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Interval finite element method based on the element for eigenvalue analysis of structures with interval parameters

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Abstract. A new method for solving the uncertain eigenvalue problems of the structures with interval parameters, interval finite element method based on the element, is presented in this paper. The calculations are done on the element basis, hence, the efforts are greatly reduced. In order to illustrate the accuracy of the method, a continuous beam system is given, the results obtained by it are compared with those obtained by Chen and Qiu (1994); in order to demonstrate that the proposed method provides safe bounds for the eigenfrequencies, an undamping spring-mass system, in which the exact interval bounds are known, is given, the results obtained by it are compared with those obtained by Qiu *et al.* (1999), where the exact interval bounds are given. The numerical results show that the proposed method is effective for estimating the eigenvalue bounds of structures with interval parameters.

Key words: interval parameters; interval eigenvalue analysis; interval finite element method.

1. Introduction

The numerical analysis of the structural dynamics characteristics is usually performed for specified structural parameters conditions. However, in most practical situations, the structural parameters are uncertain, for example, the inaccuracy of measurements, errors in manufacture, etc. Therefore, the concept of uncertainty plays an important role in the investigation of various engineering problems. The most common approach to the uncertain problems is to model the structural parameters as a random vector. Under this circumstance, all information about the structural parameters is provided by the joint probability density function (or distribution function) of the structural parameters. Unfortunately, the probabilistic approaches are not able to deliver the reliable results at the required precision without sufficient experimental data to validate the assumptions made regarding the joint probability densities of the random variables or functions involved.

At present, the majority of scientists and engineers studying problems containing uncertainty

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utilize stochastic models. They do it not because the uncertain phenomena actually have a stochastic nature. In many cases, this 'stochastization' (or 'randomization') is the result of an established scientific stereotype. According to Kuntzevichi and Lychak (1992), the assumption of stochastic nature of the uncertainty can not be accepted at least in two cases: (1) when the volume of a priori experimental data on the nature of the uncertain factors is so small that it does not allow the conclusion on the availability of stable statistic characteristics; (2) when it is known a priori that the uncertainty basically can not be considered to be produced by some probabilistic mechanism.

Since the mid-sixties, a new method called interval analysis has appeared. Moore (1979) and his co-workers, Alefeld and Herzberger (1983) have given the pioneering work. Application of interval analysis for some engineering problems was facilitated by H. U. Koyluoglu *et al.* (1995), A. D. Dimarogonas (1995), O. Dessombz *et al.* (2001), S. Nakagiri and N. Yoshikawa (1996). In terms of mathematics, linear interval equations, nonlinear interval equation and interval eigenvalue problems have been resolved partly. But due to the algorithm complex, it is difficult to apply their results into the practical engineering. Recently, Chen, Qiu and Elishakoff (1994, 1995, 1996, 1999) have used interval set models in the study of the static response and eigenvalue problems of structures with bounded uncertain parameters. In their study, using the interval analysis and matrix perturbation, several important results have been obtained. However, these results are based on the assumptions that $\Delta\boldsymbol{K}$, $\Delta\boldsymbol{f}$ are preselected in the equation $\boldsymbol{K}(\alpha)\boldsymbol{U} = \boldsymbol{f}(\alpha)$ and $\Delta\boldsymbol{K}$, $\Delta\boldsymbol{M}$ are also preselected in the equation $\boldsymbol{K}(\alpha)u = \lambda\boldsymbol{M}(\alpha)u$. In general, if the stiffness coefficients k_{ij} and the mass coefficients m_{ij} are the linear function of the parameters, it can be obtained by using the interval algorithm, if the stiffness coefficients k_{ij} and the mass coefficients m_{ij} are the nonlinear function of the parameters, the difficulties for calculating k_{ij} and m_{ij} arise. Therefore, it is very important to present an effective interval eigenvalue analysis method for structures with interval parameters. In this paper, based on the conventional finite element method, we will discuss the interval finite element for the eigenvalue analysis of structures with interval parameters by using the interval analysis. The results of the proposed method are compared with those obtained by Chen and Qiu (1994), Qiu *et al.* (1999).

2. Mathematical backgrounds

In structural analysis and design, some structural parameters have errors or uncertainties which are caused by manufacture, installation, measurement or computation. Therefore, it is very important to predict the errors resulted from the above mentioned uncertainties in the structural design. In interval mathematics, the errors or uncertainties are always denoted by intervals. From this point, we define interval parameters first.

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)^T$ be a structural parameter vector with bound errors or uncertainties, where

$$\alpha_i \in \alpha_i^I = [\alpha_i^C - \Delta\alpha_i, \ \alpha_i^C + \Delta\alpha_i]$$

then

$$\alpha \in \alpha^I = [\alpha^C - \Delta\alpha, \ \alpha^C + \Delta\alpha] \tag{1}$$

where

$$\alpha^C = (\alpha_1^C, \alpha_2^C, ..., \alpha_m^C)^T$$
$$\Delta\alpha = (\Delta\alpha_1, \Delta\alpha_2, ..., \Delta\alpha_m)^T$$

Let $\underline{\alpha}$ and $\overline{\alpha}$ be the lower and upper bound vectors of the structural parameter vector α, respectively. The bound and uncertain parameter vectors are called interval parameters.

Before we can proceed to treat static response of structures with interval parameters and interval loads, we need to introduce some results in interval analysis (Moore 1979).

In interval mathematics, a subset of real numbers R of the form $[a_1, a_2] = \{t,\ a_1 \le t \le a_2 \mid a_1, a_2 \in R\}$ is called a closed real interval or an interval, denoted by $X^I = [\underline{X}, \overline{X}]$, where $\underline{X}$ and $\overline{X}$ are the lower and upper bounds, respectively. The set of all closed real intervals is denoted by $I(R)$.

The mid-point and uncertainty (or maximum error) of an interval X^I are defined as

$$X^C = \frac{(\overline{X} + \underline{X})}{2} \tag{2}$$

and

$$\Delta X = \frac{(\overline{X} - \underline{X})}{2} \tag{3}$$

respectively.

A symmetric interval means an interval X^I in which $\underline{X} = -\overline{X}$.

We represent an n-dimensional interval vector as

$$\boldsymbol{X}^{\boldsymbol{I}} = (X_1^I, X_2^I, \cdots, X_n^I)^T \tag{4}$$

The set of all n-dimensional interval vectors is denoted by $I(R^n)$.

Similarly, the mid-vector and uncertainty of an interval vector can be defined as

$$\boldsymbol{X}^{\boldsymbol{C}} = (X_1^C, X_2^C, \cdots, X_n^C)^T \tag{5}$$

and

$$\boldsymbol{\Delta X} = (\Delta X_1, \Delta X_2, \cdots, \Delta X_n)^T \tag{6}$$

where X_i^C and ΔX_i are given by Eqs. (2) and (3) respectively.

