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Analysis of slender structural elements under unilateral contact constraints

Ricardo Azoubel da Mota Silveira†

Civil Engineering Department, Federal University of Ouro Preto (UFOP)
Morro do Cruzeiro-Campus Universitário, 35400-000 Ouro Preto, MG, Brazil

Paulo Batista Gonçalves‡

Civil Engineering Department, Catholic University, PUC-Rio
Rua Marquês de São Vicente, 225 - Gávea, 22453-900 Rio de Janeiro, RJ, Brazil

Abstract. A numerical methodology is presented in this paper for the geometrically non-linear analysis of slender uni-dimensional structural elements under unilateral contact constraints. The finite element method together with an updated Lagrangian formulation is used to study the structural system. The unilateral constraints are imposed by tensionless supports or foundations. At each load step, in order to obtain the contact regions, the equilibrium equations are linearized and the contact problem is treated directly as a minimisation problem with inequality constraints, resulting in a linear complementarity problem (LCP). After the resulting LCP is solved by Lemke's pivoting algorithm, the contact regions are identified and the Newton-Raphson method is used together with path following methods to obtain the new contact forces and equilibrium configurations. The proposed methodology is illustrated by two examples and the results are compared with numerical and experimental results found in literature.

Key words: unilateral constraints; incremental-iterative strategies; geometric non-linearity; updated Lagrangian formulation; linear complementary problem.

1. Introduction

Recent developments in structural materials, more refined design methodologies and the large amount of research on stability of structures have led to increasingly slender structural elements whose analysis necessitates a truly non-linear approach due to the presence of geometric non-linearities. These systems may exhibit multiple solutions and may loose their stability due to bifurcation or the existence of limit points along the non-linear equilibrium path.

The knowledge of the non-linear behaviour of slender structural elements, such as columns, rings and arches, is essential in the local or global stability analysis of complex structural systems. In many engineering applications these structural elements are subjected to unilateral contact constraints induced by discrete or continuous supports. In some cases the deflection is prevented from the beginning by these constraints, while in other circumstances the constraints are reached for

† Ph.D.
‡ Professor

the first time during the post-buckling process.

There are many circumstances where the inclusion of unilateral constraints in the stability analysis of structural elements is unavoidable and their presence may change substantially the non-linear behaviour of the structure and its stability characteristics (Pian *et al.* 1967, Stein and Wriggers 1984, Adan *et al.* 1994, Koo and Kwak 1995, Silveira 1995, Givoli and Doukhovni 1996, Holmes *et al.* 1999, Silveira and Gonçalves 2000). This is particularly true in the analysis of some types of foundations, railway tracks, risers and cables used in off-shore engineering, metal liners used to protect relatively stiff structures, tanks for solid-propellants and in the analysis of composite laminate structures (delamination problems). This is a non-classical type of stability problem, which is termed contact problem. Even in the range of small deformations and under linear elastic behaviour of the material, unilateral constraints introduce high non-linearities, which cannot be dealt satisfactorily by usual non-linear structural analysis methods. So, in order to study the behaviour of slender structural elements with unilateral constraints both types of non-linearities must be taken into account and a reliable and efficient solution method is necessary. The complexity of this particular class of structural problems may explain the relatively small number of papers on this subject. For a few problems involving simple geometries and loading, simplified modal solutions are possible (Adan *et al.* 1994, Holmes *et al.* 1999, Silveira and Gonçalves 2000). For more general problems where the number and location of the contact regions are not known a priori, the use of numerical techniques are usually necessary. Among various numerical techniques, the finite element method has been shown to be a very efficient tool for the analysis of complex contact problems. In recent years FE formulations for the analysis of contact problems were presented by, among others, Simo *et al.* 1986, Klarbring 1986, Joo and Kwak 1986, Belytschko and Neal 1991, Mottershead *et al.* 1992, Wriggers and Inhof 1993, Koo and Kwak 1995, 1996, Sun and Natori 1996 and Givoli and Doukhovni 1996. Additionally, efficient algorithms capable of dealing with various types of non-linearities were discussed by Endo *et al.* 1984, Nour-Omid and Wriggers 1986 and Björkman *et al.* 1995.

The finite element method has also shown to be particularly appropriate for the analysis of non-linear structural problems. The discretization process of non-linear structures by the use of finite elements leads to a system of non-linear algebraic equations that are often solved by Newton-type methodologies. Ideally, a solution method should be able to trace the entire equilibrium path, including softening and stiffening branches, and identify load or displacement limit points and bifurcations of the fundamental path. At present, the most efficient methods are the so-called incremental-iterative strategies.

A critical step in the analysis of contact problem is the selection of a numerical methodology for dealing with the contact constraints. Basically there are three major numerical approaches for this problem, namely the Lagrange multiplier method (Belytschko and Neal 1991, Mottershead *et al.* 1992), the Penalty method (Simo *et al.* 1986, Wriggers and Inhof 1993) and the Mathematical programming methods (Luenberger 1973, Joo and Kwak 1986, Klarbring 1986, Björkman *et al.* 1995, Koo and Kwak 1995, 1996, Sun and Natori 1996, Givoli and Doukhovni 1996). This last alternative enables one to solve the contact problem by directly minimising the potential energy containing explicitly moving boundary parameters and the associated inequality constraints and thus maintaining the original mathematical characteristics of the problem. Some of the optimizations techniques used for the contact problem are linear and quadratic programming, recursive quadratic programming or, alternatively, methods for the solution of linear complementary problems (LCP) such as the Dantzig's or Lemke's algorithms (Lemke 1968). The works by Koo and Kwak (1995,

1996) and Sun and Natori (1996) are examples of numerical investigations where LCPs are used to investigate buckling and post-buckling problems with contact constraints.

