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Continuous and discontinuous contact problem for a layered composite resting on simple supports

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Abstract. The frictionless contact problem for a layered composite which consists of two elastic layers having different elastic constants and heights resting on two simple supports is considered. The external load is applied to the layered composite through a rigid stamp. For values of the resultant compressive force P acting on the stamp vertically which are less than a critical value P_{cr} and for small flexibility of the layered composite, the continuous contact along the layer - the layer and the stamp - the layered composite is maintained. However, if the flexibility of the layered composite increases and if tensile tractions are not allowed on the interface, for $P > P_{cr}$, a separation may be occurred between the stamp and the layered composite or two elastic layers interface along a certain finite region. The problem is formulated and solved for both cases by using Theory of Elasticity and Integral Transform Technique. Numerical results for P_{cr}, separation initiation distance, contact stresses, distances determining the separation area, and the vertical displacement in the separation zone between two elastic layers are given.

Key words : continuous contact; discontinuous contact; separation; integral equation; elastic layer; rigid stamp; theory of elasticity; fourier transform.

1. Introduction

The contact problems in solid mechanics involving elastic layers have attracted the attention of several researchers due to its application to a great variety of important structures of practical interest. Foundation grillages, pavements in roads and runways, rolling mills, railway ballast, beams resting on supports or stamps and foundation beams are some example of the contact problems.

The general methods of the contact problems may be found in the works of Hertz (1895), Galin (1961) and Uffliand (1965). The contact problems are examined using different methods, some of which are complex variables (Muskhelishvili 1958) and Fourier transform techniques (Sneddon 1972). Problems involving contact between an elastic layer or a layered composite and a foundation which may be either elastic or rigid have been very widely studied with improvements in computer technology. The continuous and discontinuous contact problem between an elastic layer and a rigid half-plane for the case of a single load in tension is analyzed by Civelek and Erdoğan (1975). The discontinuous case of the same problem for the single load in compression is examined by the same researchers (1976). The frictionless contact problem for an infinite elastic layer lying on a horizontal rigid plane is examined by Civelek, Erdoğan and Çakıroğlu (1978), and the same problem is also analyzed by Çakıroğlu (1979) in the case of elastic rather than rigid half-plane. Similar contact

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problem is also examined by Çakıroğlu and Çakıroğlu (1991) in the case of symmetrical distributed loads and an elastic semi-infinite plane. A tensionless contact without friction between an elastic layer and elastic foundation is studied by Geçit (1980) and the same researcher (1981, 1986) analyzed axisymmetric contact problem for an elastic layer and elastic foundation and an axisymmetric contact problem for an elastic half space indented by an elastic semi-infinite circular cylinder. The general axisymmetric double contact problem for an elastic layer pressed against a half space by an elastic stamp is investigated by Civelek and Erdoğan (1974). Geçit and Yapıcı (1986) examined the contact problem for an infinite elastic layer resting on two rigid horizontal flat support, and the frictionless contact problem between a rigid stamp and a layered composite resting on simple supports is studied by Birinci and Erdöl (1999).

In the present study, the continuous and discontinuous contact problem of a layered composite which consists of two elastic layers having different elastic constants and heights is investigated. The layered composite resting on two simple supports is subjected to a concentrated load $2P$ by means of a rigid rectangular stamp of which width is $2a$. It is assumed that all surfaces are frictionless. The continuous contact problem between the elastic layered composite and the rigid stamp and between two elastic layers is examined until initial separation occurs stated contact surfaces along. In this case, the contact stress distribution, initial separation loads and distances are investigated for various dimensionless quantities. The discontinuous contact occurs either between the rigid stamp and the layered composite or between two elastic layers. Should applying external load (P) be bigger than the initial separation load (P_{cr}), the separation occurs between two elastic layers. Also, the separation may occur between the rigid stamp and the layered composite depending on the flexibility of the layered composite. In the case of the discontinuous contact, the stress distribution along the contact surface, the initial and end distances of the separation, and vertical displacement difference between two elastic layers in the separation zone are investigated for various dimensionless quantities. Finally, numerical results are analyzed and conclusions are drawn.

2. General expressions for stresses and displacements

In the absence of body forces, the two dimensional Navier equations may be written as in the following form for considered an infinite layered composite consisting of two elastic layers and resting on simple supports in Fig. 1.

$$\mu_i \nabla^2 u_i + \frac{2\mu_i}{\kappa_i - 1}\frac{\partial}{\partial x}\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right) = 0, \tag{1a}$$

$$\mu_i \nabla^2 v_i + \frac{2\mu_i}{\kappa_i - 1}\frac{\partial}{\partial y}\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right) = 0, \qquad (i=1,2). \tag{1b}$$

where u_i and v_i are the x and y-components of the displacement vector. μ_i and κ_i (i=1, 2) represent shear modules and constants of the elastic layers, respectively. $\kappa_i=(3-\nu_i)/(1+\nu_i)$ for plane stress and $\kappa_i=3-4\nu_i$ for plane strain. ν_i is the Poisson's ratio of layer. Subscript i (i=1, 2) indicates the values related to the layer.

For the case in which gravity forces are considered, the displacements are shown u_{ip} and v_{ip}, and if gravity forces are not considered, the displacements are shown as u_{ih} and v_{ih}, and total field of displacements may be expressed as,

$$u_i = u_{ih} + u_{ip}, \tag{2a}$$

$$v_i = v_{ih} + v_{ip}. \tag{2b}$$

Observing that x=0 is a plane symmetry, it is sufficient to consider the problem in the region $0 \le x \le \infty$ only. Using the symmetry consideration, the following expressions may be written

$$u_i(x,y) = -u_i(-x,y), \tag{3a}$$

$$v_i(x,y) = v_i(-x,y), \tag{3b}$$

$$u_{ih}(x,y) = \frac{2}{\pi}\int_0^\infty \Phi_i(\alpha,y)\sin(\alpha x)\,d\alpha, \tag{3c}$$

$$v_{ih}(x,y) = \frac{2}{\pi}\int_0^\infty \Psi_i(\alpha,y)\cos(\alpha x)\,d\alpha, \tag{3d}$$

where Φ_i and Ψ_i (i=1, 2) functions are inverse Fourier transforms of u_i and v_i, respectively. Taking necessary derivatives of Eqs. (3c) and (3d), and substituting them into Eqs. (1a) and (1b), and solving second order differential equations, the following expressions may be obtained for displacements

$$u_{ih}(x,y) = \frac{2}{\pi}\int_0^\infty [(A_i + B_i y)e^{-\alpha y} + (C_i + D_i y)e^{\alpha y}]\sin(\alpha x)\,d\alpha, \tag{4a}$$