A matrix whose elements are interval parameters is called an interval matrix and denoted by $\boldsymbol{A}^{\boldsymbol{I}} = [\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}}]$, where $\underline{\boldsymbol{A}}$ is a matrix composed of the lower bounds of intervals and $\overline{\boldsymbol{A}}$ is a matrix composed of the upper bounds of the intervals. The set of all interval matrices is denoted by $I(R^{m\times n})$. The mid-matrix and uncertainty of an interval matrix $\boldsymbol{A}^I$ are given as

$$\boldsymbol{A}^C = \frac{(\overline{\boldsymbol{A}} + \underline{\boldsymbol{A}})}{2} \quad \text{or} \quad a_{ij}^C = \frac{(\overline{a}_{ij} + \underline{a}_{ij})}{2}$$

and

$$\boldsymbol{\Delta A} = \frac{(\overline{\boldsymbol{A}} - \underline{\boldsymbol{A}})}{2} \quad \text{or} \quad \Delta a_{ij} = \frac{(\overline{a}_{ij} + \underline{a}_{ij})}{2}$$

where

$$\boldsymbol{A}^C = (a_{ij}^C) \quad \text{and} \quad \boldsymbol{\Delta A} = (\Delta a_{ij})$$

An arbitrary interval $X^I \in I(R)$ can be written as the sum of its mid-point X^C and a symmetric interval $\Delta X^I = [-\Delta X, \Delta X] = \Delta X\,[-1, 1]$, i.e.

$$X^I = X^C + \Delta X^I \tag{7}$$

Similar expressions exist for the interval vector and interval matrix. For $A^I = I(R^{m\times n})$ we have

$$\boldsymbol{A}^I = \boldsymbol{A}^C + \boldsymbol{\Delta A}^I \tag{8}$$

where

$$\boldsymbol{\Delta A}^I = [-\boldsymbol{\Delta A}, \boldsymbol{\Delta A}]$$

These basic quantities will play an important role in the following discussions.

Let f be a real-valued function of n real variables $x_1, x_2, \cdots, x_n$. An extension of f means that an interval value function F of n interval variables $X_1^I, X_2^I, \cdots, X_n^I$, for all $x_i \in X_i^I (i = 1, 2, \cdots, n)$ possesses the following property

$$F([x_1, x_1], [x_2, x_2], \cdots, [x_n, x_n]) = f(x_1, x_2, \cdots, x_n) \tag{9}$$

Given a rational expression in real variables, we can replace the real variables by the corresponding interval variables and replace the real arithmetic operations by the corresponding interval arithmetic operations to obtain a rational interval function which is a natural extension of the real rational function.

Let $X^I = [\underline{x}, \bar{x}]$ and $Y^I = [\underline{y}, \bar{y}]$ be interval numbers, respectively, then $X^I + Y^I$, $X^I - Y^I$, $X^I \times Y^I$ and X^I/Y^I are defined by the following formulas:

$$X^I + Y^I = [\underline{x}, \bar{x}] + [\underline{y}, \bar{y}] = [\underline{x} + \underline{y}, \ \bar{x} + \bar{y}] \tag{10}$$

$$X^I - Y^I = [\underline{x}, \bar{x}] - [\underline{y}, \bar{y}] = [\underline{x} - \bar{y}, \ \bar{x} - \underline{y}] \tag{11}$$

$$X^I \times Y^I = [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] = [\min(\underline{x}\cdot\underline{y}, \underline{x}\cdot\bar{y}, \bar{x}\cdot\underline{y}, \bar{x}\cdot\bar{y}), \max(\underline{x}\cdot\underline{y}, \underline{x}\cdot\bar{y}, \bar{x}\cdot\underline{y}, \bar{x}\cdot\bar{y})] \tag{12}$$

$$\frac{X^I}{Y^I} = \frac{[\underline{x}, \bar{x}]}{[\underline{y}, \bar{y}]} = [\underline{x}, \bar{x}] \times \left[\frac{1}{\bar{y}}, \frac{1}{\underline{y}}\right] \tag{13}$$

$$X^I \cap Y^I = [\max(\underline{x}, \underline{y}), \min(\bar{x}, \bar{y})] \tag{14}$$

$$X^I \cup Y^I = [\max(\underline{x}, \underline{y}), \min(\bar{x}, \bar{y})] \tag{15}$$

3. Matrix perturbation for eigenvalue analysis

We consider the generalized eigenproblem

$$\boldsymbol{K}_0 \boldsymbol{u}_0 = \lambda_0 \boldsymbol{M}_0 \boldsymbol{u}_0 \tag{16}$$

where $\boldsymbol{K}_0$ and $\boldsymbol{M}_0$ are, respectively, the stiffness matrix and mass matrix of the finite element assemblage. λ_0 is the eigenvalue and $\boldsymbol{u}_0$ is the eigenvector.

It should be noted that perturbation theory studies the behavior of a system subjected to small changes in its design variables. Therefore, if the system is represented by Eq. (16), the problem becomes that of determining λ and $\boldsymbol{u}$ when $\boldsymbol{K}_0$ and $\boldsymbol{M}_0$ exhibits perturbations of the form $\boldsymbol{K}_0 + \boldsymbol{\Delta K}$, and $\boldsymbol{M}_0 + \boldsymbol{\Delta M}$. Perturbation analysis is based on the solution of the original system. So, the eigenvalue problem of the perturbed system is as follows

$$(\boldsymbol{K}_0 + \boldsymbol{\Delta K})(\boldsymbol{u}_{k0} + \boldsymbol{\Delta u}_k) = (\lambda_{k0} + \Delta\lambda_k)(\boldsymbol{M}_0 + \boldsymbol{\Delta M})(\boldsymbol{u}_{k0} + \boldsymbol{\Delta u}_k) \tag{17}$$

where

$$\Delta\lambda_k = \lambda_{k1} + \lambda_{k2} + \lambda_{k3} + \cdots \tag{18}$$

$$\Delta \boldsymbol{u}_k = \boldsymbol{u}_{k1} + \boldsymbol{u}_{k2} + \boldsymbol{u}_{k3} + \cdots \tag{19}$$

$k = 1, 2, \cdots, p$. p is the number of degree of freedom. Substituting Eqs. (18)−(19) into Eq. (17), we can obtain the following equations (Chen 1993)

$$\lambda_{k1} = (\boldsymbol{u}_{k0})^T \Delta \boldsymbol{K} \boldsymbol{u}_{k0} - \lambda_{k0} (\boldsymbol{u}_{k0})^T \Delta \boldsymbol{M} \boldsymbol{u}_{k0} \tag{20}$$

where $k = 1, 2, \cdots, p$

4. Interval finite element method for structures with interval parameters

In this section, via the beam structures, we will consider the interval denotation of the structural element stiffness matrix and mass matrix when the structural parameters vary in some intervals.

