ & + [M_{xy},\Omega_{y,x}] - \frac{6}{Eh^3}\{[M_x,M_x]+[M_y,M_y]-2\mu[M_x,M_y]+2(1+\mu)[M_{xy},M_{xy}]\} \\ & - \frac{3}{5Gh}\{[Q_x,Q_x] + [Q_y,Q_y]\} + [q,w] + \frac{1}{2}k[w,w] + \frac{1}{2}G_f[(w_{,x} + w_{,y}),w] \\ & -[(w-\hat{w}),Q]_\varepsilon - [(\Omega-\hat{\Omega}),M]_\varepsilon - [w,\hat{Q}]_\sigma - [\hat{M},\Omega]_\sigma. \end{aligned} \tag{10}$$

The parenthesis with σ and ε subscripts indicate the dynamic and the geometric boundary conditions respectively and explicit expressions of boundary conditions are:

$$[Q,w] = [((Q_x - G_f w_{,x})n_x + (Q_y - G_f w_{,y})n_y),w], \tag{11a}$$

$$[M,\Omega] = [(M_x n_x + M_{xy} n_y),\Omega_x] + [(M_{xy} n_x + M_y n_y),\Omega_y]. \tag{11b}$$

A very interesting point is that the properties of foundation effect the boundary condition (Eq. 11a). As far as the author's knowledge, this boundary condition is not present in the literature.

For the dynamic analysis the functional given by Eq. (10) is valid only by letting $[q, w] = 1/2\, \rho\omega^2[w,w]$ as far as the harmonic solutions are required.

3. Mixed finite element matrix

If a four-nodded rectangular element (Fig. 4) with a parent shape function

$$\psi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i), \quad \xi_i = \pm 1, \quad \eta_i = \pm 1, \quad i = 1, \cdots, 4 \tag{12}$$

is used, the element matrix can be obtained explicitly as;

$$[k]_r = \begin{array}{c} \begin{array}{cccccccc} M_x & M_y & M_{xy} & Q_x & Q_y & w & \Omega_x & \Omega_y \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} -\gamma_1[K_1]_r & \gamma_2[K_1]_r & 0 & 0 & 0 & 0 & [K_2]_r^T & 0 \\ & -\gamma_1[K_1]_r & 0 & 0 & 0 & 0 & 0 & [K_3]_r^T \\ & & -\gamma_3[K_1]_r & 0 & 0 & 0 & [K_3]_r^T & [K_2]_r^T \\ & & & -\gamma_4[K_1]_r & 0 & [K_2]_r^T & [K_1]_r & 0 \\ & & & & -\gamma_5[K_1]_r & [K_3]_r^T & 0 & [K_1]_r \\ & & & & & [K] & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix} \end{array} \tag{13}$$

where,

$$\gamma_1 = 12/Eh^3,\ \ \gamma_2 = 2\mu/Eh^3,\ \ \gamma_3 = 24(1+\mu)/Eh^3,\ \ \gamma_4 = 12(1+\mu)/5Eh,\ \gamma_5 = 6\mu/5Eh,$$

and

$$[K] = k[K_1]_r + G_f[K_4]_r + G_f[K_5]_r.$$

The explicit form of the submatrices $[K_1]_r$, $[K_2]_r$, $[K_3]_r$, $[K_4]_r$, $[K_5]_r$ and some necessary mathematical manipulations are given in the Appendix II.

From the boundary conditions given in the functional, the boundary conditions matrix is obtained as;

$$[k]_{BCr} = \begin{array}{c} \begin{array}{cccccccc} M_x & M_y & M_{xy} & Q_x & Q_y & w & \Omega_x & \Omega_y \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & [s_1]_r & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & [s_2]_r \\ & & 0 & 0 & 0 & 0 & [s_2]_r & [s_1]_r \\ & & & 0 & 0 & [s_1]_r & 0 & 0 \\ & & & & 0 & [s_2]_r & 0 & 0 \\ & \textit{Symmetrical} & & & & [S]_r & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix} \end{array} \tag{14}$$

where,

$$[S] = -G_f[s_3]_r - G_f[s_4]_r.$$

Details and explicit forms of submatrices $[s_i]_r$, $i = 1,\cdots, 4$ are given in Appendix II. Finally the mixed finite element for Reissner plate resting on Pasternak foundation becomes,

$$[k] = [k]_r + [k]_{BCr}. \tag{15}$$

4. Dynamic analysis

The problem of determining the natural vibration frequencies of a structural system reduces to the solution of a standard eigenvalue problem,

$$[\boldsymbol{K}]-\omega^2[\boldsymbol{M}] = 0, \tag{16}$$

where $[\boldsymbol{K}]$ is the system matrix, $[\boldsymbol{M}]$ is the mass matrix for the entire domain and ω is the natural angular frequency of the system. The explicit form of Eq. (16) is

$$\left(\begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{12}]^T & [K_{22}] \end{bmatrix} - \omega^2 \begin{bmatrix} 0 & 0 \\ 0 & [M] \end{bmatrix}\right) \begin{Bmatrix} \{F\} \\ \{w\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix}, \tag{17}$$

where $\{\boldsymbol{F}\} = \{\boldsymbol{M} \;\; \boldsymbol{Q} \;\; \boldsymbol{\Omega}\}^T$, $\{\boldsymbol{w}\}$ are moments, shear forces, rotations and displacement vectors, respectively. Elimination of $\{\boldsymbol{F}\}$ from Eq. (17) gives

$$([\boldsymbol{K}^*]-\omega^2[\boldsymbol{M}])\{\boldsymbol{w}\} = \{0\}, \tag{18}$$

where

$$[\boldsymbol{K}^*] = [\boldsymbol{K}_{22}]-[\boldsymbol{K}_{12}]^T[\boldsymbol{K}_{11}]^{-1}[\boldsymbol{K}_{12}], \tag{19}$$

and $[\boldsymbol{K}^*]$ is the condensed system matrix of the problem. If there is no foundation, then $[\boldsymbol{K}_{22}] = 0$. The element mass matrix is based upon consistent mass formulation as,

$$[m] = \rho \; h \; [K_1]_r. \tag{20}$$

where h is the plate thickness, ρ is the mass density and $[K_1]_r$ is given in Appendix II.