$$v_{ih}(x,y) = \frac{2}{\pi}\int_0^\infty \left\{\left[A_i + \left(\frac{\kappa_i}{\alpha} + y\right)B_i\right]e^{-\alpha y} + \left[-C_i + \left(\frac{\kappa_i}{\alpha} - y\right)D_i\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha, \tag{4b}$$

where A_i, B_i, C_i and D_i (i=1, 2) are unknown constants which will be determined from boundary conditions of the problem. Using Hooke's law and Eq. (4), the components of the stress without gravity forces may be expressed as follows:

$$\frac{1}{2\mu_i}\sigma_{ix_h}(x,y) = \frac{2}{\pi}\int_0^\infty \left\{\left[\alpha(A_i + B_i y) - \frac{3-\kappa_i}{2}B_i\right]e^{-\alpha y} + \left[\alpha(C_i + D_i y) + \frac{3-\kappa_i}{2}D_i\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha, \tag{5a}$$

$$\frac{1}{2\mu_i}\sigma_{iy_h}(x,y) = \frac{2}{\pi}\int_0^\infty \left\{-\left[\alpha(A_i + B_i y) - \frac{1+\kappa_i}{2}B_i\right]e^{-\alpha y} + \left[\alpha(C_i + D_i y) - \frac{1+\kappa_i}{2}D_i\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha, \tag{5b}$$

$$\frac{1}{2\mu_i}\tau_{ixy_h}(x,y) = \frac{2}{\pi}\int_0^\infty \left\{-\left[\alpha(A_i + B_i y) + \frac{\kappa_i - 1}{2}B_i\right]e^{-\alpha y} + \left[\alpha(C_i + D_i y) - \frac{\kappa_i - 1}{2}D_i\right]e^{\alpha y}\right\}\sin(\alpha x)\,d\alpha. \tag{5c}$$

The components of the displacement and the stress for the case which gravity forces existing are given by (A1) and (A2) in the Appendix.

3. Continuous contact case

A layered composite consisting of two elastic layers of which heights and elastic properties are

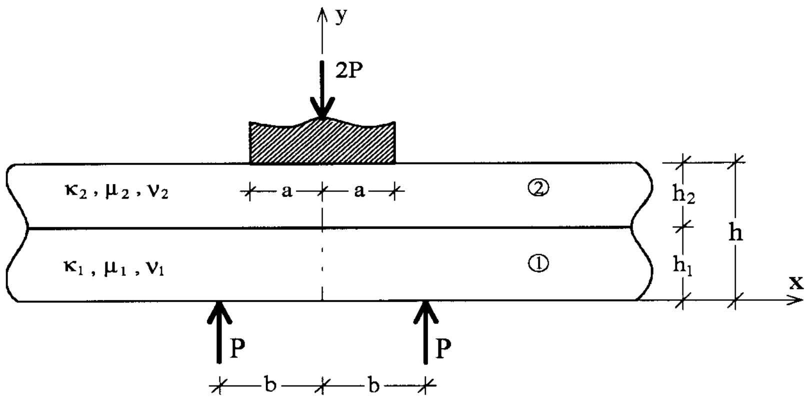

Fig. 1 Geometry of continuous contact case for the layered composite

different, resting on simple supports and subjected to a concentrated load with a magnitude 2P by means of a rigid stamp on its top surface, shown in Fig. 1, will be analyzed. Particularly, the initial separation load (λ_{cr}) and point (x_{cr}) where the elastic layers will be separated from each other, the distribution of the contact pressure between two elastic layers and under the stamp until the occurrence of the initial separation will be examined.

In this case, the continuous contact problem must be solved under the following boundary condition:

$$\tau_{2xy}(x,h)=0, \qquad (0 \le x < \infty), \tag{6a}$$

$$\tau_{2xy}(x,h_1)=0, \qquad (0 \le x < \infty), \tag{6b}$$

$$\tau_{1xy}(x,h_1)=0, \qquad (0 \le x < \infty), \tag{6c}$$

$$\sigma_{2y}(x,h_1)=\sigma_{1y}(x,h_1), \qquad (0 \le x < \infty), \tag{6d}$$

$$\tau_{1xy}(x,0)=0, \qquad (0 \le x < \infty), \tag{6e}$$

$$\sigma_{1y}(x,0)=-P\delta(x-b), \tag{6f}$$

$$\frac{\partial}{\partial x}[v_2(x,h_1)-v_1(x,h_1)]=0, \qquad (0 \le x < \infty), \tag{6g}$$

$$\sigma_{2y}(x,h)=\begin{cases} -p(x), & 0 \le x < a \\ 0, & a < x < \infty, \end{cases} \tag{6h}$$

$$\frac{\partial}{\partial x}[v_2(x,h)]=0, \qquad (0 \le x < a), \tag{6i}$$

in which subscripts 1 and 2 indicate related to the elastic layer 1 and the elastic layer 2, respectively. a, b, $p(x)$ and $\delta(x)$ are the half-width of the rigid stamp, the width of the support, the unknown contact pressure under the rigid stamp and Dirac delta function, respectively.

If a separation occurs between the elastic layers or the rigid stamp and the layered composite, this will give rise to a discontinuous contact position and the following results for former solution will no longer be valid and new solution will be attained for the latter case.

By making use of boundary conditions (6a-h), A_i, B_i, C_i and D_i (i=1, 2) constants may be calculated in terms of $p(x)$, and by substituting the values of these constants into Eq. (6i), after some routine manipulations, and using the symmetry condition, $p(x)=p(-x)$, one may obtain the following singular integral equation for $p(x)$.

$$\int_{-a}^{a}\left[\frac{1}{t-x}+\frac{1}{h}k_1(x,t)\right]p(t)dt=\frac{P}{h}k_2(x), \qquad (-a<x<a), \tag{7}$$

where the kernels $k_1(x, t)$ and $k_2(x)$ are given by (A3) and (A4) in Appendix. In (7), the kernel $k_1(x, t)$ is bounded in the closed interval $-a\le(x,t)\le a$, and the index of the integral equation is +1 (Erdoğan and Gupta 1972). The equilibrium condition of the problem is written as,

$$\int_{-a}^{a} p(t)dt=2P. \tag{8}$$

In order to investigate the separation between two elastic layers, the contact stress $\sigma_y(x,h_1)$ needs to be evaluated. Substituting the values of A_i, B_i, C_i and D_i (i=1, 2) as evaluated in terms of $p(x)$ into (5b), after some algebra manipulations, the contact stress is found to be,

$$\sigma_y(x,h_1)=-\rho_2 g h_2-\frac{1}{\pi h}\int_{-a}^{a} k_3(x,t)p(t)dt-\frac{P}{\pi h}k_4(x), \quad (0\le x<\infty), \tag{9}$$

where ρ and g_2 are gravity acceleration and mass density of the layer 2, respectively. The kernels $k_3(x, t)$ and $k_4(x)$ are given by (A5) and (A6) in Appendix.