The stiffness matrix and mass matrix of the beam element can be respectively denoted by

$$\boldsymbol{K_i} = \begin{bmatrix} \frac{E_iA_i}{L_i} & 0 & 0 & \frac{-E_iA_i}{L_i} & 0 & 0 \\ 0 & \frac{12E_iI_i}{L_i^3} & \frac{6E_iI_i}{L_i^2} & 0 & \frac{-12E_iI_i}{L_i^3} & \frac{6E_iI_i}{L_i^2} \\ 0 & \frac{6E_iI_i}{L_i^2} & \frac{4E_iI_i}{L_i} & 0 & \frac{-6E_iI_i}{L_i^2} & \frac{2E_iI_i}{L_i} \\ \frac{-E_iA_i}{L_i} & 0 & 0 & \frac{E_iA_i}{L_i} & 0 & 0 \\ 0 & \frac{-12E_iI_i}{L_i^3} & \frac{-6E_iI_i}{L_i^2} & 0 & \frac{12E_iI_i}{L_i^3} & \frac{-6E_iI_i}{L_i^2} \\ 0 & \frac{6E_iI_i}{L_i^2} & \frac{2E_iI_i}{L_i} & 0 & \frac{-6E_iI_i}{L_i^2} & \frac{4E_iI_i}{L_i} \end{bmatrix} \tag{21}$$

and

$$\boldsymbol{M_i} = \rho A L_i \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & \frac{13}{35} & \frac{11L_i}{210} & 0 & \frac{9}{70} & \frac{-13L_i}{420} \\ 0 & \frac{11L_i}{210} & \frac{L_i^2}{105} & 0 & \frac{13L_i}{420} & \frac{-L_i^2}{140} \\ \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{9}{70} & \frac{13L_i}{420} & 0 & \frac{13}{35} & \frac{-11L_i}{210} \\ 0 & \frac{-13L_i}{420} & \frac{-L_i^2}{140} & 0 & \frac{-11L_i}{210} & \frac{L_i^2}{105} \end{bmatrix} \tag{22}$$

Supposing that the cross-section of the beam is rectangle, whose height is H_i, width is B_i, one has $A_i = B_i H_i$ and $I_i = B_i H_i^3/12$. If E_i, ρ_i, B_i and H_i are interval variables; i.e., $E_i = E_i^C + \Delta E_i e_{i1}$, $\rho_i = \rho_i^C + \Delta\rho_i e_{i4}$, $B_i = B_i^C + \Delta B_i e_{i2}$, $H_i = H_i^C + \Delta H_i e_{i3}$, Ignoring the terms of higher order, using the natural interval extension of the real function and the Taylor expansion for the stiffness coefficients of the beam element, the interval stiffness matrix and mass matrix of the beam element can be respectively written as follows:

$$\begin{aligned}\boldsymbol{K}_i(E_i^I, B_i^I, H_i^I) &= \boldsymbol{K}_i(E_i^C, B_i^C, H_i^C) + \frac{\partial \boldsymbol{K}_i(E_i^C, B_i^C, H_i^C)}{\partial E_i}(E_i^I - E_i^C)\\ &\quad + \frac{\partial \boldsymbol{K}_i(E_i^C, B_i^C, H_i^C)}{\partial B_i}(B_i^I - B_i^C) + \frac{\partial \boldsymbol{K}_i(E_i^C, B_i^C, H_i^C)}{\partial H_i}(H_i^I - H_i^C)\\ &= \boldsymbol{K}_i(E_i^C, B_i^C, H_i^C) + \frac{\partial \boldsymbol{K}_i(E_i^C, B_i^C, H_i^C)}{\partial E_i}\Delta E_i e_{i1}\\ &\quad + \frac{\partial \boldsymbol{K}_i(E_i^C, B_i^C, H_i^C)}{\partial B_i}\Delta B_i e_{i2} + \frac{\partial \boldsymbol{K}_i(E_i^C, B_i^C, H_i^C)}{\partial H_i}\Delta H_i e_{i3}\end{aligned} \tag{23}$$

$$\begin{aligned}\boldsymbol{M}_i(\rho_i^I, B_i^I, H_i^I) &= \boldsymbol{M}_i(\rho_i^C, B_i^C, H_i^C) + \frac{\partial \boldsymbol{M}_i(\rho_i^C, B_i^C, H_i^C)}{\partial \rho_i}(\rho_i^I - \rho_i^C)\\ &\quad + \frac{\partial \boldsymbol{M}_i(\rho_i^C, B_i^C, H_i^C)}{\partial B_i}(B_i^I - B_i^C) + \frac{\partial \boldsymbol{M}_i(\rho_i^C, B_i^C, H_i^C)}{\partial H_i}(H_i^I - H_i^C)\\ &= \boldsymbol{M}_i(\rho_i^C, B_i^C, H_i^C) + \frac{\partial \boldsymbol{M}_i(\rho_i^C, B_i^C, H_i^C)}{\partial \rho_i}\Delta\rho_i e_{i4}\\ &\quad + \frac{\partial \boldsymbol{M}_i(\rho_i^C, B_i^C, H_i^C)}{\partial B_i}\Delta B_i e_{i2} + \frac{\partial \boldsymbol{M}_i(\rho_i^C, B_i^C, H_i^C)}{\partial H_i}\Delta H_i e_{i3}\end{aligned} \tag{24}$$

where

$$\frac{\partial \boldsymbol{K}_i(E_i^C, B_i^C, H_i^C)}{\partial E_i} = \begin{bmatrix} \frac{B_i^C H_i^C}{L_i} & 0 & 0 & \frac{-B_i^C H_i^C}{L_i} & 0 & 0 \\ 0 & \frac{B_i^C (H_i^C)^3}{L_i^3} & \frac{B_i^C (H_i^C)^3}{2L_i^2} & 0 & \frac{-B_i^C (H_i^C)^3}{L_i^3} & \frac{B_i^C (H_i^C)^3}{2L_i^2} \\ 0 & \frac{B_i^C (H_i^C)^3}{2L_i^2} & \frac{B_i^C (H_i^C)^3}{3L_i} & 0 & \frac{-B_i^C (H_i^C)^3}{2L_i^2} & \frac{B_i^C (H_i^C)^3}{6L_i} \\ \frac{-B_i^C H_i^C}{L_i} & 0 & 0 & \frac{B_i^C H_i^C}{L_i} & 0 & 0 \\ 0 & \frac{-B_i^C (H_i^C)^3}{L_i^3} & \frac{-B_i^C (H_i^C)^3}{2L_i^2} & 0 & \frac{B_i^C (H_i^C)^3}{L_i^3} & \frac{-B_i^C (H_i^C)^3}{2L_i^2} \\ 0 & \frac{B_i^C (H_i^C)^3}{2L_i^2} & \frac{B_i^C (H_i^C)^3}{6L_i} & 0 & \frac{-B_i^C (H_i^C)^3}{2L_i^2} & \frac{B_i^C (H_i^C)^3}{3L_i} \end{bmatrix} \tag{25}$$