5. Numerical examples

Free vibration of plates has been investigated intensively in literature and an excellent review is given for free vibration of plates in (Leissa 1969). In order to check the performance of the new method, various problems are solved and results are compared with some existing studies in the literature (Omurtag *et al.* 1997, Leissa 1969 & 1973, Bardell 1991, Yuan and Miller 1988, Yuan and Miller 1992). Example problems will display properties of the new element for plate vibration.

The following dimensions and numerical properties of plate are considered for all problems:

L=2a=2b=10m, h=0.15m, E=25GPa, n=0.15, ρ =24 kN/m^3.

5.1 Convergence test

Convergence refers to the accuracy of solution as the number of elements in the mesh is increased. The convergence test is performed for simply supported plate of which dimensions and material properties are cited above. The calculation is accomplished for the different mesh Ω^h, beginning with 2×2 elements in the complete plate and the element numbers increased as the sequence 3×3, 4×4, ···, n×n. Fig. 5 shows plots of first three frequency parameters versus number of

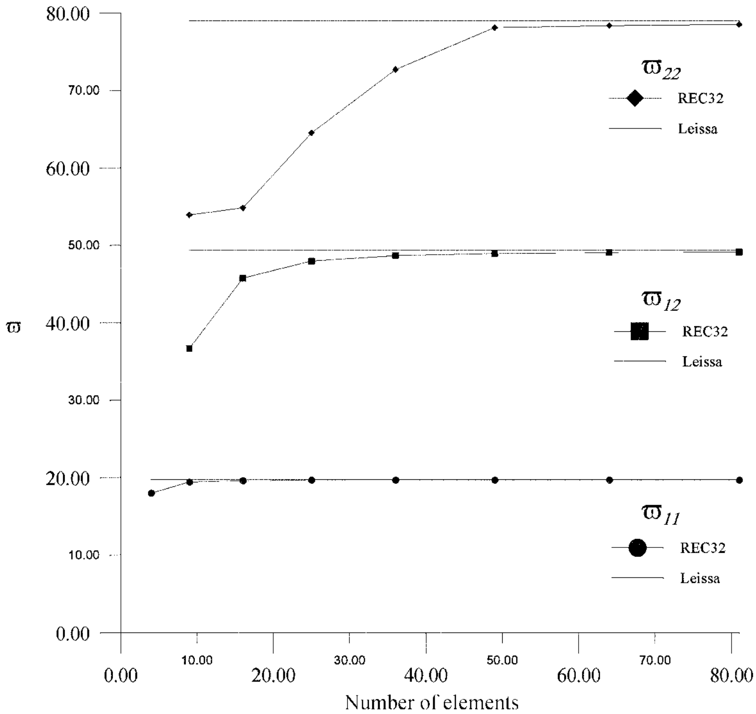

Fig. 5 Frequency parameters of S-S-S-S plate for different meshes

elements.

The error in the energy norm satisfy the inequality (Reddy 1993)

$$e \le c\ h^p \tag{21}$$

where c is a constant, h is the characteristic length of element. p is called the rate of convergence. p depends on the derivative of w in the functional and the degree of polynominals k used to approximate w. Therefore, the error can be reduced by reducing the size of elements. This reduction is called h-convergence. In order to provide the means to quantitatively estimate the error in approximate solution, the numerical results is plotted in the axes log(h) versus log(e) in Fig. 6. Inspection of Fig. 6 shows that data is well interpolated since we have straight lines on log-log plot. This numerical experiments verify the error Eq. (21) and also the rate of convergence p for the first three frequency parameters ($\varpi = \omega L^2\sqrt{\rho/D}$) as $p_1 = 4$, $p_2 = 4.27$, $p_3 = 4.28$. To establish Fig. 6, we utilized the knowledge of the exact solution value of ϖ. An error estimation procedure can be obtained without knowing the exact solution on mesh refinements. N_1, N_2, e_1, e_2 are the numerical solutions and errors, respectively corresponding h_1, h_2 mesh size. They satisfy $e_1+N_1=e_2+N_2$ equality. If $h_1 = h_2/2$, then quantitative error estimate is (Baker and Pepper 1991)

$$e_1 = \frac{N_1 - N_2}{2^{2k} - 1} \tag{22}$$

This error estimate is valid on Ω provided the numerical solutions N_1 and N_2 lie on the convergence line where k is the order of polynomial.