To simplify the numerical analysis, the following dimensionless quantities are introduced:

$$x=a\xi,\ \ t=a\eta,\ \ k_2(\xi)=\frac{k_2(a\xi)}{P/h},\ \ g(\eta)=\frac{p(a\eta)}{P/h},\ \ \lambda=\frac{P}{\rho_2 g h h_2}. \tag{10a-e}$$

Substituting from (10), Eqs. (7), (8) and (9) may be expressed as,

$$\int_{-1}^{1}\left[\frac{1}{\eta-\xi}+\frac{a}{h}k_1(\xi,\eta)\right]g(\eta)d\eta=k_2(\xi), \qquad (-1<\xi<1), \tag{11a}$$

$$\frac{a}{h}\int_{-1}^{1} g(\eta)d\eta=2, \tag{11b}$$

$$\frac{\sigma_y(x,h_1)}{P/h}=-\frac{1}{\lambda}-\frac{1}{\pi}\frac{a}{h}\int_{-1}^{1}k_3(x,a\eta)g(\eta)d\eta-\frac{1}{\pi}k_4(x), \qquad (0\le x<\infty) \tag{11c}$$

Noting that the index of the integral Eq. (11a) is +1, its solution may be expressed as;

$$g(\eta)=G(\eta)(1-\eta^2)^{-1/2}, \qquad (-1<\eta<1), \tag{12}$$

where $G(\eta)$ is bounded in interval $(-1\le\eta\le1)$. Then, using the appropriate integration formula (Erdoğan and Gupta 1972), Eqs. (14a) and (14b) are replaced by

$$\sum_{i=1}^{n} W_i\left[\frac{1}{\eta_i-\xi_j}+\frac{a}{h}k_1(\xi_j,\eta_i)\right]G(\eta_i)=\frac{1}{\pi}k_2(\xi_j), \quad (j=1, ..., n-1), \tag{13a}$$

$$\frac{a}{h}\sum_{i=1}^{n} W_i G(\eta_i)=\frac{2}{\pi}, \tag{13b}$$

where

$$W_1=W_n=\frac{1}{2(n-1)},\quad W_i=\frac{1}{n-1},\qquad (i=2,\ ...,\ n-1) \tag{14a}$$

$$\eta_i=\cos\left(\frac{i-1}{n-1}\pi\right),\qquad (i=1,\ ...,\ n), \tag{14b}$$

$$\xi_j=\cos\left(\frac{\pi}{2}\frac{2j-1}{n-1}\right),\qquad (j=1,\ ...,\ n-1). \tag{14c}$$

The unknowns $G(\eta_i)$, $(i = 1,\ ...,\ n)$, are determined from the system of Eqs. (13a) and (13b). By using (12), substituting the results into (11c), and using a Gaussian integration formula, the contact stress $\sigma_y(x,\ h_1)$ is evaluated.

It should be observed that the integral Eq. (11a) is valid provided the contact stress obtained from (11c) is compressive everywhere. For given values of a/h, b/h, h_1/h and elastic properties of the layers by evaluating the contact stress, one may obtain both the location x_η (x_{cr}) at which the interface separation starts between two elastic layers and the corresponding critical load factor (λ_{cr}). This factor is related to the separation load P_{cr} by

$$\lambda_{cr}=\frac{P_{cr}}{\rho_2 g h h_2}. \tag{15}$$

4. Discontinuous contact case

The discontinuous contact may occur in two cases. Firstly, while there is the continuous contact between two elastic layers, the separation may occur between the rigid stamp and the layered composite. Secondly, while there is the continuous contact between the rigid stamp and the layered composite, the separation may occur between two elastic layers. Let us examine two cases, respectively.

4.1 The discontinuous contact between the rigid stamp and the layered composite

The discontinuous contact between the rigid stamp and the layered composite may also occur in two cases. Depending on the flexibility of the layered composite, the separation between the stamp and the layered composite starts from either x=0 symmetry axis (Fig. 2a) or the edges of the rigid stamp (Fig. 2b).

4.1.1 The case of the separation starting from on the symmetry axis

The boundary conditions of the continuous contact case is valid for this case except for the fact that $(0\le x<a)$ and $(a<x<\infty)$ in Eqs. (6h) and (6i) must be replaced by $(f<x<a)$ and $(0\le x<f,\ a<x<\infty)$, respectively. Therefore, the integral Eq. (7) for the continuous contact case become as following form for this case.

$$\int_f^a\left[\frac{1}{t+x}-\frac{1}{t-x}+\frac{1}{h}k_1^*(x,t)\right]p(t)dt=-\frac{P}{h}k_2(x),\qquad (f<x<a), \tag{16}$$

where the kernel $k_1^*(x,t)$ is given by (A7) in the Appendix. The equilibrium condition (8) may be expressed as follows for this case:

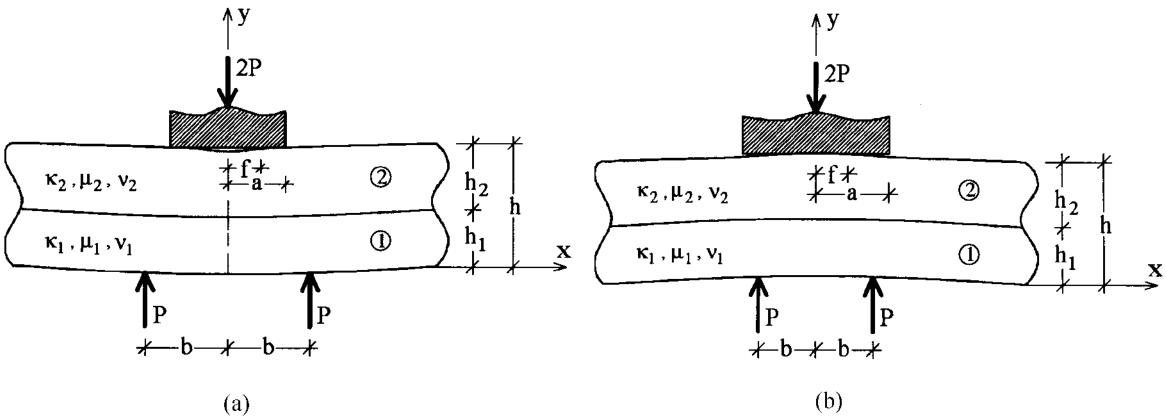

Fig. 2 The discontinuous contact between the rigid stamp and the layered composite