The aim of the present work is to develop a numerical methodology for the geometrically non-linear analysis of uni-dimensional slender structural elements with unilateral contact constraints. The finite element method together with an updated Lagrangian formulation is used for the beam. In order to solve the resulting algebraic non-linear equations and obtain non-linear equilibrium paths, the Newton-Raphson method is used together with an arc-length iteration procedure (Crisfield 1991, 1997). This incremental-iterative strategy allows limit points to be passed and, consequently, snap buckling phenomena to be identified. Additionally, the use of very small random nodal imperfections enables one to identify bifurcations points and the associated post-bifurcation solution. The unilateral contact is due to the presence of tensionless supports and foundations. The influence of friction in the contact area is neglected in this paper. At each load step, in order to obtain the contact regions, the equilibrium equations are linearized and the contact problem is treated directly as a minimisation problem, involving only the original variables, subjected to inequality constraints. Then, the resulting linear programming problem is solved by the Lemke's algorithm, the contact regions are identified and the Newton-Raphson method is used to obtain the new contact forces and equilibrium configurations. At this point the constraint equations are checked. If they are satisfied the optimum solution has been obtained; otherwise the procedure is repeated and improved contact regions are identified.

In view of the complexity of buckling and post-buckling analysis of structural elements with contact constraints, validation of the adopted model is an essential task. Here, the local one-way buckling of rigidly confined rings under inertial loading and the buckling and post-buckling behaviour of a column under contact constraint are analysed and compared with results found in literature (Pian *et al*. 1967, Stein and Wriggers 1984, Adan *et al*. 1994). These results demonstrate the accuracy and versatility of the present methodology in the solution of stability problems with unilateral constraints.

2. Problem formulation

Consider the structural system shown in Fig. 1 consisting of an elastic body and an elastic tensionless foundation and assume that both bodies may undergo large deflections and rotations but small strains, within the elastic range of the material. Also, the contact surface is assumed unbonded and frictionless.

The structure is defined as a solid elastic continuum which occupies a domain ${}^{i}V$ ($i = 0$, ω and $\omega + \Delta\omega$). Its boundary ${}^{i}S$ is considered to be regular and composed of three different regions: ${}^{i}S_u$, ${}^{i}S_f$ and ${}^{i}S_c$, where the surface forces are specified on ${}^{i}S_f$ and the displacements are specified on ${}^{i}S_u$. The remaining part, ${}^{i}S_c$, corresponds to the region where contact is likely to occur, which is not known a priori.

Assume now that the kinematic and static variables are known for the equilibrium configurations 0, $\Delta\omega$, 2 $\Delta\omega$, ... and ω, and that the solution for the adjacent configuration $\omega + \Delta\omega$ is required. Since the relevant physical and geometrical variables are known in the last configuration ω, it's advisable to refer all these quantities to this configuration and use an updated Lagrangian formulation to study the structural problem. This, as it will be shown subsequently, is vital for the development of the proposed numerical methodology.

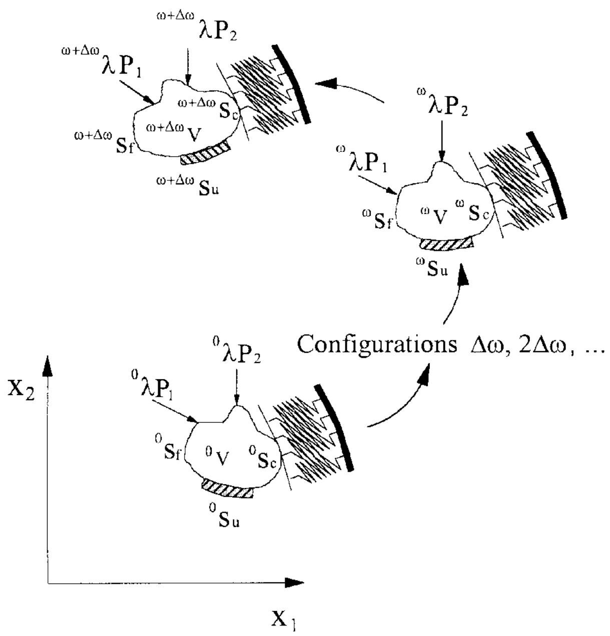

Fig. 1 Equilibrium configurations

For the structural system, the equilibrium equations, the kinematic relations and the constitutive law are given respectively by:

$$\Delta S_{ij,j}+(\Delta u_{i,j}\,{}^{\omega+\Delta\omega}S_{jk,i})_{,k}=0 \tag{1}$$

$$\Delta\varepsilon_{ij}=\frac{1}{2}(\Delta u_{i,j}+\Delta u_{j,i}+\Delta u_{k,i}\Delta u_{k,j}) \tag{2}$$

$$\Delta S_{ij}=C_{ijkl}\Delta\varepsilon_{kl} \tag{3}$$

where the customary summation convention is used. In Eq. (1), ΔS_{ij} are the components of the 2nd Piola-Kirchhoff stress increment tensor, the unknowns of the problem, and ${}^{\omega+\Delta\omega}S_{ij}$, are the Cartesian components of this same tensor at state $\omega+\Delta\omega$, which are measured from the previous equilibrium configuration ω. The stress components ΔS_{ij} and ${}^{\omega+\Delta\omega}S_{ij}$ are related by the equation:

$${}^{\omega+\Delta\omega}S_{ij}={}^{\omega}\tau_{ij}+\Delta S_{ij} \tag{4}$$

where ${}^{\omega}\tau_{ij}$ are the components of the Cauchy stress tensor at the reference configuration. In Eq. (1), the unknown increments in the displacements Δu_i from configurations ω to $\omega+\Delta\omega$ are defined as:

$$\Delta u_i={}^{\omega+\Delta\omega}x_i-{}^{\omega}x_i, \qquad i=1,\,2 \tag{5}$$

In many engineering applications, the designer is interested only on the response of the foundation at the contact area and not on the stresses and displacements inside the foundation. So it is possible to construct simple mathematical models for describing the response of the foundation at the contact zone with a reasonable degree of accuracy. Using the well-known Winkler model (Kerr 1964) or the formulation for an elastic half-space, the following constitutive equation can be written to describe the elastic foundation:

$$\Delta r_b = C_b \Delta u_b \tag{6}$$

where Δr_b and Δu_b are, respectively, the incremental compressive reaction and deflection of the foundation and C_b is the foundation elastic modulus.