$$\frac{\partial \boldsymbol{K}_i(E_i^C, B_i^C, H_i^C)}{\partial B_i} = \begin{bmatrix} \frac{E_i^C H_i^C}{L_i} & 0 & 0 & \frac{-E_i^C H_i^C}{L_i} & 0 & 0 \\ 0 & \frac{E_i^C (H_i^C)^3}{L_i^3} & \frac{E_i^C (H_i^C)^3}{2L_i^2} & 0 & \frac{-E_i^C (H_i^C)^3}{L_i^3} & \frac{E_i^C (H_i^C)^3}{2L_i^2} \\ 0 & \frac{E_i^C (H_i^C)^3}{2L_i^2} & \frac{E_i^C (H_i^C)^3}{3L_i} & 0 & \frac{-E_i^C (H_i^C)^3}{2L_i^2} & \frac{E_i^C (H_i^C)^3}{6L_i} \\ \frac{-E_i^C H_i^C}{L_i} & 0 & 0 & \frac{E_i^C H_i^C}{L_i} & 0 & 0 \\ 0 & \frac{-E_i^C (H_i^C)^3}{L_i^3} & \frac{-E_i^C (H_i^C)^3}{2L_i^2} & 0 & \frac{E_i^C (H_i^C)^3}{L_i^3} & \frac{-E_i^C (H_i^C)^3}{2L_i^2} \\ 0 & \frac{E_i^C (H_i^C)^3}{2L_i^2} & \frac{E_i^C (H_i^C)^3}{6L_i} & 0 & \frac{-E_i^C (H_i^C)^3}{2L_i^2} & \frac{E_i^C (H_i^C)^3}{3L_i} \end{bmatrix} \tag{26}$$

$$\frac{\partial \boldsymbol{K}_i(E_i^C, B_i^C, H_i^C)}{\partial H_i} = \begin{bmatrix} \frac{E_i^C B_i^C}{L_i} & 0 & 0 & \frac{-E_i^C B_i^C}{L_i} & 0 & 0 \\ 0 & \frac{3E_i^C B_i^C (H_i^C)^2}{L_i^3} & \frac{3E_i^C B_i^C (H_i^C)^2}{2L_i^2} & 0 & \frac{-3E_i^C B_i^C (H_i^C)^2}{L_i^3} & \frac{3E_i^C B_i^C (H_i^C)^2}{2L_i^2} \\ 0 & \frac{3E_i^C B_i^C (H_i^C)^2}{2L_i^2} & \frac{E_i^C B_i^C (H_i^C)^2}{L_i} & 0 & \frac{-3E_i^C B_i^C (H_i^C)^2}{2L_i^2} & \frac{E_i^C B_i^C (H_i^C)^2}{2L_i} \\ \frac{-E_i^C B_i^C}{L_i} & 0 & 0 & \frac{E_i^C B_i^C}{L_i} & 0 & 0 \\ 0 & \frac{-3E_i^C B_i^C (H_i^C)^2}{L_i^3} & \frac{-3E_i^C B_i^C (H_i^C)^2}{2L_i^2} & 0 & \frac{3E_i^C B_i^C (H_i^C)^2}{L_i^3} & \frac{-3E_i^C B_i^C (H_i^C)^2}{2L_i^2} \\ 0 & \frac{3E_i^C B_i^C (H_i^C)^2}{2L_i^2} & \frac{E_i^C B_i^C (H_i^C)^2}{2L_i} & 0 & \frac{-3E_i^C B_i^C (H_i^C)^2}{2L_i^2} & \frac{E_i^C B_i^C (H_i^C)^2}{L_i} \end{bmatrix} \tag{27}$$

$$\frac{\partial \boldsymbol{M}_i(\rho_i^C, B_i^C, H_i^C)}{\partial \rho_i} = B_i^C H_i^C L_i \times \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & \frac{13}{35} & \frac{11L_i}{210} & 0 & \frac{9}{70} & \frac{-13L_i}{420} \\ 0 & \frac{11L_i}{210} & \frac{L_i^2}{105} & 0 & \frac{13L_i}{420} & \frac{-L_i^2}{140} \\ \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{9}{70} & \frac{13L_i}{420} & 0 & \frac{13}{35} & \frac{-11L_i}{210} \\ 0 & \frac{-13L_i}{420} & \frac{-L_i^2}{140} & 0 & \frac{-11L_i}{210} & \frac{L_i^2}{105} \end{bmatrix} \tag{28}$$

$$\frac{\partial \boldsymbol{M}_i(\rho_i^C, B_i^C, H_i^C)}{\partial B_i} = \rho_i^C H_i^C L_i \times \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & \frac{13}{35} & \frac{11L_i}{210} & 0 & \frac{9}{70} & \frac{-13L_i}{420} \\ 0 & \frac{11L_i}{210} & \frac{L_i^2}{105} & 0 & \frac{13L_i}{420} & \frac{-L_i^2}{140} \\ \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{9}{70} & \frac{13L_i}{420} & 0 & \frac{13}{35} & \frac{-11L_i}{210} \\ 0 & \frac{-13L_i}{420} & \frac{-L_i^2}{140} & 0 & \frac{-11L_i}{210} & \frac{L_i^2}{105} \end{bmatrix} \tag{29}$$

$$\frac{\partial \boldsymbol{M}_i(\rho_i^C, B_i^C, H_i^C)}{\partial H_i} = \rho_i^C B_i^C L_i \times \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & \frac{13}{35} & \frac{11L_i}{210} & 0 & \frac{9}{70} & \frac{-13L_i}{420} \\ 0 & \frac{11L_i}{210} & \frac{L_i^2}{105} & 0 & \frac{13L_i}{420} & \frac{-L_i^2}{140} \\ \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{9}{70} & \frac{13L_i}{420} & 0 & \frac{13}{35} & \frac{-11L_i}{210} \\ 0 & \frac{-13L_i}{420} & \frac{-L_i^2}{140} & 0 & \frac{-11L_i}{210} & \frac{L_i^2}{105} \end{bmatrix} \tag{30}$$