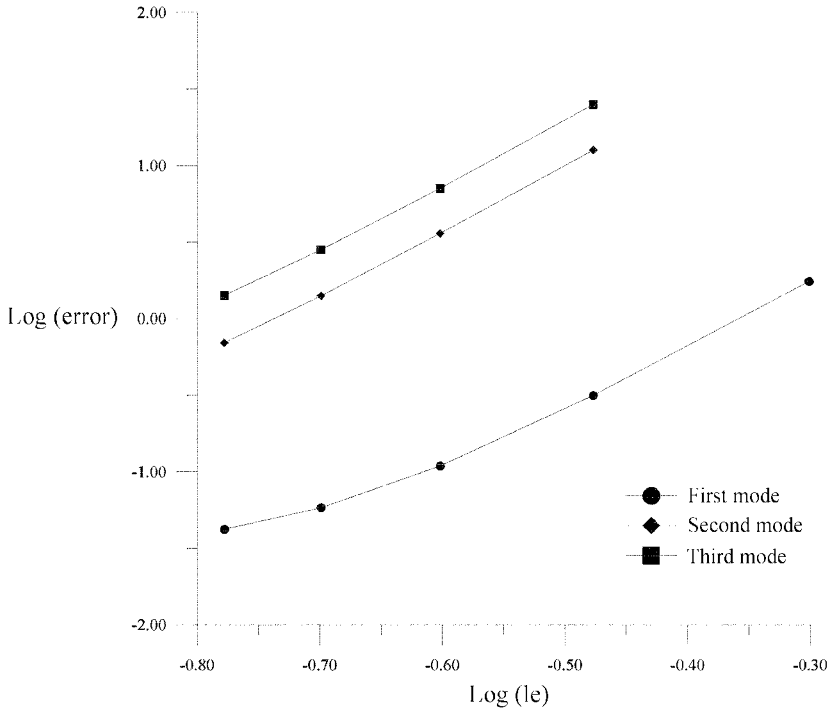

Fig. 6 Error analysis for frequency parameters of S-S-S-S plate

5.2 The validity limits of Kirchhoff plate theory and shear locking

The plate thickness effects the behavior of the plate. If the plate thickness increases, the transverse shear gains importance on the plate behavior. For example, it is well known that if transverse shear is neglected, results overestimate the plate frequency in the literature (Yuan and Miller 1988, Yuan and Miller 1992). It is accepted that whenever a plate is relatively thick. i.e., the ratio of $h/2a$ is greater than 0.05 and shear deformation theories must be considered such as Reissner, Mindlin theories. Otherwise shear deformations are negligible. The comparison of first frequency variations for plates with different supporting conditions S-S-S-S, C-C-C-C, S-C-S-C and for different plate thicknesses in Fig. 7 helps to create a criterion for the validity limits of Kirchhoff plate theory. Dependency of vibration frequencies on plate thicknesses is presented in Table 1 for different supporting conditions.

One of the important problems encountered in finite element method is shear locking phenomena for which as plate thickness decreases plate behaves more rigidly than actually it does. Bhashyam and Gallagher (1984) reported that they can not obtain reliable result below to $h/2a$=0.005. Using REC32, frequency parameter has been obtained as ϖ_{11}=19.734 for simply supported rectangular plate having thickness-to-length ratio $h/2a=10^{-3} \div 10^{-6}$. Although this thickness is meaningless in the physical viewpoint, in order to demonstrate the performance of REC32, we determined the frequency parameters for such a thin plate.

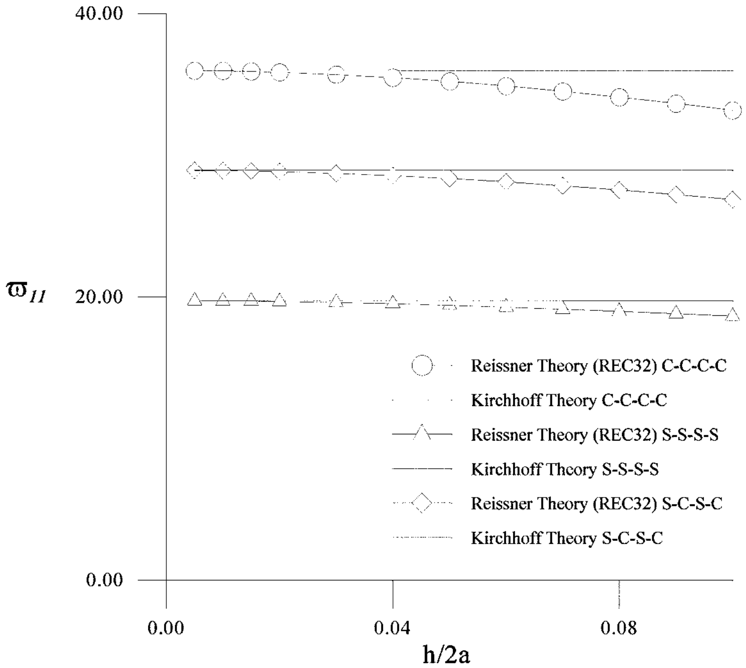

Fig. 7 Validity limits of Kirchhoff plate theory for different edge conditions

Table 1 Dependency of vibration frequencies on plate thickness for different edge conditions

$h/2a$	ϖ_{11} S-S-S-S	ϖ_{11} C-C-C-C	ϖ_{11} S-C-S-C
0.0001	19.734	36.208	30.257
0.0005	19.734	36.046	28.883
0.001	19.734	36.031	28.932
0.005	19.730	35.996	28.941
0.010	19.720	35.972	28.924
0.015	19.703	35.932	28.893
0.020	19.679	35.875	28.851
0.030	19.612	35.715	28.733
0.040	19.522	35.496	28.571
0.050	19.411	35.221	28.368
0.060	19.282	34.895	28.128
0.070	19.139	34.524	27.854
0.080	18.983	34.111	27.551
0.090	18.819	33.664	27.224
0.100	18.647	33.188	26.875

5.3 All edges supported

The three cases having S-S-S-S, C-C-C-C and S-C-S-C edges are considered in the following dimensions and material properties of the plate for each case; h=0.15 m, E=25 GPa, ν=0.15,

L=2a=2b=10 m, ρ=24 kN/m^3. The first four frequency parameters ($\varpi = \omega L^2\sqrt{\rho/D}$) for each of these plates are presented in Tables 2-4 respectively. Inspection of tables show that frequency parameters obtained by REC32 is a little smaller than given compared studies (Omurtag *et al.* 1997, Leissa 1969). This difference in the results is expected, because frequency parameters are obtained employing two different plate theories. The contour lines and mode shapes of ω_{11}, ω_{12}, ω_{21}, ω_{22} are shown in Figs. 8-10.

5.4 Completely free plate

Validity of formulation in this study can be checked for completely free plates (F-F-F-F). The non-zero three frequency parameters ($\varpi = \omega\ L^2\sqrt{\rho/D}$) for F-F-F-F are presented in Table 5. The contour lines and mode shapes of non-zero frequency parameters are shown in Fig. 11. Results are obtained using 9×9 elements.