For the two bodies under investigation, the following conditions must be satisfied on S_c:

1. The gap in the potential contact area, β, after the increment of the displacements, must satisfy the following inequality constraint at configuration $\omega + \Delta\omega$:

$$\beta \geq 0, \quad \text{on} \quad S_c \tag{7}$$

where

$$\beta = \psi - (\Delta u_i{}^{\omega+\Delta\omega} n_i + \Delta u_b{}^{\omega+\Delta\omega} n_b) \tag{8}$$

with ψ defining the gap between the bodies at configuration ω. Here n_i and n_b are respectively the unit outward normal vector on the structure and foundation boundaries;

2. Under the assumption of a tensionless foundation model, contact pressure must be compressive, i.e.:

$$\Delta r_b \geq 0 \tag{9}$$

3. The complementary relation between β and Δr_b is given by:

$$\int_{^{\omega+\Delta\omega}S_c} \Delta r_b \beta^{\omega+\Delta\omega} dS_c = 0 \tag{10}$$

For a given load increment, the solution of the unilateral contact problem can be obtained by solving Eqs. (1) and (6), together with relations (2) and (3), and by satisfying the appropriate boundary conditions on S_u and S_f, as well as the restrictions (7), (9) and (10) on S_c.

The non-linearity due to the unilateral constraints and the non-linear strain-displacement relations make it difficult to solve this problem directly. For this reason, an equivalent minimisation problem is formulated, which is particularly suitable for numerical analysis. It can be shown that the optimization's problem (Joo and Kwak 1986, Silveira 1995):

$$\text{Min } J\,(\Delta u, \Delta u_b) \tag{11}$$

$$\text{Subject to: } -\beta \leq 0\,, \quad \text{on} \quad S_c \tag{12}$$

where,

$$J = \int_{^{\omega}V} \left({}^{\omega}\tau_{ij} + \frac{1}{2}\Delta S_{ij}\right) \Delta\varepsilon_{ij}{}^{\omega} dV + \int_{^{\omega+\Delta\omega}S_c} \left({}^{\omega}r_b + \frac{1}{2}\Delta r_b\right) \Delta u_b{}^{\omega+\Delta\omega} dS_c - \int_{^{\omega+\Delta\omega}S_f} F_i \Delta u_i{}^{\omega+\Delta\omega} dS_f \tag{13}$$

is equivalent to the problem described above by Eqs. (1) to (10).

3. Numerical methodology

The numerical strategy adopted here for the minimisation problem described by Eqs. (11)-(13), has as starting point an approximate solution based on the known displacements and stresses obtained at conclusion of the previous load step, ω. It is assumed that perfect convergence has been achieved at the previous step so that the solution satisfies the equilibrium equations and all constraints. A cycle of the proposed incremental-iterative strategy can be summarised as follows.

Considering the previous updated Lagrangian solution, the initial increment of the load parameter $\Delta\lambda^0$ is selected and used to calculate the initial increment of the nodal displacements $\Delta\boldsymbol{u}^0$. The approximations $\Delta\lambda^0$ and $\Delta\boldsymbol{u}^0$ are termed here "tangent incremental solution". This is followed by two corrections:

1. The first correction deals only with the non-linearity associated with the unilateral constraints and is used to correct the dimension of the contact zone, which was assumed equal to ${}^{\omega}S_c$ at the previous step. After the solution of the standard linear complementary problem, an improved solution ${}^{\omega+\Delta\omega}S_c$ for the contact regions is obtained.
2. The second correction deals with the geometric non-linearity of the structural system. Here the Newton-Raphson method is used to solve the discretized equilibrium equations. This solution was obtained using the improved contact solution found in the previous step. Now a new contact region ${}^{\omega+\Delta\omega}S_c$ is obtained and compared with the solution obtained in the previous step. If the convergence criterion for the contact zone is not satisfied, a new incremental solution is obtained and the correction procedure is repeated until the convergence criteria are satisfied.

3.1 Solution procedure

The solution procedure summarised previously will be detailed herein. In the following, the left superscript c is the iteration counter associated with the contact problem and the right superscript k, the iteration counter used in the Newton-Raphson procedure. The tangent incremental solution, ${}^{c}\Delta\lambda^0$ and ${}^{c}\Delta\boldsymbol{u}^0$, corresponding to k=0 and k=1, 2, .., denotes successive iterative cycles; λ, $\Delta\lambda$, $\delta\lambda$ and $\boldsymbol{u}$, $\Delta\boldsymbol{u}$, $\delta\boldsymbol{u}$ are, respectively, the total, incremental and iterative load parameter and nodal displacements.

The first step begins with the computation of the tangent stiffness matrix ${}^{c}\boldsymbol{K}_{\boldsymbol{T}}$ of the beam-foundation system, based upon the known equilibrium solution and contact region of the previous load step. The tangent nodal displacements ${}^{c}\delta\boldsymbol{u}_{\boldsymbol{T}}$ are then computed as the solution of:

$$ {}^{c}\boldsymbol{K}_{\boldsymbol{T}}\,{}^{c}\delta\boldsymbol{u}_{\boldsymbol{T}}=\boldsymbol{R}_{\mathbf{ref}} \tag{14}$$

where $\boldsymbol{R}_{\mathbf{ref}}$ is the reference external load vector. This vector has an arbitrary amplitude, since only its direction is important at the present step.