$$\int_f^a p(t)dt=P. \tag{17}$$

Defining the following dimensionless quantities,

$$x=\frac{a-f}{2}\xi+\frac{a+f}{2}, \tag{18a}$$

$$t=\frac{a-f}{2}\eta+\frac{a+f}{2}, \tag{18b}$$

and making use of Eqs. (10c, d), the integral Eq. (16) and the equilibrium condition (17) may be written as follows:

$$\int_{-1}^{1}\left[\frac{1}{\eta+\xi+2\frac{a+f}{a-f}}-\frac{1}{\eta-\xi}+\frac{a-f}{2h}k_1^*(\xi,\eta)\right]g(\eta)d\eta=-k_2(\xi), \qquad (-1<\xi<1), \tag{19a}$$

$$\frac{a-f}{2h}\int_{-1}^{1}g(\eta)d\eta=1. \tag{19b}$$

The separation starts at $|x|=f$ and the contact between the rigid stamp and the layered composite will be smooth at this point and will be infinite at the end of the stamp. Therefore, $g(-1)$ vanishes and consequently, the index of the integral Eq. (19a) will be zero (Erdoğan and Gupta 1972). Hence, the solution will be in the following form

$$g(\eta)=G(\eta)(1+\eta)^{1/2}(1-\eta)^{-1/2}, \qquad (-1<\eta<1), \tag{20}$$

where again, $G(\eta)$ is bounded in interval $(-1\le\eta\le 1)$. The use of Gauss-Chebyshev integration formula (Erdoğan and Ratwani 1974) reduces Eqs. (19a) and (19b) to

$$\sum_{i=1}^{n}W_i\left[\frac{1}{\eta_i+\xi_j+2\frac{a+f}{a-f}}-\frac{1}{\eta_i-\xi_j}+\frac{a-f}{2h}k_1^*(\xi_j,\eta_i)\right]G(\eta_i)=-\frac{1}{\pi}k_2(\xi_j), \qquad (j=1,\ldots,n), \tag{21a}$$

$$\frac{a-f}{2h}\sum_{i=1}^{n} W_i G(\eta_i)=\frac{1}{\pi}, \tag{21b}$$

where

$$W_i=\frac{2(1+\eta_i)}{2n+1}, \qquad (i=1, ..., n), \tag{22a}$$

$$\eta_i=\cos\left(\frac{2i-1}{2n+1}\pi\right), \qquad (i=1, ..., n), \tag{22b}$$

$$\xi_i=\cos\left(\frac{2j}{2n+1}\pi\right), \qquad (j=1, ..., n). \tag{22c}$$

Note that the system given by Eqs. (21a) and (21b) contains n+1 equations for n+1 unknowns, $G(\eta_i)$, (i= 1, ..., n), and f. The system is nonlinear in f and an interpolation scheme is required to determine these unknowns.

4.1.2 The case of the separation starting from the edges of the stamp

In this case, the integral Eq. (7) is valid except for the fact that a must be replaced by f. At $|x|=f$ the separation starts and the contact between the stamp and the layered composite will be smooth near these points. Therefore, $g(\pm1)=0$ and consequently, the index of the integral Eq. (7) for this case is −1 (Erdoğan and Gupta 1972), its solution may be expressed as

$$g(\eta)=G(\eta)(1-\eta^2)^{1/2}, \qquad (-1<\eta<1), \tag{23}$$

where also again, $G(\eta)$ is bounded in interval $(-1\le\eta\le1)$. The Eqs. (11a) and (11b) may be replaced by

$$\sum_{i=1}^{n} W_i\left[\frac{1}{\eta_i-\xi_j}+\frac{f}{h}k_1(\xi_j,\eta_i)\right]G(\eta_i)=\frac{1}{\pi}k_2(\xi_j), \qquad (j=1, .., n+1), \tag{24a}$$

$$\frac{f}{h}\sum_{i=1}^{n} W_i G(\eta_i)=\frac{2}{\pi}, \tag{24b}$$

where

$$W_i=\frac{1-\eta_i^2}{n+1}, \qquad (i=1, ..., n), \tag{25a}$$

$$\eta_i=\cos\left(\frac{i\pi}{n+1}\right), \qquad (i=1, ..., n), \tag{25b}$$

$$\xi_j=\cos\left(\frac{\pi}{2}\frac{2j-1}{n+1}\right), \qquad (j=1, ..., n+1). \tag{25c}$$

It may be shown that the $(n/2+1)^{th}$ equation in (24) is automatically satisfied. Thus, the equations given by (24) constitute a system of n+1 equations for n+1 unknowns, $G(\eta_i)$, (i=1, ..., n), and f. Note that the system is also nonlinear in f (Geçit and Gökpınar 1985) and an interpolation scheme is required as being Eqs. (21).

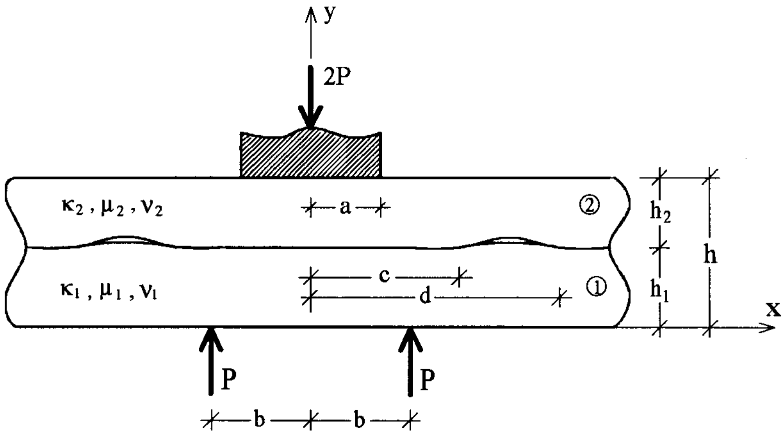

Fig. 3 The discontinuous contact between two elastic layers

4.2 The discontinuous contact between two elastic layers

Since the interface cannot carry tensile tractions, for $P>P_{cr}$ or $\lambda>\lambda_{cr}$ there will be separation between two elastic layers (Fig. 3). Assuming that the separation area is described by $c<x<d$, $y=h_1$, where c and d are unknown and are functions of P or λ.