Generally speaking, the interval stiffness matrix and mass matrix of the element of the structures with interval parameters can be respectively expressed as

$$\boldsymbol{K}_i(\alpha^I) = \boldsymbol{K}_i(\alpha^C) + \sum_{j=1}^{m} \left(\frac{\partial \boldsymbol{K}_i}{\partial \alpha_j} \Delta \alpha_j e_{ij} \right) \tag{31}$$

$$\boldsymbol{M}_i(\alpha^I) = \boldsymbol{M}_i(\alpha^C) + \sum_{j=1}^{m} \left(\frac{\partial \boldsymbol{M}_i}{\partial \alpha_j} \Delta \alpha_j e_{ij} \right) \tag{32}$$

where

$$\alpha^I = [\alpha^C - \boldsymbol{\Delta}\alpha,\ \alpha^C + \boldsymbol{\Delta}\alpha]$$

$$\alpha^C = (\alpha_1^C,\ \alpha_2^C \ldots,\ \alpha_m^C)$$

$$\boldsymbol{\Delta}\alpha = (\boldsymbol{\Delta}\alpha_1, \boldsymbol{\Delta}\alpha_2 \ldots, \boldsymbol{\Delta}\alpha_m),\ \ e_{ij} = [-1,\ 1]$$

The stiffness matrix and mass matrix of the structure are assembled by using the element stiffness matrix

$$\boldsymbol{K}(\alpha) = \sum_{i=1}^{n} \boldsymbol{K}_i(\alpha) \tag{33}$$

$$\boldsymbol{M}(\alpha) = \sum_{i=1}^{n} \boldsymbol{M}_i(\alpha) \tag{34}$$

Using the natural interval extension, and substituting the Eqs. (31) and (32) into the Eqs. (33) and (34), we can obtain

$$\begin{aligned} \boldsymbol{K}(\alpha^I) &= \sum_{i=1}^{n} \boldsymbol{K}_i(\alpha^I) \\ &= \sum_{i=1}^{n} \left[\boldsymbol{K}_i(\alpha^C) + \sum_{j=1}^{m} \left(\frac{\partial \boldsymbol{K}_i}{\partial \alpha_j} \Delta\alpha_j e_{ij} \right) \right] \\ &= \boldsymbol{K}(\alpha^C) + \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\frac{\partial \boldsymbol{K}_i}{\partial \alpha_j} \Delta\alpha_j e_{ij} \right) \end{aligned} \tag{35}$$

$$\begin{aligned} \boldsymbol{M}(\alpha^I) &= \sum_{i=1}^{n} \boldsymbol{M}_i(\alpha^I) \\ &= \sum_{i=1}^{n} \left[\boldsymbol{M}_i(\alpha^C) + \sum_{j=1}^{m} \left(\frac{\partial \boldsymbol{M}_i}{\partial \alpha_j} \Delta\alpha_j e_{ij} \right) \right] \\ &= \boldsymbol{M}(\alpha^C) + \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\frac{\partial \boldsymbol{M}_i}{\partial \alpha_j} \Delta\alpha_j e_{ij} \right) \end{aligned} \tag{36}$$

The eigenvalue equation in the finite element system becomes

$$\boldsymbol{K}(\alpha)\boldsymbol{u} = \lambda \boldsymbol{M}(\alpha)\boldsymbol{u} \tag{37}$$

subject to

$$\underline{\alpha} \le \alpha \le \overline{\alpha}$$

where $\boldsymbol{K}(\alpha)$ and $\boldsymbol{M}(\alpha)$ are, respectively, the stiffness matrix and mass matrix of $n \times n$ order, λ and $\boldsymbol{u}$ are, respectively, the eigenvalue and eigenvector, α is the vector of the interval structural parameters. Using the natural interval extension, we have

$$\boldsymbol{K}(\alpha^I)\boldsymbol{u} = \lambda \boldsymbol{M}(\alpha^I)\boldsymbol{u} \tag{38}$$

One views

$$\Delta \boldsymbol{K}^I = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\frac{\partial \boldsymbol{K}_i}{\partial \alpha_j} \Delta\alpha_j e_{ij} \right)$$

and

$$\Delta \boldsymbol{M}^I = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\frac{\partial \boldsymbol{M}_i}{\partial \alpha_j} \Delta\alpha_j e_{ij} \right)$$

as an interval perturbation around $\boldsymbol{K}^C$ and $\boldsymbol{M}^C$, respectively. From the matrix perturbation as given in the above section, we can obtain the interval extension of the eigenvalue

$$\lambda_k^I = \lambda_{k0} + \Delta\lambda_k^I \tag{39}$$

where

$$\Delta\lambda_k^I = \lambda_{k1}^I = [\Delta\underline{\lambda}_k, \Delta\overline{\lambda}_k] \tag{40}$$

$k = 1, 2, \cdots, p.$

$$\begin{aligned}\lambda_{k1}^I &= (\boldsymbol{u}_{k0})^T \Delta\boldsymbol{K}^I \boldsymbol{u}_{k0} - \lambda_{k0}(\boldsymbol{u}_{k0})^T \Delta\boldsymbol{M}^I \boldsymbol{u}_{k0} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} (\bar{\boldsymbol{u}}_{k0}^i)^T \frac{\partial \boldsymbol{K}_i}{\partial \alpha_j} \Delta\alpha_j \bar{\boldsymbol{u}}_{k0}^i e_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_{k0} (\bar{\boldsymbol{u}}_{k0}^i)^T \frac{\partial \boldsymbol{M}_i}{\partial \alpha_j} \Delta\alpha_j \bar{\boldsymbol{u}}_{k0}^i e_{ij} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \left[(\bar{\boldsymbol{u}}_{k0}^i)^T \frac{\partial \boldsymbol{K}_i}{\partial \alpha_j} \Delta\alpha_j \bar{\boldsymbol{u}}_{k0}^i - \lambda_{k0} (\bar{\boldsymbol{u}}_{k0}^i)^T \frac{\partial \boldsymbol{M}_i}{\partial \alpha_j} \Delta\alpha_j \bar{\boldsymbol{u}}_{k0}^i \right] e_{ij} \\ &= \left[\sum_{i=1}^{n}\sum_{j=1}^{m} \left| (\bar{\boldsymbol{u}}_{k0}^i)^T \frac{\partial \boldsymbol{K}_i}{\partial \alpha_j} \Delta\alpha_j \bar{\boldsymbol{u}}_{k0}^i - \lambda_{k0} (\bar{\boldsymbol{u}}_{k0}^i)^T \frac{\partial \boldsymbol{M}_i}{\partial \alpha_j} \Delta\alpha_j \bar{\boldsymbol{u}}_{k0}^i e_{ij} \right| \right] [-1, 1] \end{aligned} \tag{41}$$