5.5 Natural frequency of plate on elastic foundation

In order to demonstrate the efficiency of REC32 elements, numerical results are presented for

Table 2 Frequency parameters ϖ for a S-S-S-S plate (b/a = 1)

Frequency Parameters	Leissa 1969	Omurtag *et al.* 1997	REC32
ϖ_{11}	19.738	19.911	19.703
ϖ_{12}	49.349	50.112	49.069
ϖ_{21}	49.349	50.112	49.069
ϖ_{22}	78.958	80.090	78.354

Table 3 Frequency parameters ϖ for a C-C-C-C plate (b/a = 1)

Frequency Parameters	Leissa 1969	Omurtag *et al.* 1997	REC32
ϖ_{11}	35.999	36.018	35.931
ϖ_{12}	73.405	74.497	73.823
ϖ_{21}	73.405	74.497	73.823
ϖ_{22}	108.237	108.949	110.14

Table 4 Frequency parameters ϖ for a S-C-S-C plate (b/a = 1)

Frequency Parameters	Leissa 1969	Bardell 1991	REC32
ϖ_{11}	28.946	28.950	28.893
ϖ_{12}	54.743	54.740	54.531
ϖ_{21}	69.320	69.330	69.658
ϖ_{22}	94.584	94.590	95.222

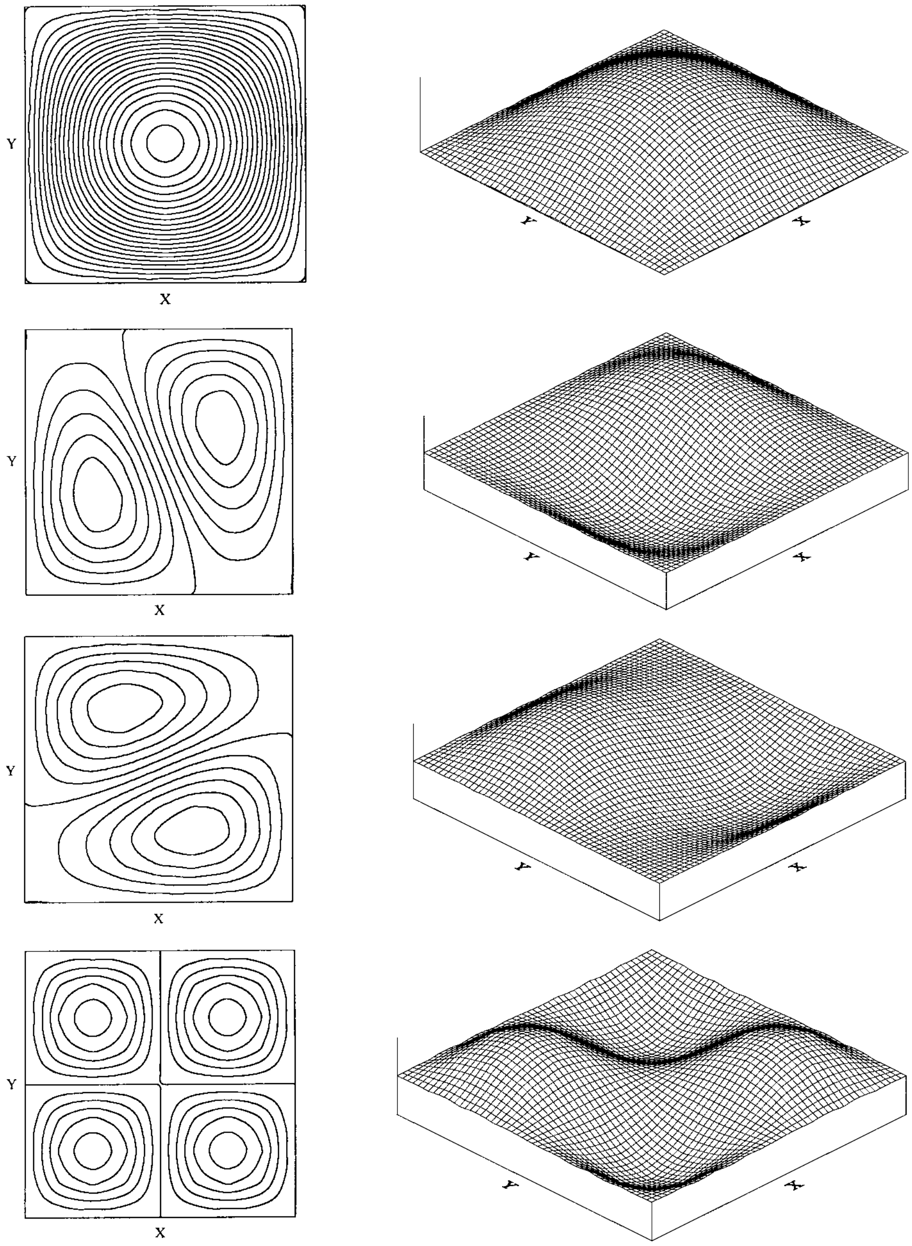

Fig. 8 The contour line and mode shape of ω_{11}, ω_{12}, ω_{21}, ω_{22} for S-S-S-S

series of problems involving free vibration of rectangular plates on Pasternak foundation for different foundation coefficients (k, G_f) and various support conditions.

First, calculations were performed for Winkler foundation that is $k = 100000$ kN/m^3, $G_f = 0$ and a uniform plate, dimensions L=2a=2b=10 m, h=0.15 m. We assume E=25 GPa, ν=0.15 and ρ=24 kN/m^3. We determined the natural frequency parameters ϖ_{11}, ϖ_{12}, ϖ_{22} associated with transverse deflection w for simply supported (S-S-S-S) and clamped supported (C-C-C-C). All frequency