In this case, the separation problem must be solved under the following boundary conditions:

$$\tau_{2xy}(x,h)=0, \qquad (0\le x<\infty), \tag{26a}$$

$$\tau_{2xy}(x,h_1)=0, \qquad (0\le x<\infty), \tag{26b}$$

$$\tau_{1xy}(x,h_1)=0, \qquad (0\le x<\infty), \tag{26c}$$

$$\tau_{1xy}(x,0)=0, \qquad (0\le x<\infty), \tag{26d}$$

$$\sigma_{1y}(x,0)=-P\,\delta(x-b), \tag{26e}$$

$$\frac{\partial}{\partial x}[v_2(x,h_1)-v_1(x,h_1)]=\begin{cases}\varphi(x), & c<x<d\\ 0, & 0\le x<c,\ d<x<\infty\end{cases}, \tag{26f}$$

$$\sigma_{2y}(x,h)=\begin{cases}-p(x), & 0\le x<a\\ 0, & a<x<\infty\end{cases}, \tag{26g}$$

$$\sigma_{2y}(x,h_1)=\sigma_{1y}(x,h_1), \qquad (0\le x<c,\ d<x<\infty), \tag{26h}$$

$$\sigma_{2y}(x,h_1)=\sigma_{1y}(x,h_1)=0, \qquad (c<x<d), \tag{26i}$$

$$\frac{\partial}{\partial x}[v_2(x,h)]=0, \qquad (0\le x<a). \tag{26j}$$

Utilising the boundary conditions defined in Eqs. (26a)-(26h), the functions A_i, B_i, C_i and D_i (i=1, 2) which appear in (4) and (5) may be obtained in terms of $p(x)$ and $\varphi(x)$. The new unknown functions $p(x)$ and $\varphi(x)$ are then determined from the conditions (26i) and (26j) which have not yet been satisfied. These conditions give the following system of integral equations:

$$\frac{1}{\pi}\int_{-a}^{a}\left[\frac{1}{t-x}+\frac{1}{h}k_1(x,t)\right]p(t)dt+\frac{1}{\pi h}\frac{4\mu_2}{(1+\kappa_2)}\int_c^d k_5(x,t)\varphi(t)dt-\frac{1}{\pi}\frac{P}{h}k_2(x)=0, \quad (-a<x<a), \quad (27a)$$

$$\frac{1}{\pi}\frac{4\mu_2}{(1+\kappa_2)(1+\beta)}\int_c^d\left[\frac{1}{t-x}+\frac{1}{t+x}+\frac{1}{h}k_7(x,t)\right]\varphi(t)dt+\frac{1}{\pi h}\int_{-a}^{a}k_6(x,t)p(t)dt-\frac{1}{\pi}\frac{P}{h}k_4(x)-\rho_2 g h_2=0, \quad (c<x<d), \quad (27b)$$

where β is the ratio of the elastic constants given in the Appendix, and the kernels $k_1(x, t)$, $k_2(x)$, $k_4(x, t)$, $k_5(x, t)$, $k_6(x, t)$ and $k_7(x, t)$ are given by (A3), (A4), (A6), (A8), (A9) and (A10) in the Appendix, respectively.

The index of the integral Eq. (27a) is +1. On the other hand, because of the smooth contact at the end points c and d, the function $\varphi(x)$ is zero at the ends and the index of the integral Eq. (27b) is −1. In this case, the two relations which are needed to determine the unknown constants c and d are the consistency condition of the integral Eq. (27b) and the single-valuedness condition:

$$\int_c^d \varphi(t)dt=0. \quad (28)$$

Designating the variables (x, t) on $y=h$ and $y=h_1$ by (x_1, t_1) and (x_2, t_2) respectively, and defining the following dimensionless quantities,

$$\eta_1=t_1/a, \quad \eta_2=\frac{2t_2}{d-c}-\frac{d+c}{d-c}, \quad (29a)$$

$$\xi_1=x_1/a, \quad \xi_2=\frac{2x_2}{d-c}-\frac{d+c}{d-c}, \quad (29b)$$

$$g_1(\eta_1)=\frac{p(t_1)}{P/h}, \quad g_2(\eta_2)=\frac{4\mu_2}{1+\kappa_2}\frac{\varphi(t_2)}{P/h}, \quad (29c)$$

the system of integral Eqs. (27) may be expressed as follows:

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{\eta-\xi}+\frac{a}{h}m_1(\xi,\eta)\right]g_1(\eta)d\eta+\frac{1}{\pi}\frac{d-c}{2h}\int_{-1}^{1}m_2(\xi,\eta)g_2(\eta)d\eta-\frac{1}{\pi}m_3(\xi)=0, \quad (-1<\xi<1), \quad (30a)$$

$$\frac{1}{\pi}\frac{1}{(1+\beta)}\int_{-1}^{1}\left[\frac{1}{\eta-\xi}+\frac{1}{\eta+\xi+2\frac{d+c}{d-c}}+\frac{d-c}{2h}m_5(\xi,\eta)\right]g_2(\eta)d\eta+\frac{1}{\pi}\frac{a}{h}\int_{-1}^{1}m_4(\xi,\eta)g_1(\eta)d\eta-\frac{1}{\pi}m_6(\xi)-\frac{1}{\lambda}=0, \quad (-1<\xi<1), \quad (30b)$$

where

$$m_1(\xi_1,\eta_1)=k_1(x_1,t_1), \quad (31a)$$

$$m_2(\xi_1,\eta_2)=k_5(x_1,t_2), \quad (31b)$$

$$m_3(\xi_1)=k_2(x_1), \quad (31c)$$

$$m_4(\xi_2,\eta_1)=k_6(x_2,t_1), \quad (31d)$$

$$m_5(\xi_2,\eta_2)=k_7(x_2,t_2), \quad (31e)$$

$$m_6(\xi_2)=k_4(x_2). \tag{31f}$$

In Eqs. (30), the subscripts have been deleted since the variables ξ_1, η_1, ξ_2 and η_2 all vary between −1 and +1. Similarly, the additional conditions (8) and (28) may be expressed as

$$\frac{a}{h}\int_{-1}^{1} g_1(\eta)d\eta=2, \tag{32a}$$

$$\int_{-1}^{1} g_2(\eta)d\eta=0. \tag{32b}$$

In order to solve the system of integral equations, it is found to be more convenient to assume that (30b) as well as (30a) has an index +1 (Civelek *et al.* 1978) and let

$$g_1(\eta)=G_1(\eta)/(1-\eta^2)^{1/2}, \qquad (-1<\eta<1), \tag{33a}$$

$$g_2(\eta)=G_2(\eta)/(1-\eta^2)^{1/2}, \qquad (-1<\eta<1). \tag{33b}$$

To insure smooth contact at the end points of the separation area, it is imposed the following condition on G_2:

$$G_2(-1)=0, \qquad G_2(+1)=0. \tag{34a,b}$$

Using appropriate Gauss-Chebyshev integration formula (Erdoğan and Gupta 1972), Eqs. (30a), (30b), (32a) and (32b) are reduced following algebraic expressions.