After the value of the initial load increment is determined by the use of a particular load incrementation arc-length strategy (Crisfield 1991, 1997), the incremental displacements ${}^{c}\Delta\boldsymbol{u}^0$ are evaluated scaling the tangent displacements as follows:

$$ {}^{c}\Delta\boldsymbol{u}^0={}^{c}\Delta\lambda^0\;{}^{c}\delta\boldsymbol{u}_{\boldsymbol{T}} \tag{15}$$

At this stage, the load parameter and the total displacements are updated:

$$ {}^{c}\lambda^{0} = {}^{\omega}\lambda + {}^{c}\Delta\lambda^{0} \quad \text{and} \quad {}^{c}\boldsymbol{u}^{0} = {}^{\omega}\boldsymbol{u} + {}^{c}\Delta\boldsymbol{u}^{0} \tag{16} $$

Here, ${}^{\omega}\lambda$ are ${}^{\omega}\boldsymbol{u}$ the load and displacements obtained at the conclusion of the previous step. This solution rarely satisfies the equilibrium equations and the contact constraints and so additional iterative cycles are required.

Now the contact region ${}^{\omega}S_c$ must be corrected. To estimate the new region the following linear complementary problem (LCP) must be solved (Silva *et al.* 2001):

$$ \boldsymbol{K_T}\,{}^{c}\Delta\boldsymbol{u} + \boldsymbol{A}^{T}\,{}^{c}\Delta\boldsymbol{r_b} - \Delta\boldsymbol{R} = \boldsymbol{0} \tag{17} $$

$$ \boldsymbol{A}\,{}^{c}\Delta\boldsymbol{u} - \boldsymbol{T}\,{}^{c}\Delta\boldsymbol{r_b} \le \boldsymbol{0} \tag{18a} $$

$$ {}^{c}\Delta\boldsymbol{r_b} \ge \boldsymbol{0} \tag{18b} $$

$$ (\boldsymbol{A}\,{}^{c}\Delta\boldsymbol{u} - \boldsymbol{T}\,{}^{c}\Delta\boldsymbol{r_b})^{T}\,{}^{c}\Delta\boldsymbol{r_b} = 0 \tag{18c} $$

where, the Eq. (17) is the system of beam-foundation equilibrium equations while Eqs. (18a)-(18c) are the constraints that characterise the unilateral contact problem. The constraint (18a) represents, physically, the impenetrability condition between the bodies; (18b) defines the positivity condition of ${}^{c}\Delta\boldsymbol{r_b}$; and (18c) is the complementarity relation between the gap and ${}^{c}\Delta\boldsymbol{r_b}$. The vectors ${}^{c}\Delta\boldsymbol{u}$, ${}^{c}\Delta\boldsymbol{r_b}$ and $\Delta\boldsymbol{R}$ are, respectively, the incremental nodal displacements of the structure, the incremental nodal reaction of the elastic foundation and the incremental nodal load. The matrices $\boldsymbol{K_T}$, $\boldsymbol{T}$ and $\boldsymbol{A}$ represents, respectively, the stiffness matrix of the beam, the flexibility matrix of the elastic foundation and the joining matrix between the bodies (Ascione and Grimaldi 1984, Silva *et al.* 2001).

The solution of the Eq. (17), considering the constraints, is reached through the use of mathematical programming methods, in particular, pivoting techniques developed for complementary problems (Lemke 1968). Before the use of Lemke's algorithms, however, it is necessary to reduce the previous relationships to a "standard" form (${}^{c}\boldsymbol{w} = {}^{c}\boldsymbol{q} + {}^{c}\boldsymbol{M}\,{}^{c}\boldsymbol{z}$; ${}^{c}\boldsymbol{z} \ge \boldsymbol{0}$; ${}^{c}\boldsymbol{w} \ge \boldsymbol{0}$; ${}^{c}\boldsymbol{w}^{T}\ {}^{c}\boldsymbol{z} = 0$) as presented in Silveira (1995) and Silva *et al.* (2001), where alternative formulations for the contact problem are developed. The solution of any "standard" form of the LCP enables one to compute from its variables the new contact region, ${}^{\omega+\Delta\omega}S_c$.

Next, using the new contact region, the Newton-Raphson method is employed to restore equilibrium. If the non-linear equilibrium path is to be obtained and possible limit points are to be overcome, the load parameter must be allowed to vary whilst iterating to convergence. Following the general solution strategy initially proposed by Batoz and Dhatt (1979), one can compute the incremental change in the nodal displacements by solving the equation:

$$ {}^{c}\boldsymbol{K}_{\boldsymbol{T}}^{(k-1)}\,{}^{c}\delta\boldsymbol{u}^{k} = -{}^{c}\boldsymbol{g}(\boldsymbol{u}^{(k-1)}, \lambda^{k}), \qquad k \ge 1 \tag{19} $$

where ${}^{c}\boldsymbol{g}$ is a gradient vector to be minimised during the present iterative cycle. In Eq. (19), ${}^{c}\boldsymbol{g}$ is a function of the nodal displacements (calculated at the previous iteration) and of the current total load parameter, ${}^{c}\lambda^{k}$, which is an additional unknown. ${}^{c}\lambda^{k}$ can be written as:

$$ {}^{c}\lambda^{k} = {}^{c}\lambda^{(k-1)} + {}^{c}\delta\lambda^{k} \tag{20} $$

where $\delta\lambda^{k}$ is the desired correction or increment of the load parameter. Substituting Eq. (20) into