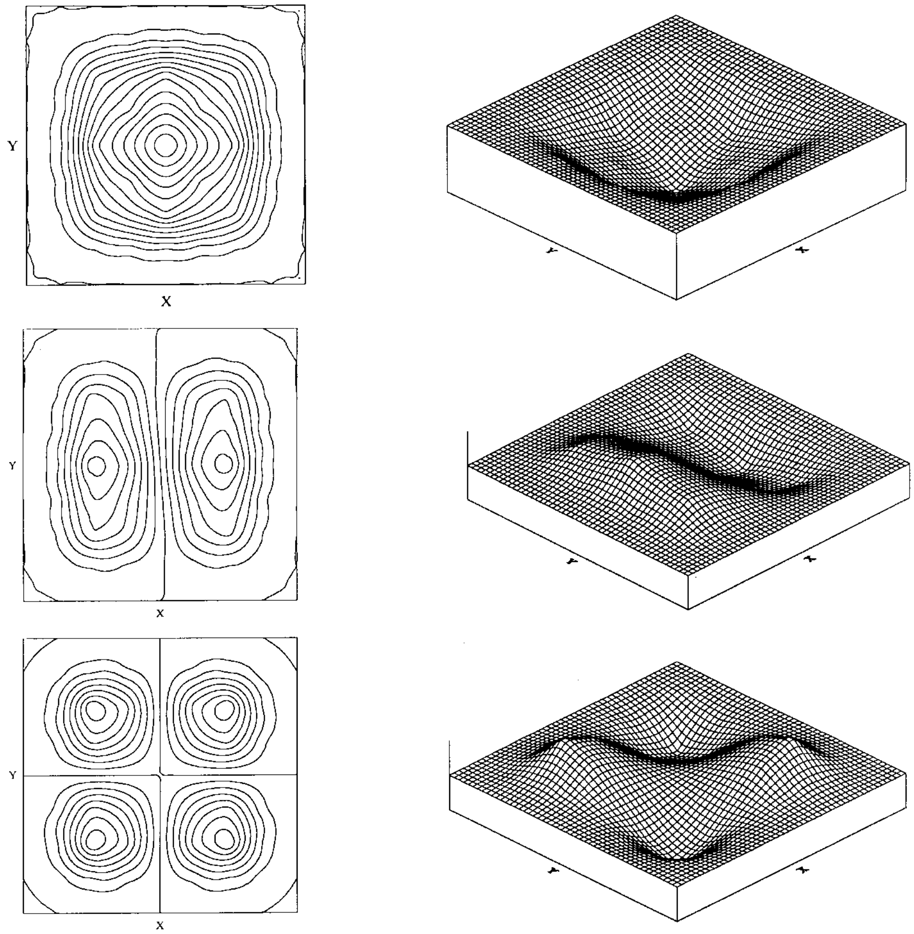

Fig. 9 The contour line and mode shape of ω_{11}, ω_{12}, ω_{22} for C-C-C-C

parameters are obtained using the 9×9 elements mesh for complete plate.

Second, calculations were performed for Pasternak foundation that is k=100000 kN/m^3, G_f =120 MPa. All the other parameters are the same with Winkler foundation. Results are tabulated in Tables 6-7.

Third, the effect of variation of k on the frequency is inspected. This problem is solved only for the simply supported plate. In this problem G_f is kept constant (G_f = 120 GPa). Results are shown in Fig. 12.

The last, the effect of variation of shear modulus G_f on the foundation is inspected. This problem is also solved only for the simply supported plate. In this problem k is kept constant as k=100000 kN/m^3. Results are shown in Fig. 13.

Inspection of Figs. show that:

· The variation of k on the frequency parameters has similar effects for the Winkler and Pasternak foundation.

· G_f effects the frequecy parameters more than k does.

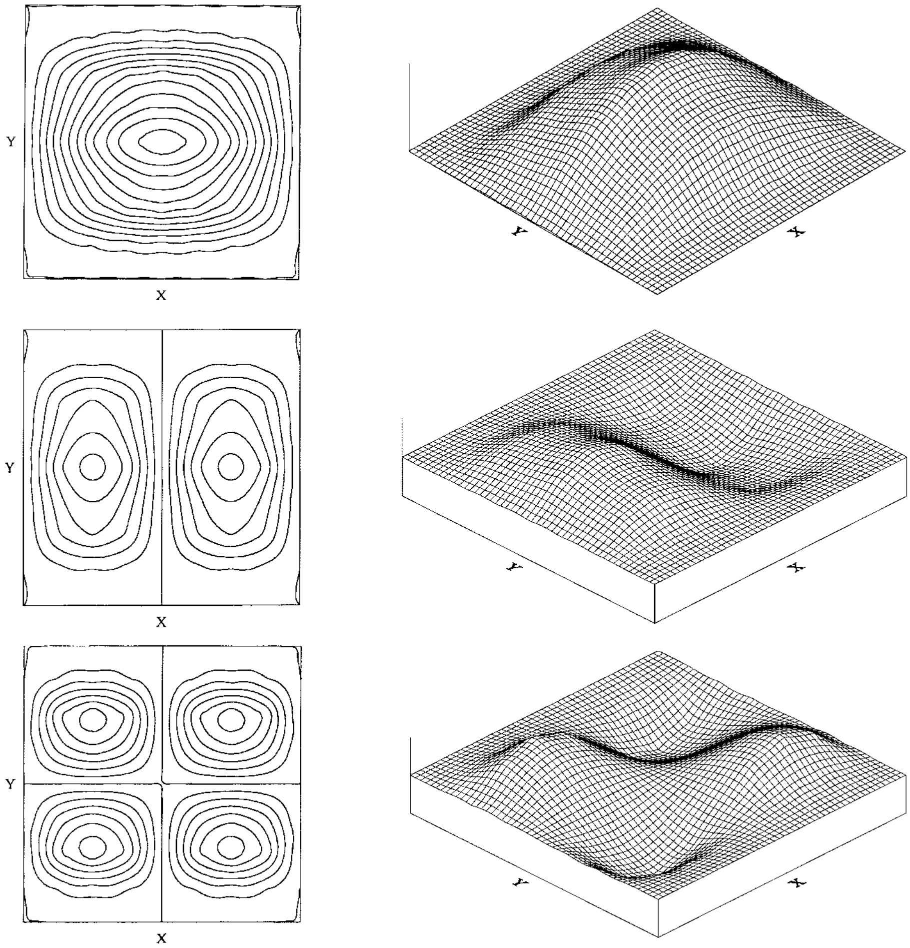

Fig. 10 The contour line and mode shape of ω_{11}, ω_{12}, ω_{22} for S-C-S-C

Table 5 Frequency parameters ϖ for a F-F-F-F plate (b/a = 1)

Non-Zero Frequency Parameters	Leissa 1969	Bardell 1991	REC32
1	13.473	13.468	12.861
2	19.596	19.596	19.549
3	24.270	24.270	24.157

6. Conclusions

In this study, REC32 is developed to analyze free vibration of Reissner-Mindlin plates on Pasternak foundation. Eight independent variables such as displacement, rotations, shear forces, bending and twisting moments are free parameters and linear interpolation functions satisfy the continuity and completeness requirements. The explicit expression for rectangular plate (REC32) is