$$\sum_{i=1}^{n} W_i\left\{G_1(\eta_i)\left[\frac{1}{\eta_i-\xi_j}+\frac{a}{h}m_1(\xi_j,\eta_i)\right]+\frac{d-c}{2h}G_2(\eta_i)m_2(\xi_j,\eta_i)\right\}=\frac{1}{\pi}m_3(\xi_j), \quad (j=1,\ ...,\ n-1), \tag{35a}$$

$$\sum_{i=1}^{n} W_i\left\{G_2(\eta_i)\frac{1}{1+\beta}\left[\frac{1}{\eta_i-\xi_j}+\frac{1}{\eta_i+\xi_j+2\frac{d+c}{d-c}}+\frac{d-c}{2h}m_5(\xi_j,\eta_i)\right]+\frac{a}{h}G_1(\eta_i)m_4(\xi_j,\eta_i)\right\}=\frac{1}{\pi}m_6(\xi_j)+\frac{1}{\lambda},$$

$$(j=1,\ ...,\ n-1), \tag{35b}$$

$$\frac{a}{h}\sum_{i=1}^{n} W_iG_1(\eta_i)=\frac{2}{\pi}, \tag{36a}$$

$$\sum_{i=1}^{n} W_iG_2(\eta_i)=0, \tag{36b}$$

where W_i, ξ_j and η_i are given by (14a)-(14c). Eqs. (34), (35) and (36) give $2n$+2 algebraic equations to determine the $2n$+2 unknowns $G_1(\eta_i)$, $G_2(\eta_i)$, (i=1, ..., n), c and d. The system is nonlinear. So, an interpolation scheme is required for the solution. If c and d are selected for known $\lambda>\lambda_{cr}$ and they are substituted into (35a,b), $G_1(\eta_i)$ and $G_2(\eta_i)$, (i=1, ..., n) are obtained. But, at the same time these values must also satisfy Eqs. (36a,b). If these equations are not satisfied, c and d must be changed and the solution must be repeated until the Eqs. (35) and (36) are satisfied at the same time. After $G_1(\eta_i)$, $G_2(\eta_i)$, (i=1, ..., n), c and d are determined, $\sigma_y(x, h_1)$ contact stress out of (c, d) can be calculated by making use of Eq. (27b).

The separation between two elastic layers may be expressed as

$$\frac{\partial}{\partial x}[v_2(x,h_1)-v_1(x,h_1)]=\varphi(x), \qquad (c<x<d), \tag{37a}$$

or

$$\bar{v}(x,h_1)=v_2(x,h_1)-v_1(x,h_1)=\int_b^x \varphi(t)dt, \quad (c<x<d), \tag{37b}$$

If the values of $\varphi(t)$ is calculated from Eq. (29c), Eq. (37b) may be written as

$$\frac{\bar{v}(x,h_1)}{P}\frac{4\mu_2}{1+\kappa_2}=\frac{d-c}{2h}\int_{-1}^{\xi} g_2(\eta)d\eta, \qquad (-1<\xi<1), \tag{38}$$

where,

$$\xi=\frac{2x}{d-c}-\frac{d+c}{d-c}. \tag{39}$$

Also using appropriate Gauss-Chebyshev integration formula and taking +1 the index of Eq. (38), the following expression may be written for the separation.

$$\frac{4\mu_2}{\pi(1+\kappa_2)}\frac{\bar{v}(x,h_1)}{P}=\frac{d-c}{2h}\sum_{i=1}^{k-1} W_i G_2(\eta_i), \qquad (k=2, ..., n-1), \tag{40}$$

where W_i and η_i are given by (14a,b).

5. Results and discussion

Some of the calculated results obtained from the solution of the continuous and discontinuous

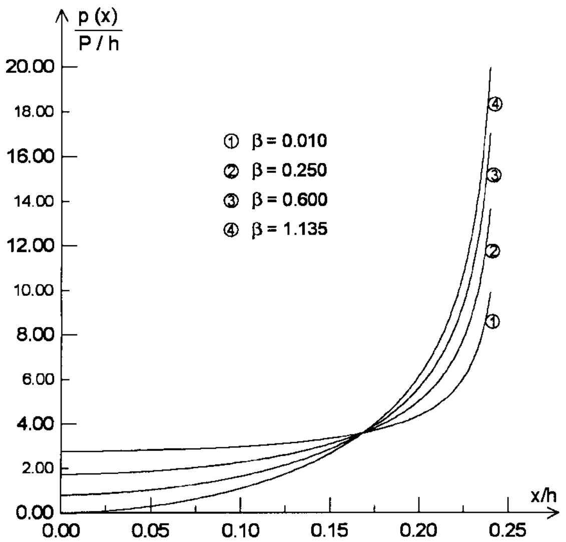

Fig. 4 Contact pressure distribution under the stamp for the case of continuous contact (b/h=1.0, h_1/h=0. 50, a/h=0.25)

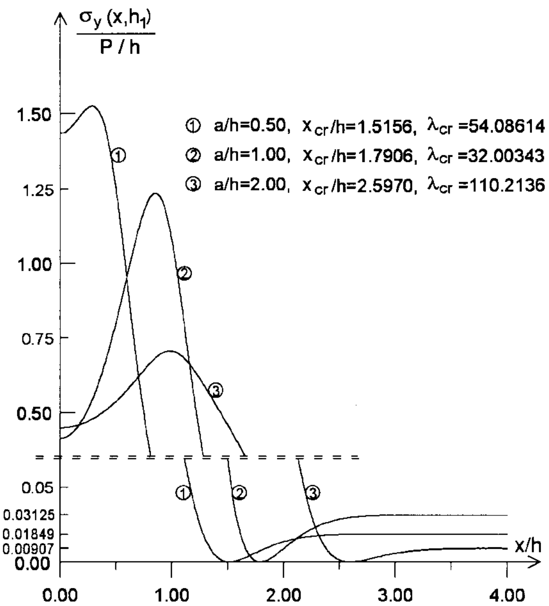

Fig. 5 Contact stress distribution between two elastic layers for continuous contact (b/h=1.0, β= 0.05, h_1/h=0. 50)