Eq. (19), one obtains:

$$ {}^{c}\boldsymbol{K}_{T}^{(k-1)}\ {}^{c}\delta\boldsymbol{u}^{k} = -[{}^{c}\boldsymbol{F}_{\mathbf{int}}^{(k-1)} - ({}^{c}\lambda^{(k-1)} + {}^{c}\delta\lambda^{k})\boldsymbol{R}_{\mathbf{ref}}] \tag{21} $$

where ${}^{c}\boldsymbol{F}_{\mathbf{int}}^{(k-1)}$ is the internal force vector acting on the structure. Eq. (21) can be rewritten as:

$$ {}^{c}\boldsymbol{K}_{T}^{(k-1)}\ {}^{c}\delta\boldsymbol{u}^{k} = -{}^{c}\boldsymbol{g}^{(k-1)} + {}^{c}\delta\lambda^{k}\boldsymbol{R}_{\mathbf{ref}} \tag{22} $$

This is the desired constraint equation that controls the increment of the structure variables at each iterative cycle. From Eq. (22), the displacements can be decomposed into two groups:

$$ {}^{c}\delta\boldsymbol{u}^{k} = {}^{c}\delta\boldsymbol{u}_{g}^{k} + {}^{c}\delta\lambda^{k}\ {}^{c}\delta\boldsymbol{u}_{R}^{k} \tag{23} $$

where

$$ {}^{c}\delta\boldsymbol{u}_{g}^{k} = -{}^{c}\boldsymbol{K}_{T}^{-1(k-1)}\ {}^{c}\boldsymbol{g}^{(k-1)} \quad \text{and} \quad {}^{c}\delta\boldsymbol{u}_{R}^{k} = {}^{c}\boldsymbol{K}_{T}^{-1(k-1)}\boldsymbol{R}_{\mathbf{ref}} \tag{24} $$

The correction of the load parameter, ${}^{c}\delta\lambda^{k}$, is obtained from one of the several different iterative strategies available (Crisfield 1991, 1997). Using the corrected load parameter ${}^{c}\delta\lambda^{k}$, the corrections of the displacements can be obtained from Eq. (23). The incremental variables are finally updated:

$$ {}^{c}\Delta\lambda^{k} = {}^{c}\Delta\lambda^{(k-1)} + {}^{c}\delta\lambda^{k} \quad \text{and} \quad {}^{c}\Delta\boldsymbol{u}^{k} = {}^{c}\Delta\boldsymbol{u}^{(k-1)} + {}^{c}\delta\boldsymbol{u}_{g}^{k} + {}^{c}\delta\lambda^{k}\ {}^{c}\delta\boldsymbol{u}_{R}^{k} \tag{25} $$

$$ {}^{c}\lambda^{k} = {}^{\omega}\lambda + {}^{c}\Delta\lambda^{k} \quad \text{and} \quad {}^{c}\boldsymbol{u}^{k} = {}^{\omega}\boldsymbol{u} + {}^{c}\Delta\boldsymbol{u}^{k} \tag{26} $$

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1. Initial state: ${}^{\omega}\boldsymbol{u}$. ${}^{\omega}\lambda$ and ${}^{\omega}S_c$
2. Iterations - *contact problem*: c = 1, 2,…, $I_{c,max}$
3. Compute the "tangent incremental solution": ${}^{c}\Delta\lambda^{0}$ and ${}^{c}\Delta\boldsymbol{u}^{0}$
4. "Standard" LCP solution: ${}^{c}\boldsymbol{w} = {}^{c}\boldsymbol{q} + {}^{c}\boldsymbol{M}\,{}^{c}z$; ${}^{c}z \geq \mathbf{0}$; ${}^{c}w \geq \mathbf{0}$; ${}^{c}w^{T}\,{}^{c}z = \mathbf{0}$
5. Estimate the new contact region: ${}^{\omega+\Delta\omega}S_c$
6. Iterations - *equilibrium problem*: k = 1, 2,…, I_{max}
7. Check convergence: $\left\| {}^{c}\boldsymbol{g}^{(k-1)} \right\| / \left\| {}^{c}\Delta\lambda^{(k-1)}\boldsymbol{R}_{ref} \right\| \leq \zeta$?

<u>Yes</u>: Go to step 8

<u>No</u>: Corrections: ${}^{c}\delta\boldsymbol{u}^{k}$ and ${}^{c}\delta\lambda^{k}$; updated variables: ${}^{c}\Delta\lambda^{k}$ and ${}^{c}\Delta\boldsymbol{u}^{k}$, go to step 6

8. Compute the new contact region: ${}^{\omega+\Delta\omega}S_c^{\boldsymbol{n}}$
9. Check convergence: ${}^{\omega+\Delta\omega}S_c^{\boldsymbol{n}}$ - ${}^{\omega+\Delta\omega}S_c \leq \zeta_c$?

<u>Yes</u>: New load increment. Go to step 1.

<u>No</u>: Change ${}^{c}\boldsymbol{K}_{T}$, make ${}^{\omega}S_c = {}^{\omega+\Delta\omega}S_c^{\boldsymbol{n}}$, updated variables, go to step 2

Fig. 2 Iterative solution strategy

After the convergence of the Newton-Raphson procedure is achieved, a new contact region ${}^{\omega+\Delta\omega}S_c^n$ is evaluated. Here n indicates that this solution was obtained after the equilibrium cycle was completed. Next ${}^{\omega+\Delta\omega}S_c^n$ is compared with its previous value, i.e., ${}^{\omega+\Delta\omega}S_c$. If the desired accuracy is attained, a new initial load increment is computed and the whole process is repeated and a new equilibrium configuration is obtained. If convergence is not achieved, an improved contact region must be calculated. But, before of this, the following calculations are required: the stress state for the structure and foundation at the present configuration is obtained and a new initial stress matrix is evaluated. Also the contact nodes based on the solution ${}^{\omega+\Delta\omega}S_c^n$ are identified and the variables ${}^c\lambda$, ${}^c\boldsymbol{u}$, ${}^c\Delta\lambda$ and ${}^c\Delta\boldsymbol{u}$ are re-initiated. The solution procedure described previously is summarised in Fig. 2.