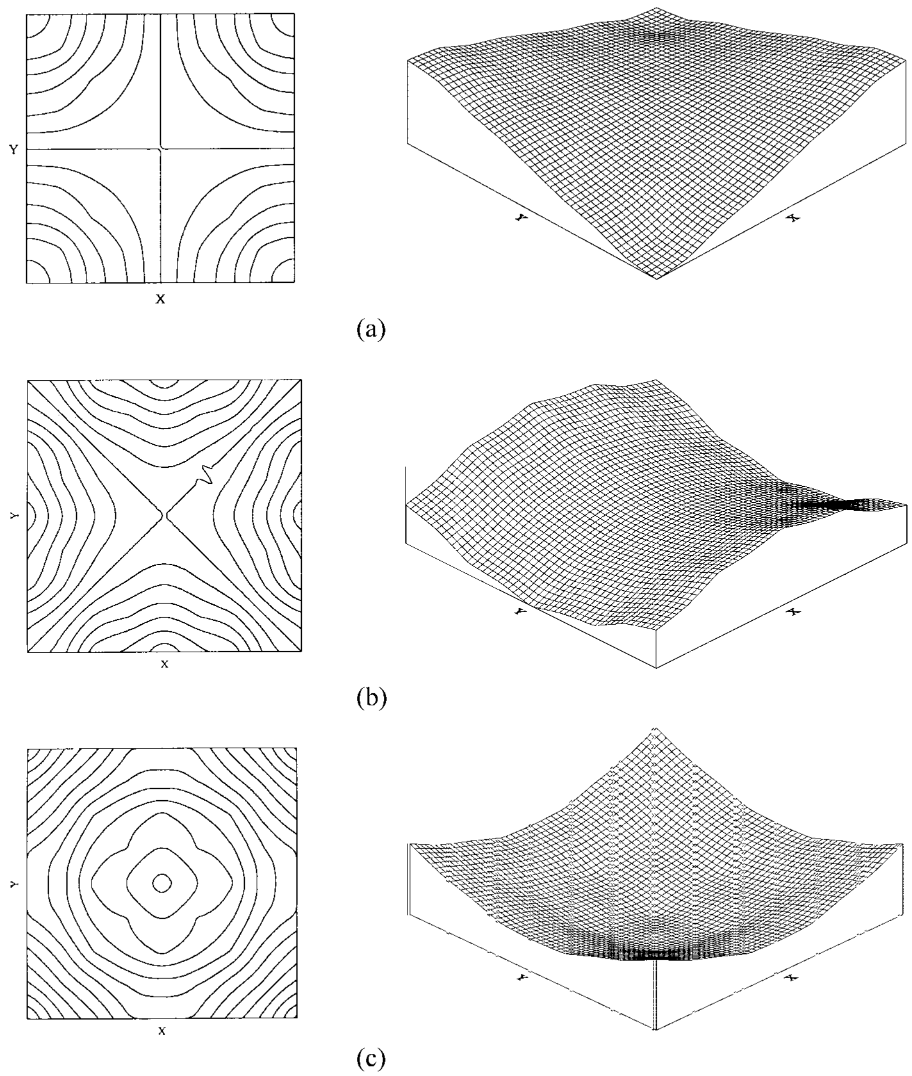

Fig. 11 The contour lines and mode shapes of free plate

Table 6 Frequency parameters ϖ of S-S-S-S plate on foundation (b/a = 1)

Frequency Parasmeters *s*	Omurtag *et al.* 1997 Winkler	REC32 Winkler	Omurtag *et al.* 1997 Pasternak	REC32 Pasternak
ϖ_{11}	42.282	42.169	71.365	71.512
ϖ_{12}	62.417	61.624	110.963	111.340
ϖ_{22}	88.144	86.768	145.863	146.260

Table 7 Frequency parameters ϖ of C-C-C-C plate on foundation (b/a = 1)

Frequency Parasmeters s	Omurtag *et al.* 1997 Winkler	REC32 Winkler	Omurtag *et al.* 1997 Pasternak	REC32 Pasternak
ϖ_{11}	51.902	51.779	80.314	85.414
ϖ_{12}	83.222	82.701	127.294	137.520
ϖ_{22}	115.214	116.270	167.116	181.780