contact problems described in the previous section for various dimensionless quantities such as a/h, b/h, h_1/h, β and λ are shown in Figs. 4-10 and Table 1. Fig. 4 shows the normalized contact pressure $p(x)/P/h$ for the continuous contact case. The contact pressure becomes infinitely large at the corners of the rigid stamp. As it can be seen in Fig. 4, if β defined in Appendix is sufficiently increased (i.e., β=1.135), the contact pressure $p(x)$ becomes zero around x=0. For bigger values of β, $p(x)$ changes sign and a separation of the contacting surface of the stamp and the layered composite may take place around x=0. The solution given continuous contact case, of course, would not be valid for this case. For fixed values of b/h, h_1/h and β, Fig. 5 shows variation of the contact stress $\sigma_y(x, h_1)$ with a/h for continuous contact between two elastic layers described in Section 3. As a/h increases, the initial separation point x_{cr} seems to increase and the contact stress $\sigma_y(x, h_1)$ seems to decrease.

Fig. 6 and Table 1 show the variations of starting point of the separation with h_1/h and β for discontinuous contact between the stamp and the layered composite. As it can be seen in Fig. 6 and Table 1, the discontinuous contact area between the stamp and the layered composite increases as flexibility of the layered composite depending on b/h, h_1/h and β increases. If the flexibility decreases, i.e. if β decreases and h_1/h increases, the discontinuous contact area decreases, and if β sufficiently small and h_1/h sufficiently big, the discontinuous contact between the stamp and the layered composite is replaced by continuous contact. This case can be seen in Fig. 6 for β=0.10 and $h_1/h \geq 0.5532$ for the case of the separation starting from edges of the stamp and it can be seen in Table 1 for either β=0.10 and $h_1/h \geq 0.5392$ or β=0.20 and $h_1/h \geq 0.7112$ for the case of the separation starting from x=0 symmetry axis. Fig. 7 shows the contact pressure $p(x)/P/h$ under the stamp for the discontinuous contact case. For small b/h values (i.e., b/h=0.10 and 0.35), the separation between the stamp and the layered composite starts from edges of the stamp, and for larger b/h values (i.e., b/h=0.90, 1.00 and 1.20), the separation starts from x=0 symmetry axis. For

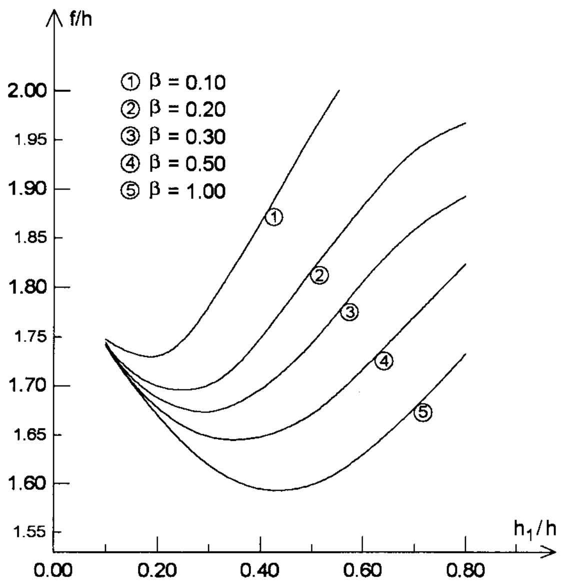

Fig. 6 Variations of starting point of the separation with h_1/h and β for the case of the separation starting from edge of the stamp (a/h=2.0, b/h=1.0)

Table 1 Variations of starting point of the separation with h_1/h and β for the case of the separation starting from symmetry axis (a/h=0.50, b/h=2.0)

h_1/h ⇓	f/h $\beta=0.10$	$\beta=0.20$	$\beta=0.30$	$\beta=1.00$
0.20	0.4108	0.4181	0.4205	0.4238
0.30	0.4036	0.4280	0.4357	0.4462
0.40	0.3541	0.4165	0.4349	0.4592
0.50	0.2133	0.3769	0.4157	0.4635
0.60	0.0000	0.2914	0.3719	0.4592
0.70	0.0000	0.0903	0.2913	0.4459
0.80	0.0000	0.0000	0.1776	0.4211

$0.40 < b/h < 0.80$, a separation would not occur under the rigid stamp and the discontinuous contact case is no longer valid in this case.

Fig. 8 shows important results giving the distances c and d which define the separation zone between two elastic layers. It appears that, for a fixed value of β and increasing load factor λ, c/h approaches a constant asymptotic value (of approximately $1.5h$). However, d/h keeps increasing with increasing λ. Sharp point in this graphic are corresponding to the initial separation loads and the initial separation points. In Fig. 9, the variation of the normalized contact stress $\sigma_y(x, h_1)/P/h$ at the interface two elastic layers with a/h is given for discontinuous contact case. As it can be seen in graphic, there are three regions in the discontinuous contact between two elastic layers. These are the continuous contact region, separation zone, and also the continuous contact region where the effect of the external load (P) decreases and disappears infinitely. The separation zone (d–c) is equal to $1.0246h$ for a/h=0.5, $1.5407h$ for a/h=1.0, $1.1856h$ for a/h=1.5 and $0.4883h$ for a/h=2.0. These

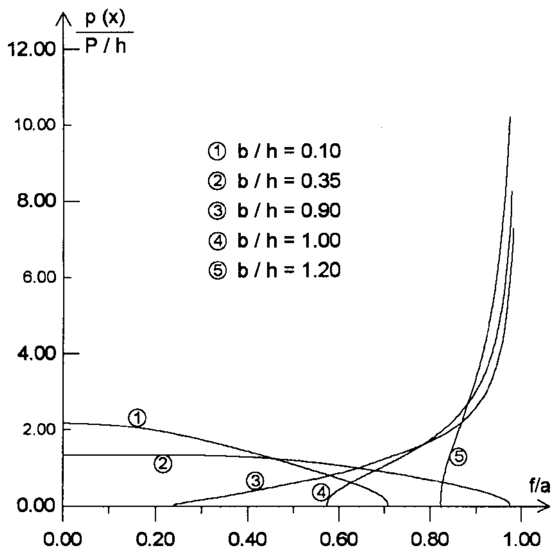

Fig. 7 Contact pressure distribution under the stamp for the case of discontinuous contact (a/h=1.0, h_1/h=0.50, β=0.50)