4. Examples

The first numerical example used to test the present methodology is shown in Fig. 3. It consists of a radially constrained circular ring subjected to inertial loading. The surrounding medium is considered to be a tensionless rigid foundation, so that the ring can only deform inward. Also shown in Fig. 3 are the geometrical and physical parameters used in the analysis. This is a typical example of a structure that is subjected from the beginning to unilateral constraints. Here the ring deforms locally inward and is subjected to limit point instability when the length of the contact region reaches a critical value. It is used for the ring the non-linear finite element model developed by Alves (1995), while the rigid foundation is described by discrete springs with a high stiffness value (here, $K=36\times10^5$), as illustrated in Fig. 4. Lemke's algorithm (Lemke 1968) is used to solve the LCP (Eqs. 17 and 18), while the Newton-Raphson method, the iterative technique devised by Chan (1988) and the automatic load increment strategy proposed by Crisfield (1991) are used to solve the equilibrium equations and trace the non-linear equilibrium path.

It is shown in Fig. 5 the variation of the non-dimensional load parameter pR^3/EI with the central deflection v divided by R. This problem was analysed previously by Pian *et al.* (1967) using the finite difference method and, more recently, by Stein and Wriggers (1984) who, using the finite

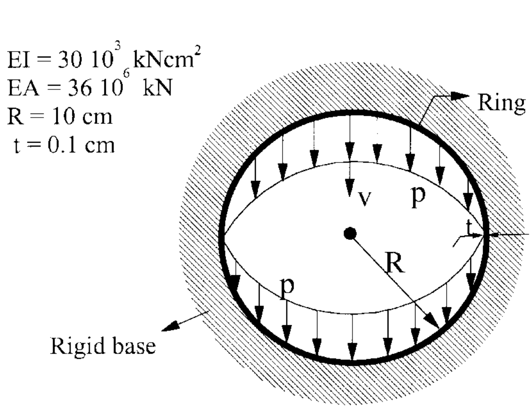

Fig. 3 Ring in rigid confinement

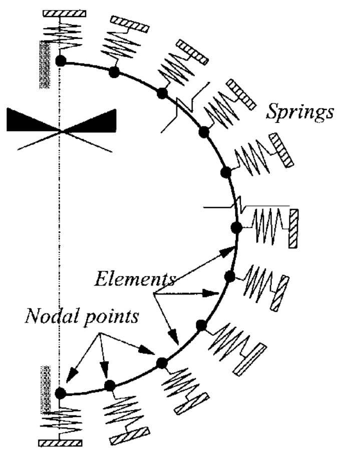

Fig. 4 Finite element model

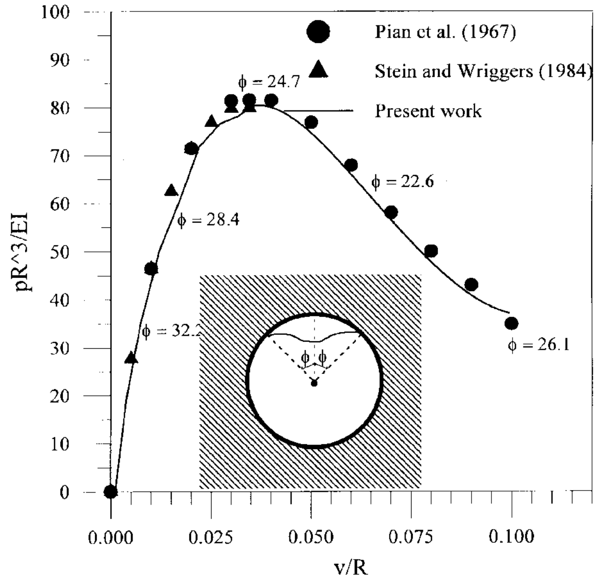

Fig. 5 Load-deflection curve for inertial loading

element method, obtained the non-linear equilibrium path up to the critical load. The results obtained by these authors are compared in Fig. 5 with the results obtained using the present formulation and numerical methodology. Their values for the critical load parameter are compared with the present result in Table 1. The computed values of the separation angle ϕ for different load levels, corresponding to stable, neutral and unstable configurations, are compared with those reported by Pian *et al.* (1967) in Table 2. As observed, the present results compare well with those found in literature. In order to investigate the influence of the foundation stiffness on the non-linear response and stability of the ring, the same structure is now analysed considering increasing values for the foundation stiffness *K*. The results are presented in Fig. 6, where again the non-dimensional

Table 1 Comparison of critical load parameter, pR^3/EI

	Solutions		
	Pian *et al.*	Stein and Wriggers	Present Work
pR^3/EI	81.5	79.9	80.3

Table 2 Comparison of separation angle ϕ for different load levels

pR^3/EI	ϕ (Pian *et al.*)	ϕ (Present Work)
34	32.9°	32.2°
57	30.1°	28.4°
81.5	24.6°	24.7°
62	23.9°	22.6°
28	27.2°	26.1°

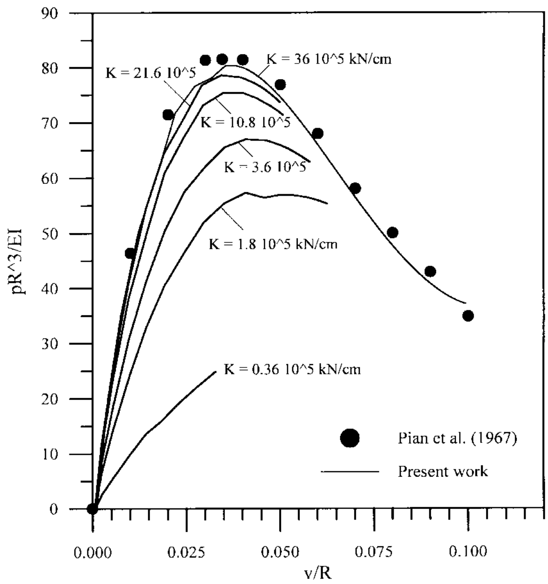

Fig. 6 Influence of the foundation stiffness on the non-linear response of the ring