$$\Delta\underline{\lambda}_k = -\left[\sum_{i=1}^{n}\sum_{j=1}^{m} \left| (\bar{\boldsymbol{u}}_{k0}^i)^T \frac{\partial \boldsymbol{K}_i}{\partial \alpha_j} \Delta\alpha_j \bar{\boldsymbol{u}}_{k0}^i - \lambda_{k0} (\bar{\boldsymbol{u}}_{k0}^i)^T \frac{\partial \boldsymbol{M}_i}{\partial \alpha_j} \Delta\alpha_j \bar{\boldsymbol{u}}_{k0}^i \right| \right] \tag{42}$$

$$\Delta\overline{\lambda}_k = \left[\sum_{i=1}^{n}\sum_{j=1}^{m} \left| (\bar{\boldsymbol{u}}_{k0}^i)^T \frac{\partial \boldsymbol{K}_i}{\partial \alpha_j} \Delta\alpha_j \bar{\boldsymbol{u}}_{k0}^i - \lambda_{k0} (\bar{\boldsymbol{u}}_{k0}^i)^T \frac{\partial \boldsymbol{M}_i}{\partial \alpha_j} \Delta\alpha_j \bar{\boldsymbol{u}}_{k0}^i \right| \right] \tag{43}$$

Therefore, the upper and lower bounds of eigenvalues, $\Delta\underline{\lambda}_k$ and $\Delta\overline{\lambda}_k$, are given by

$$\underline{\lambda}_k = \lambda_{k0} + \Delta\underline{\lambda}_k \tag{44}$$

and

$$\overline{\lambda}_k = \lambda_{k0} + \Delta\overline{\lambda}_k \tag{45}$$

It should be noted that in the above formulas (42) and (43), the calculations for $\Delta\underline{\lambda}_k$ and $\Delta\overline{\lambda}_k$ are on the element basis, hence, the calculations are greatly simplified, and the over bar signifies that the eigenvector, $\bar{\boldsymbol{u}}_{k0}$, contains only the components needed for the *i*th element.

5. Numerical examples

In this section, two numerical examples will be given to illustrate that the proposed method is effective for estimating the eigenvalue bounds of structures with interval parameters.

5.1 Example 1

Consider a continuous beam system shown in Fig. 1. The finite element model of the given structure consists of 7 nodes and 6 elements. We suppose that the height and width of the cross-

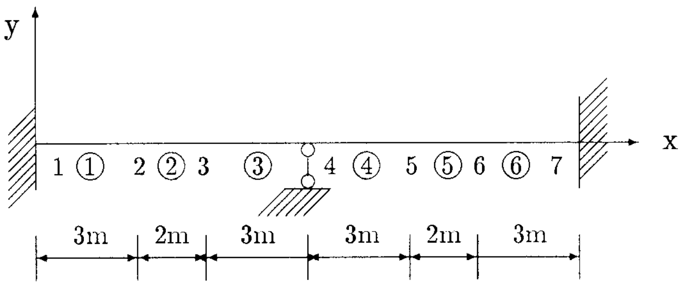

Fig. 1 Continuous beam system with 7 nodes and 6 elements

section are 0.3 m and 0.2 m respectively; the Young's modulus of the element is $E_0 = 2.1\times10^{11}$ N/m^2; the mass density for all the elements is $\rho = 7.8\times10^3$ kg/m^3. The analysis results obtained by Chen and Qiu (1994) are listed in Tables 1a, 2a and 3a; the results obtained by the present method are listed in Tables 1b, 2b and 3b. In the tables, i is the number of elements; k is the number of modes; $\underline{\lambda}_k$ is the lower bound of the kth eigenvalue; λ_k is the kth eigenvalue; $\overline{\lambda}_k$ is the upper bound of the kth eigenvalue; and $\Delta\lambda_k$ is the interval half width of the kth eigenvalue.

From the results listed in Tables, it can be seen that the present method is more available than that

Table 1a The lower and upper bounds of eigenvalues obtained by Chen and Qiu (1994) $\left(\Delta\rho_i = \frac{1}{100}\rho_i\right)$

k	$\underline{\lambda}_k$	λ_k	$\overline{\lambda}_k$	$\Delta\lambda_k/\Delta\rho_1$
1	0.827031E+05	0.836168E+05	0.845306E+05	0.117148E+02
2	0.174383E+06	0.176401E+06	0.178420E+06	0.258739E+02
3	0.920498E+06	0.931264E+06	0.942030E+06	0.138025E+03
4	0.516086E+07	0.525109E+07	0.534132E+07	0.115682E+04
5	0.143194E+07	0.145010E+07	0.146826E+07	0.232816E+03
6	0.849903E+07	0.858800E+07	0.867698E+07	0.114070E+04
7	0.192414E+08	0.199215E+08	0.206016E+08	0.871918E+04
8	0.629607E+08	0.677518E+08	0.725428E+08	0.614235E+05
9	0.439905E+08	0.459903E+08	0.479901E+08	0.256382E+05

Table 1b The lower and upper bounds of eigenvalues obtained by the present method $\left(\Delta\rho_i = \frac{1}{100}\rho_i\right)$

k	$\underline{\lambda}_k$	λ_k	$\overline{\lambda}_k$	$\Delta\lambda_k/\Delta\rho_1$
1	0.827806E+05	0.836168E+05	0.844530E+05	0.107201E+02
2	0.174637E+06	0.176401E+06	0.178165E+06	0.226156E+02
3	0.921951E+06	0.931264E+06	0.940576E+06	0.119393E+03
4	0.519858E+07	0.525109E+07	0.530360E+07	0.673217E+03
5	0.143560E+07	0.145010E+07	0.146460E+07	0.185910E+03
6	0.850212E+07	0.858800E+07	0.867388E+07	0.110103E+04
7	0.197223E+08	0.199215E+08	0.201207E+08	0.255404E+04
8	0.670742E+08	0.677518E+08	0.684293E+08	0.868612E+04
9	0.455304E+08	0.459903E+08	0.464502E+08	0.589619E+04