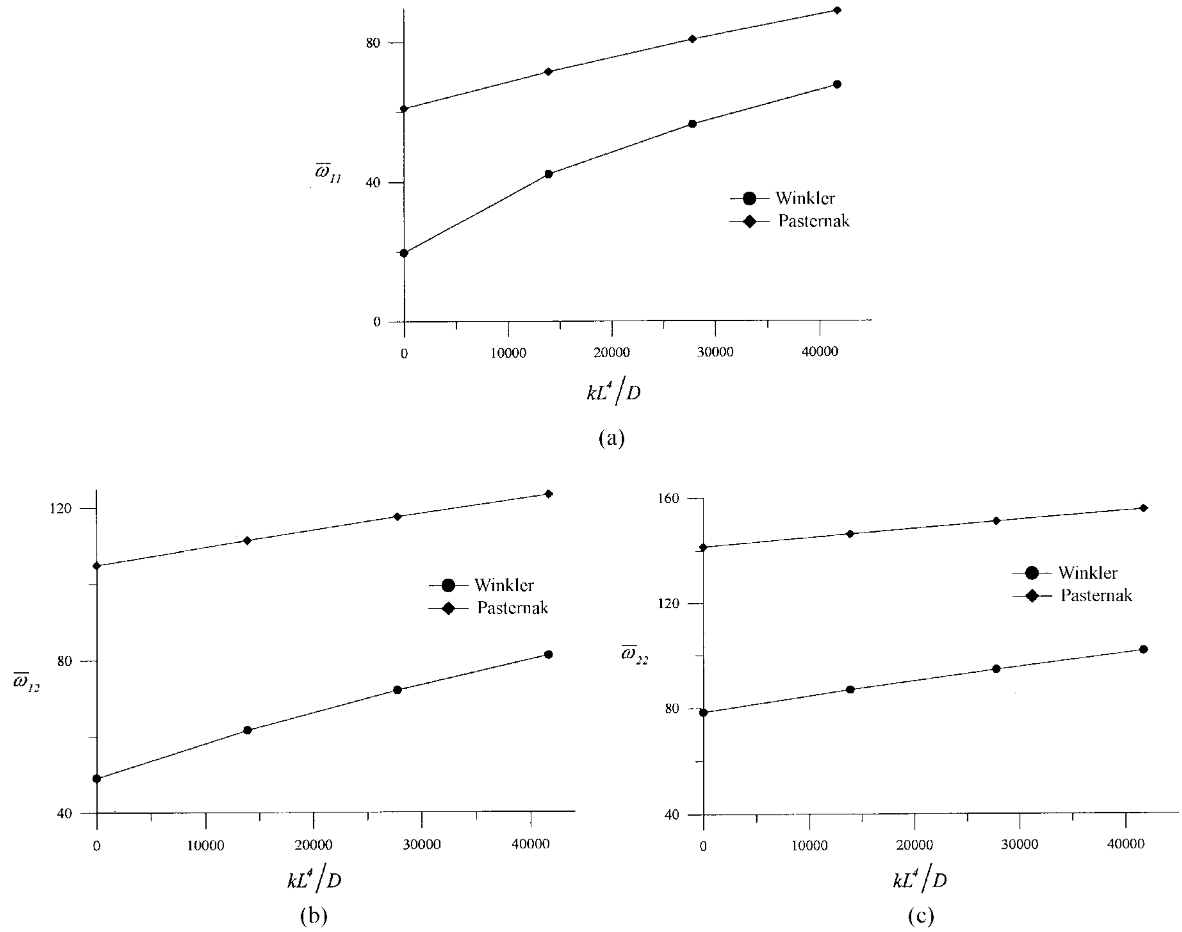

Fig. 12 The effect of variation of k on the frequency parameters for Winkler and Pasternak foundation (G_f= 120 MPa)

obtained. The performance of the elements has also been investigated through the representative problems. From the numerical assessments of the element, the following remarks can be made:

· For the Reissner plate on Pasternak foundation, a new functional is obtained based on Gâteaux differential and new dynamic boundary conditions are established.

· In this functional, only the first order derivatives present therefore, bilinear shape functions are

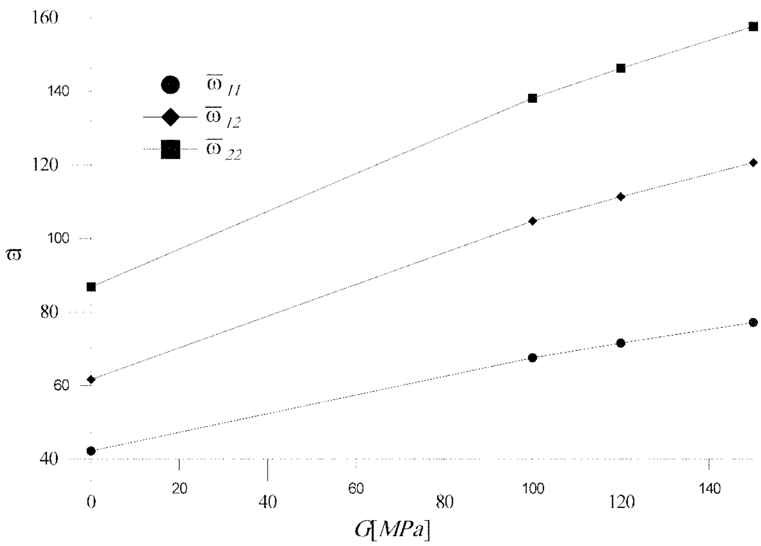

Fig. 13 The effect of variation of G_f on the frequency parameters (k = 100000 kN/m^3)

used and a mixed finite element REC32 are obtained in an explicit form.

· REC32 avoids the shear locking and converges to the Kirchhoff solution as the plate thickness decreases (REC32 is tested for as far as $h/2a=10^{-6}$).

· REC32 provides accurate and stable solution.

· To assess the performance of REC32, S-S-S-S, C-C-C-C, S-C-S-C and F-F-F-F square plates are solved. Frequencies of free vibration of plates are compared by theoretical results and excellent agreement is achieved.

· The validity limits of Kirchhoff plate theory is tested and established that this limit is not dependent only the ratio of $h/2a$ but also the supporting conditions.

· The free vibration analysis of thick plates resting on Pasternak foundation is performed and reasonable results are obtained. Winkler foundation is obtained as a special case of Pasternak foundation.

· The variation of k on the frequency parameters has similar effects for the Winkler and Pasternak foundation.

· G_f effects the frequecy parameters more than k does.

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Appendix I

Notation

M_x, M_y, M_{xy}	: internal moment components
Q_x, Q_y	: shear forces
q	: distributed load
k, G_f	: spring and shear coefficient of foundation, respectively
w	: deflection of plate
Ω_x, Ω_y	: cross-sectional rotation of plates about x and y axes, respectively
$2a$,$2b$	: dimensions of the rectangular element
h	: thickness of plate
E, ν, G, D	: modulus of elasticity, Poisson's ratio, shear modulus of elasticity and flexural rigidity of the plate, respectively
$I(\boldsymbol{y})$	: functional
[,]	: inner product
$[\ ,\ ]_\varepsilon$	: geometric boundary condition
$[\ ,\ ]_\sigma$	: dynamic boundary condition
ψ_i	: shape functions (i = 1, ..., 4 for REC32)
ξ, η	: nondimensional coordinates of a master element
n_x, n_y	: directional cosines
$[k]_{\mathrm{r}}$	: rectangular finite element matrix
$[K_i]_{\mathrm{r}}$, $[s_i]_{\mathrm{r}}$	: submatrices of the rectangular finite element (i = 1, ..., 4)
$[k]_{\mathrm{BCr}}$	: boundary condition matrix of rectangular finite element
$\boldsymbol{L}$	: coefficient matrix
$\boldsymbol{f}$	: load vector
$\boldsymbol{y}$	: unknown vectors
$[\boldsymbol{K}]$, $[\boldsymbol{M}]$, $[\boldsymbol{K}^*]$	: system matrix, mass matrix and condensed matrix vectors, respectively
$[m]$	: mass matrix of element
ρ, ρ	: mass density per unit volume and per unit area, respectively
ω, ϖ	: natural angular frequency and frequency parameter, respectively.