load parameter pR^3/EI is plotted as a function of v/R. As the foundation stiffness increases the limit point load increases and the results approach from below the results obtained by Pian *et al.* (1967), considering a rigid foundation. As observed a foundation stiffness of 36×10^5 practically reproduces the non-linear response obtained by Pian *et al*. It should be pointed out that the use of an elastic foundation with very large stiffness (here, for example, $K > 10^6$) leads to numerical difficulties and the response presents, as one approaches the limit point, spurious oscillations. This is a problem typical of penalty-type methods, so care should be taken in choosing the foundation stiffness in order to represent a rigid foundation. For soft foundations no numerical difficulties were encountered during the analysis.

The buckling and post-buckling behaviour of a beam under contact constraint is illustrated through a simple, but representative example, shown in Fig. 7. The model consists of an imperfect beam located near a rigid smooth surface. The imperfection function is represented by $\bar{w}(x)$ and d defines the distance from the perfect beam to this surface. The beam is subjected to compression induced by an axial load P. Experimental and semi-analytical results obtained by Adan *et al*. (1994)

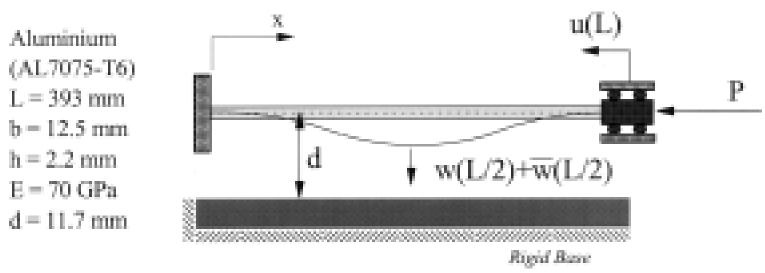

Fig. 7 Column model for buckling under contact constraints

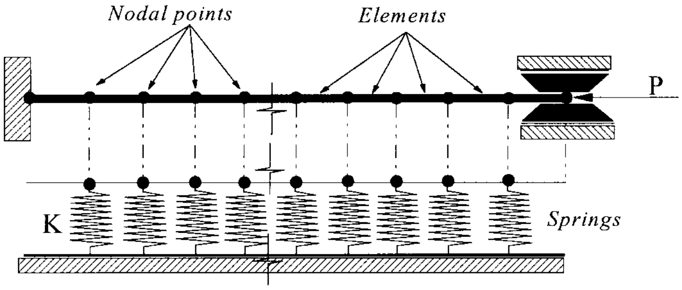

Fig. 8 Finite element model

revealed two distinct types of post-buckling behaviour, namely symmetric and asymmetric responses.

The finite element model used to solve this problem is shown in Fig. 8. Here 40 elements were used for the column, while once more the rigid foundation was described by discrete springs with a high stiffness value. The iterative technique proposed by Gierlinski and Graves Smith (1985) and the automatic load increment method proposed by Crisfield (1991) have been used in the non-linear solution strategy. The form of the initial asymmetric imperfection is based on the experimental results obtained by Adan *et al.* (1994).

The non-linear equilibrium paths characterising the non-linear structural response are shown in Figs. 9 and 10, where, respectively, the variations of the midspan displacement and end shortening are plotted as a function of the non-dimensional load parameter, *P*/*Pcr.* The experimental and numerical results obtained by Adan *et al.* (1994) are compared with the results obtained in this work. Excellent agreement is observed between the present results and those obtained by Adan *et al.*, confirming the accuracy and efficiency of the present methodology.

In these figures, the points a0-a5 indicate transitions between characteristic stages of response. The first part of both equilibrium curves (a0-a1) follow the familiar non-linear path of an imperfect beam under compression. The contact between the bodies occurs at position a1 ($P/Pcr \cong 0.93$).

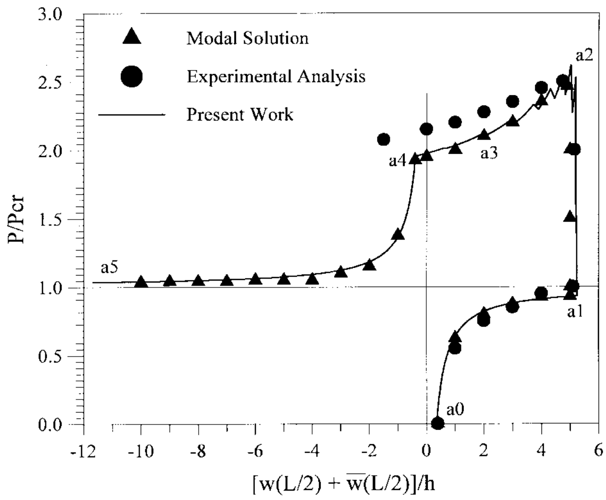

Fig. 9 Non-linear response of the column (Variation of the load parameter, *P*/*Pcr*, with the mid-span displacement, $(\boldsymbol{w}+\bar{\boldsymbol{w}})/\boldsymbol{h}$)

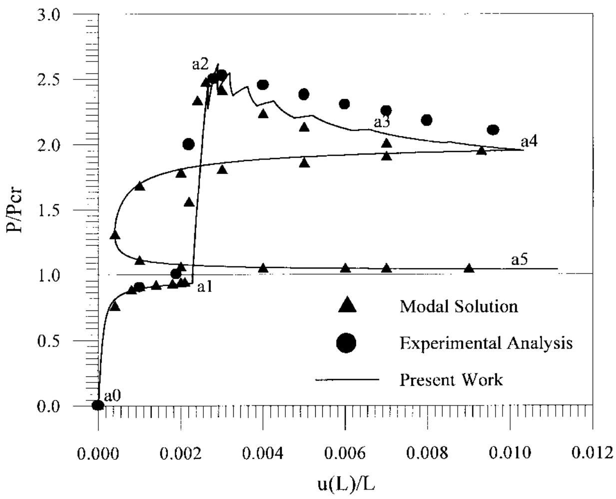

Fig. 10 Variation of the load parameter *P*/*Pcr* as a function of the column end shortening, *u*/*L*