Table 2a The lower and upper bounds of eigenvalues obtained by Chen and Qiu (1994) $\left(\Delta B_i = \frac{5}{100} B_i\right)$

k	$\underline{\lambda}_k$	λ_k	$\overline{\lambda}_k$	$\Delta\lambda_k/\Delta B_1$
1	−0.844067E+05	0.836168E+05	0.251640E+06	0.672094E+07
2	−0.115475E+05	0.176401E+06	0.364350E+06	0.751796E+07
3	0.610549E+06	0.931264E+06	0.125198E+07	0.128286E+08
4	0.408864E+07	0.525109E+07	0.641354E+07	0.464980E+08
5	0.108613E+07	0.145010E+07	0.181406E+07	0.145587E+08
6	0.732812E+07	0.858800E+07	0.984788E+07	0.503952E+08
7	0.151984E+08	0.199215E+08	0.246446E+08	0.188923E+09
8	0.403200E+08	0.677518E+08	0.951835E+08	0.109727E+10
9	0.336919E+08	0.459903E+08	0.582887E+08	0.491937E+09

Table 2b The lower and upper bounds of eigenvalues obtained by the present method $\left(\Delta B_i = \frac{5}{100} B_i\right)$

k	$\underline{\lambda}_k$	λ_k	$\overline{\lambda}_k$	$\Delta\lambda_k/\Delta B_1$
1	0.752551E+05	0.836168E+05	0.919785E+05	0.334467E+06
2	0.158761E+06	0.176401E+06	0.194042E+06	0.705605E+06
3	0.838137E+06	0.931264E+06	0.102439E+07	0.372505E+07
4	0.472598E+07	0.525109E+07	0.577620E+07	0.210044E+08
5	0.130509E+07	0.145010E+07	0.159511E+07	0.580039E+07
6	0.772920E+07	0.858800E+07	0.944680E+07	0.343520E+08
7	0.179293E+08	0.199215E+08	0.219136E+08	0.796860E+08
8	0.609766E+08	0.677518E+08	0.745269E+08	0.271007E+09
9	0.413913E+08	0.459903E+08	0.505893E+08	0.183961E+09

Table 3a The lower and upper bounds of eigenvalues obtained by Chen and Qiu (1994) $\left(\Delta H_i = \frac{10}{100} H_i\right)$

k	$\underline{\lambda}_k$	λ_k	$\overline{\lambda}_k$	$\Delta\lambda_k/\Delta H_1$
1	−0.906249E+06	0.836168E+05	0.107348E+07	0.123733E+08
2	−0.910929E+06	0.176401E+06	0.126373E+07	0.135916E+08
3	−0.777705E+06	0.931264E+06	0.264023E+07	0.213621E+08
4	0.810320E+05	0.525109E+07	0.104211E+08	0.646257E+08
5	−0.370507E+06	0.145010E+07	0.327070E+07	0.227576E+08
6	0.280822E+07	0.858800E+07	0.143678E+08	0.722473E+08
7	0.518493E+07	0.199215E+08	0.346580E+08	0.184207E+09
8	−0.101795E+07	0.677518E+08	0.136521E+09	0.859621E+09
9	0.121954E+08	0.459903E+08	0.797852E+08	0.422436E+09

of Chen and Qiu (1994) for estimating the lower and upper bounds of eigenvalues, for example, if $\Delta B_i = (5/100)\,B_i$, the lower and upper bounds of λ_1 and λ_2 are respectively $\underline{\lambda}_1 = -0.844067E+05$, $\overline{\lambda}_1 = 0.251640E+06$, $\underline{\lambda}_2 = -0.115475E+05$, $\overline{\lambda}_2 = 0.364350E+06$ obtained by Chen and Qiu (1994), and the corresponding results obtained by the present method are $\underline{\lambda}_1 = 0.752551E+05$, $\overline{\lambda}_1 =$

Table 3b The lower and upper bounds of eigenvalues obtained by the present method $\left(\Delta H_i = \frac{10}{100}H_i\right)$

k	$\underline{\lambda}_k$	λ_k	$\bar{\lambda}_k$	$\Delta\lambda_k/\Delta H_1$
1	0.501701E+05	0.836168E+05	0.117064E+06	0.418084E+06
2	0.105841E+06	0.176401E+06	0.246962E+06	0.882007E+06
3	0.558758E+06	0.931264E+06	0.130377E+07	0.465632E+07
4	0.315065E+07	0.525109E+07	0.735153E+07	0.262555E+08
5	0.870059E+06	0.145010E+07	0.203014E+07	0.725049E+07
6	0.515280E+07	0.858800E+07	0.120232E+08	0.429400E+08
7	0.119529E+08	0.199215E+08	0.278901E+08	0.996074E+08
8	0.406511E+08	0.677518E+08	0.948525E+08	0.338759E+09
9	0.275942E+08	0.459903E+08	0.643864E+08	0.229952E+09

$0.919785E+05$, $\underline{\lambda}_2 = 0.158761E+06$, $\bar{\lambda}_2 = 0.194042E+06$. These results indicate that if the interval range of the structural parameters is large, the method proposed by Chen and Qiu (1994) can not be used and the present method can be still used to estimate the lower and upper bounds of eigenvalues of structures with interval parameters. The reason is that the $\Delta\boldsymbol{K}$ is the increment of the global stiffness matrix in the approach presented by Chen and Qiu (1994), and the corresponding calculations of the present method are done on the element basis, thus simplifying the complex interval algorithm and ensuring the reliability of the algorithm.

5.2 Example 2

Fig. 2 is an undamping spring-mass system with 5 nodes and 5 degrees of freedom. Its physical parameters are as follows: $m_1^C = 30.0$ kg, $m_2^C = 27.0$ kg, $m_3^C = 27.0$ kg, $m_4^C = 25.0$ kg, $m_5^C = 18.0$ kg; $k_1^C = 2010.0$ N/m, $k_2^C = 1825.0$ N/m, $k_3^C = 1615.0$ N/m, $k_4^C = 1410.0$ N/m, $k_5^C = 1205.0$ N/m. The uncertain parameters are in the following:

Case I

$m_j^I = m_j^C + 1.0 \times e_\Delta$, $k_1^I = k_1^C + 10.0 \times e_\Delta$, $k_2^I = k_2^C + 25.0 \times e_\Delta$, $k_3^I = k_3^C + 15.0 \times e_\Delta$, $k_4^I = k_4^C + 10.0 \times e_\Delta$, $k_5^I = k_5^C + 5.0 \times e_\Delta$

Case II

$m_j^I = m_j^C + \beta m_j^C e_\Delta$, $k_j^I = k_j^C + \beta k_j^C e_\Delta$

where $e_\Delta = [-1, 1]$, β is a small parameter. The results obtained by Qiu *et al.* (1999) are listed in Tables 4a, 5a, 6a and 7a; and the corresponding ones obtained by the present method are listed in Tables 4b, 5b, 6b and 7b. Where k is the number of modes; $\underline{\lambda}_k$ is the lower bound of the kth