Appendix II

Operator form of the field equations $\boldsymbol{Q} = \boldsymbol{Ly} - \boldsymbol{f}$:

$$
\begin{bmatrix}
P_{11} & 0 & 0 & 0 & 0 & 0 & P_{17} & P_{18} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P_{24} & 0 & P_{26} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_{35} & P_{36} & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & P_{42} & 0 & P_{44} & P_{45} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & P_{53} & P_{54} & P_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & P_{62} & P_{63} & 0 & 0 & P_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\
P_{71} & 1 & 0 & 0 & 0 & 0 & P_{77} & 0 & 0 & 0 & 0 & 0 \\
P_{81} & 0 & 1 & 0 & 0 & 0 & 0 & P_{88} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} w \\ \Omega_x \\ \Omega_y \\ M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \\ w \\ \Omega \\ M \\ Q \end{bmatrix}
=
\begin{bmatrix} q \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hat{Q} \\ \hat{M} \\ -\hat{\Omega} \\ -\hat{w} \end{bmatrix}
\tag{A.1}
$$

where

$$
\begin{aligned}
& P_{11} = -k + G_f\frac{\partial^2}{\partial x^2} + G_f\frac{\partial^2}{\partial y^2},\ P_{17} = -\frac{\partial}{\partial x},\ P_{18} = -\frac{\partial}{\partial y},\ P_{24} = -\frac{\partial}{\partial x},\ P_{26} = -\frac{\partial}{\partial y} \\
& P_{35} = -\frac{\partial}{\partial y},\ P_{36} = -\frac{\partial}{\partial x},\ P_{42} = -\frac{\partial}{\partial x},\ P_{44} = -\frac{12}{Eh^3},\ P_{45} = -\frac{12}{Eh^3}\mu \\
& P_{53} = \frac{\partial}{\partial y},\ P_{54} = \frac{12}{Eh^3}\mu,\ P_{55} = -\frac{12}{Eh^3} \\
& P_{62} = \frac{\partial}{\partial y},\ P_{63} = \frac{\partial}{\partial x},\ P_{66} = -\frac{12}{Gh^3} \\
& P_{71} = \frac{\partial}{\partial x},\ P_{77} = -\frac{6}{5Gh},\ P_{81} = \frac{\partial}{\partial y},\ P_{88} = -\frac{6}{5Gh}
\end{aligned}
\tag{A.2}
$$

Submatrices for quadrilateral element;

$$
[K_1]_r = \int_A \psi_i \psi_j dA = \begin{bmatrix} 4ab/9 & 4ab/18 & 4ab/18 & 4ab/36 \\ 4ab/18 & 4ab/9 & 4ab/36 & 4ab/18 \\ 4ab/18 & 4ab/36 & 4ab/9 & 4ab/18 \\ 4ab/36 & 4ab/18 & 4ab/18 & 4ab/9 \end{bmatrix} \tag{A.3}
$$

$$
[K_2]_r = \int_A \frac{\partial \psi_i}{\partial x} \psi_j dA = \begin{bmatrix} -b/3 & -b/6 & -b/3 & -b/6 \\ -b/6 & -b/3 & -b/6 & -b/3 \\ b/3 & b/6 & b/3 & b/6 \\ b/6 & b/3 & b/6 & b/3 \end{bmatrix} \tag{A.4}
$$

$$
[K_3]_r = \int_A \frac{\partial \psi_i}{\partial y} \psi_j dA = \begin{bmatrix} -a/3 & -a/3 & -a/6 & -a/6 \\ a/3 & a/3 & a/6 & a/6 \\ -a/6 & -a/6 & -a/3 & -a/3 \\ a/6 & a/6 & a/3 & a/3 \end{bmatrix} \tag{A.5}
$$

$$
[K_4]_r = \int_A \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} dA = \begin{bmatrix} b/3a & b/6a & -b/3a & -b/6a \\ b/6a & b/3a & -b/6a & -b/3a \\ -b/3a & -b/6a & b/3a & b/6a \\ -b/6a & -b/3a & b/6a & b/3a \end{bmatrix} \tag{A.6}
$$

$$[K_5]_r = \int_A \frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y} dA = \begin{bmatrix} a/3b & -a/3b & a/6b & -a/6b \\ -a/3b & a/3b & -a/6b & a/6b \\ a/6b & -a/6b & a/3b & -a/3b \\ -a/6b & a/6b & -a/3b & a/3b \end{bmatrix} \tag{A.7}$$

$$[s_1]_r = \begin{bmatrix} -2b/3 & -b/3 & 0 & 0 \\ -b/3 & -2b/3 & 0 & 0 \\ 0 & 0 & 2b/3 & b/3 \\ 0 & 0 & b/3 & 2b/3 \end{bmatrix} \tag{A.8}$$

$$[s_2]_r = \begin{bmatrix} -2a/3 & 0 & -a/3 & 0 \\ 0 & 2a/3 & 0 & a/3 \\ -a/3 & 0 & -2a/3 & 0 \\ 0 & a/3 & 0 & 2a/3 \end{bmatrix} \tag{A.9}$$

$$[s_3]_r = \begin{bmatrix} b/3a & b/6a & -b/3a & -b/6a \\ b/6a & b/3a & -b/6a & -b/3a \\ -b/3a & -b/6a & b/3a & b/6a \\ -b/6a & -b/3a & b/6a & b/3a \end{bmatrix} \tag{A.10}$$

$$[s_4]_r = \begin{bmatrix} a/3b & -a/3b & a/6b & -a/6b \\ -a/3b & a/3b & -a/6b & a/6b \\ a/6b & -a/6b & a/3b & -a/3b \\ -a/6b & a/6b & -a/3b & a/3b \end{bmatrix} \tag{A.11}$$