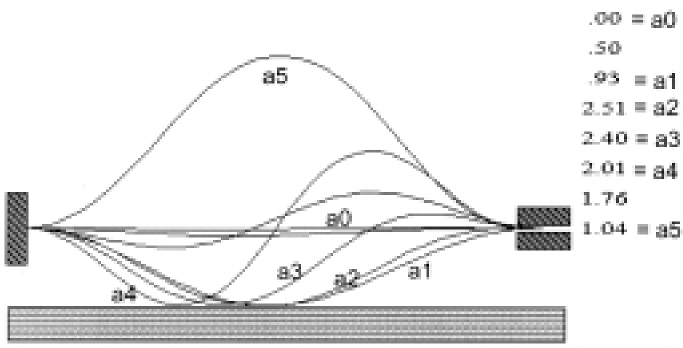

Fig. 11 Evolution of the deformation pattern of the imperfect column

After this point, as the load increases the contact region increases and moves slightly to the left. Notice that the constrained response is considerably stiffer than the free one and that a limit load occurs at $P/Pcr \cong 2.5$ (point a2), followed by a decrease in the axial force (a2-a3). Along this path the contact region decreases and moves to the left, with the beam assuming an asymmetric form. As the contact constraints are deactivated ($P/Pcr \cong 1.94$, point a4) the beam jumps to a non-constrained configuration. The point a5 exemplifies this unconstrained configuration, which is identical to the large displacement response of a column without constraints. The deformation pattern associated with the five critical points mentioned above is illustrated in Fig. 11, where the load levels associated with these points are also shown. The numerical solution shown in Figs. 9 and 10 presents some oscillation at a small region between points a2 and a3. This is due to the representation of the continuous rigid foundation by discrete springs with high stiffness. Along this portion there is a fast variation of the contact region that simultaneously decreases and moves to the left as shown in Fig. 11. This leads to sudden and discontinuous variations in the foundation reaction and consequently in the effective stiffness of the beam.

5. Conclusions

In this work a numerical methodology for the non-linear analysis of slender structural elements with unilateral constraints is derived. The results show good agreement with those found in literature and validate the formulation and the proposed numerical methodology. The examples analysed showed that the non-linear proposed formulation can be used successfully in many engineering phenomena involving contact.

The use of an updated Lagrangian approach enables one to linearize the contact problem at each equilibrium configuration and solve the resulting problem as a linear complementary problem by Lemke's algorithm. This step is essential for the success of the present formulation. The simultaneous use of an updated Lagrangian formulation, Lemke's algorithm and an efficient incremental-iterative strategy minimises the errors along the non-linear path and enables one to trace convoluted non-linear paths with a varying number of contact regions. Also, it is shown that the use of programming methods allows the development of logically and numerically simple algorithms for the solution of non-classical stability problems with unilateral constraints.

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Notation

$\boldsymbol{A}$ = joining matrix between the bodies
C_b = foundation elastic modulus
C_{ijkl} = constitutive tensor components
$\boldsymbol{F}_{\mathbf{int}}$ = internal force vector
$\boldsymbol{g}$ = gradient vector
$I_{\max}$ = maximum iteration number
J = energy functional
$\boldsymbol{K}_{\boldsymbol{\tau}}$ = initial stress matrix
$\boldsymbol{K}_{\boldsymbol{T}}$ = tangent stiffness matrix
n_b = outward normal vector on the foundation comp.
n_i = outward normal vector on the structure comp.
r_b = compressive reaction of the foundation
$\boldsymbol{R}_{\mathbf{ref}}$ = reference external load vector

S	= boundary of body (m^2)
S_c,	= contact region (m^2)
S_{jk}	= 2nd Piola-Kirchhoff stress comp.
S_u, S_f	= boundary where the displacements and forces are specified (m^2)
$\boldsymbol{T}$	= flexibility matrix of the elastic foundation
$\boldsymbol{u}$	= total nodal displacement vector (m)
V	= body domain (m^3)
x_j	= coord. of the structures comp.(m)

Greek Symbols:

ψ	= gap between the bodies at conf. ω
β	= gap between the bodies
ω	= reference equilibrium configuration
ξ	= tolerance
$\Delta\lambda^0$	= initial increment load parameter increment
$\Delta\varepsilon_{ij}$	= strain increment tensor components
$\lambda,\Delta\lambda,\delta\lambda$	= total, incremental and iterative load parameter
τ_{ij}	= Cauchy stress tensor comp.
Δr_b	= incremental compressive reaction
ΔS_{ij}	= 2nd Piola-Kirchhoff stress increment comp.
$\Delta\boldsymbol{u},\delta\boldsymbol{u}$	= incremental and iterative nodal displac.
Δu_b	= incremental deflection of the foundation
Δu_i	= incremental displacement comp.

Subscripts:

b	= elastic foundation index
g	= iterative change in $\boldsymbol{u}$ due to $\boldsymbol{g}$ index
i,j,k,l	= summation indexes
R	= iterative change in $\boldsymbol{u}$ due to $\boldsymbol{R}_{\mathbf{ref}}$ index
u,f,c	= boundary indexes

Superscripts:

i	= equilibrium configuration index
T	= transpose
n	= new contact region
c	= iteration counter (contact problem)
k	= iteration counter (equilibrium problem)
$k-1$	= last iteration