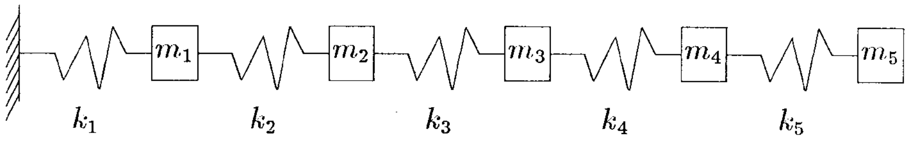

Fig. 2 Undamping spring-mass system with 5 nodes and 5 degrees of freedom

Table 4a The lower and upper bounds of eigenvalues obtained by the Qiu *et al*. (1999) (Case I)

k	$\underline{\lambda}_k$	$\overline{\lambda}_k$	$w(\overline{\lambda}_k - \underline{\lambda}_k)$
1	0.585810E+01	0.650204E+01	0.643932E+00
2	0.420293E+02	0.463088E+02	0.427947E+01
3	0.988564E+02	0.108689E+03	0.983262E+01
4	0.158052E+03	0.173778E+03	0.157262E+02
5	0.209514E+03	0.230084E+03	0.205696E+02

Table 4b The lower and upper bounds of eigenvalues obtained by the present method (Case I)

k	$\underline{\lambda}_k$	$\overline{\lambda}_k$	$w(\overline{\lambda}_k - \underline{\lambda}_k)$
1	0.584486E+01	0.648760E+01	0.642740E+00
2	0.419423E+02	0.462137E+02	0.427141E+01
3	0.986593E+02	0.108475E+03	0.981541E+01
4	0.157740E+03	0.173442E+03	0.157013E+02
5	0.209149E+03	0.229691E+03	0.205423E+02

Table 5a The lower and upper bounds of eigenvalues obtained by the Qiu *et al.* (1999) (Case II, $\beta = 0.05$)

k	$\underline{\lambda}_k$	$\overline{\lambda}_k$	$w(\overline{\lambda}_k - \underline{\lambda}_k)$
1	0.557897E+01	0.681531E+01	0.123634E+01
2	0.398801E+02	0.487178E+02	0.883769E+01
3	0.937035E+02	0.114469E+03	0.207653E+02
4	0.149820E+03	0.183021E+03	0.332011E+02
5	0.198523E+03	0.242517E+03	0.439941E+02

Table 5b The lower and upper bounds of eigenvalues obtained by the present method (Case II, $\beta = 0.05$)

k	$\underline{\lambda}_k$	$\overline{\lambda}_k$	$w(\overline{\lambda}_k - \underline{\lambda}_k)$
1	0.554961E+01	0.678286E+01	0.123325E+01
2	0.396702E+02	0.484858E+02	0.881561E+01
3	0.932103E+02	0.113924E+03	0.207134E+02
4	0.149032E+03	0.182150E+03	0.331182E+02
5	0.197478E+03	0.241362E+03	0.438839E+02

Table 6a The lower and upper bounds of eigenvalues obtained by the Qiu *et al.* (1999) (Case II, $\beta = 0.10$)

k	$\underline{\lambda}_k$	$\overline{\lambda}_k$	$w(\overline{\lambda}_k - \underline{\lambda}_k)$
1	0.504510E+01	0.753651E+01	0.249141E+01
2	0.360638E+02	0.538731E+02	0.178093E+02
3	0.847367E+02	0.126582E+03	0.418453E+02
4	0.135484E+03	0.202389E+03	0.669054E+02
5	0.179525E+03	0.268180E+03	0.886544E+02

Table 6b The lower and upper bounds of eigenvalues obtained by the present method (Case II, $\beta = 0.10$)

k	$\underline{\lambda}_k$	$\bar{\lambda}_k$	$w(\bar{\lambda}_k - \underline{\lambda}_k)$
1	0.493299E+01	0.739948E+01	0.246649E+01
2	0.352624E+02	0.528936E+02	0.176312E+02
3	0.828536E+02	0.124280E+03	0.414268E+02
4	0.132473E+03	0.198709E+03	0.662364E+02
5	0.175536E+03	0.263304E+03	0.877679E+02

Table 7a The lower and upper bounds of eigenvalues obtained by the Qiu *et al.* (1999) (Case II, $\beta = 0.15$)

k	$\underline{\lambda}_k$	$\bar{\lambda}_k$	$w(\bar{\lambda}_k - \underline{\lambda}_k)$
1	0.455765E+01	0.834255E+01	0.378490E+01
2	0.325794E+02	0.596350E+02	0.270556E+02
3	0.765496E+02	0.140120E+03	0.635706E+02
4	0.122393E+03	0.224035E+03	0.101642E+03
5	0.162180E+03	0.296862E+03	0.134682E+03

Table 7b The lower and upper bounds of eigenvalues obtained by the present method (Case II, $\beta = 0.15$)

k	$\underline{\lambda}_k$	$\bar{\lambda}_k$	$w(\bar{\lambda}_k - \underline{\lambda}_k)$
1	0.431636E+01	0.801610E+01	0.369974E+01
2	0.308546E+02	0.573014E+02	0.264468E+02
3	0.724969E+02	0.134637E+03	0.621402E+02
4	0.115914E+03	0.215268E+03	0.993546E+02
5	0.153594E+03	0.285246E+03	0.131652E+03

eigenvalue; $\bar{\lambda}_k$ is the upper bound of the kth eigenvalue; and $w(\bar{\lambda}_k - \underline{\lambda}_k)$ is the interval width of the kth eigenvalue.

From the results listed in tables, it can be seen that the present method is the same effective as one in Qiu *et al.* (1999) for estimating the lower and upper bounds of eigenvalues when the interval range of the structural parameters is not large. The results indicate that the proposed method provides safe bounds for the eigenfrequencies.

6. Conclusions

Using the interval analysis method, a new method, the interval finite element method, which is on the element basis and matrix perturbation, to solve uncertain eigenvalue problems of the structures with interval parameters is presented. The calculations of the present method are done on the element basis, thus, the calculations are greatly simplified and the reliability of the algorithm is ensured. The present method has been applied to a continuous beam and an undamping spring-mass system with interval parameters. The numerical results show that the proposed approach is more effective than that of Chen and Qiu (1994) and the same effective as one of Qiu *et al.* (1999). It should be noted that although the method is based on the beam structures, if one replace matrix differentiation with matrix sensitivity, it can be used to estimate the lower and upper bounds of the eigenvalues of the plates and shells.

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