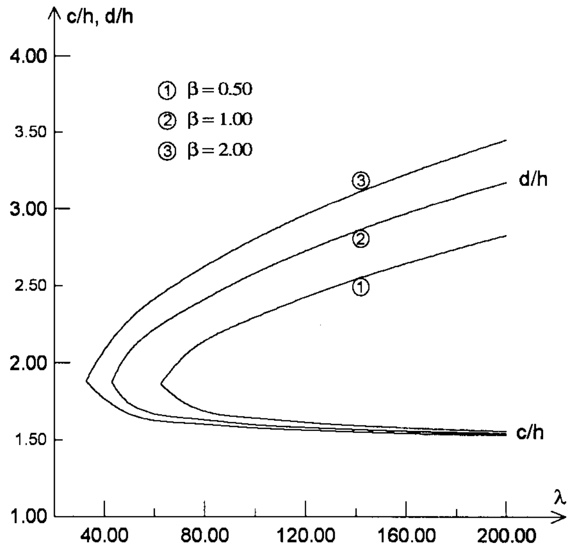

Fig. 8 Separation distances c and d between two elastic layers as a function of load factor λ for various values of β (b/h=1.0, a/h=0.10, h_1/h=0.50)

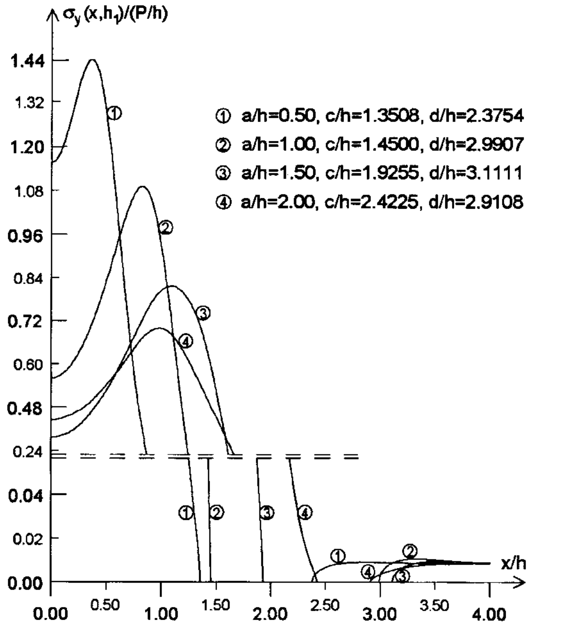

Fig. 9 Contact stress distribution between two elastic layers for discontinuous contact (b/h=1.0, β=0.05, h_1/h=0.50, λ=150>λ_{cr})

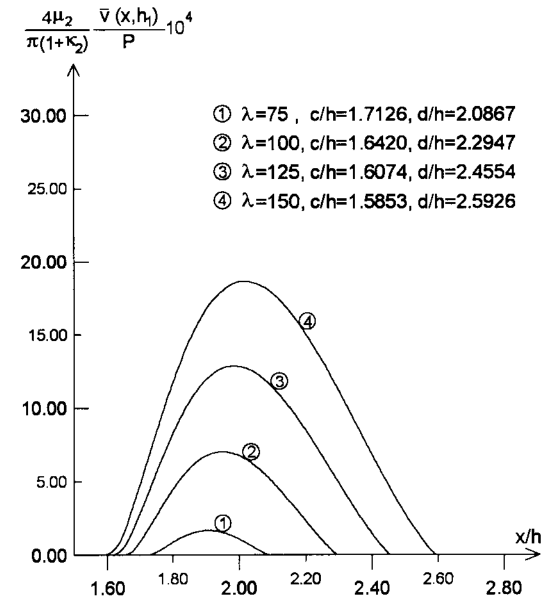

Fig. 10 Separation displacement $\bar{v}(x,h_1)$ between two elastic layers as a function x for various values of the load factor λ (b/h=1.0, a/h=0.10, h_1/h=0.50, β=0.50)

values show that the separation zone increases until a certain value of a/h (i.e., $a/h \cong 1.0$), then for a/h>1.0, it becomes decreasing.

Some sample results calculated from Eq. (40) giving the displacement $\bar{v}(x,h_1)$ in the separation zone $c<x<d$, $y=h_1$, are shown in Fig. 10 as function x for various values of λ. As expected, The separation zone and the separation displacement $\bar{v}(x,h_1)$ increase with increasing load factor λ.

6. Conclusions

It has been demonstrated that the support width, the rigid stamp width, and the elastic properties and the thickness of the layers play a very important role in the formation of the continuous and the discontinuous contact area, the initial separation point, the separation displacement, and the stress distribution on the contact surface. The separation both between the layered composite and the rigid stamp, and between two elastic layers occur in various dimensionless values for various dimensionless quantities as mentioned in section 5. However, generally, in order for the separation not to occur or to be more difficult :

· The rigid stamp width and the support width must be sufficiently small (a/h<1.0, b/h<2.0).
· The lower layer must be more rigid than the upper layer (β<1.0).
· The thickness of the layers must be close to each other.

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Appendix

For the case in which gravity forces exist, i.e., special solution of the Navier equations for each strips of which heights are h_1 and h_2, respectively, the components of the displacements and the stresses are given following expressions.

$$u_{1p}(x)=\frac{3-\kappa_1}{8\mu_1}\frac{\rho_1 g h_1}{2}x, \tag{A1a}$$

$$v_{1p}(y)=\frac{\kappa_1-1}{\kappa_1+1}\frac{\rho_1 g y}{2\mu_1}(y-h_1)-\frac{1+\kappa_1}{8\mu_1}y(\rho_2 g h_2+\rho_1 g h_1/2), \tag{A1b}$$

$$u_{2p}(x)=\frac{3-\kappa_2}{8\mu_2}\frac{\rho_2 g h_2}{2}x, \tag{A1c}$$

$$v_{2p}(y)=-\frac{\rho_2 g y}{2\mu_2}\left[\frac{1+\kappa_2}{8}h_2-\frac{\kappa_2-1}{\kappa_2+1}(h_1+h-y)\right], \tag{A1d}$$

$$\sigma_{1xp}(y)=\frac{3-\kappa_1}{\kappa_1+1}\frac{\kappa_1-1}{\kappa_1+1}\frac{\rho_1 g}{2}(2y-h_1), \tag{A2a}$$

$$\sigma_{1yp}(y)=-\rho_2 g h_2+\rho_1 g(y-h_1), \qquad (0\le y\le h_1), \tag{A2b}$$

$$\sigma_{2xp}(y)=\frac{3-\kappa_2}{\kappa_2+1}\frac{\kappa_2-1}{\kappa_2+1}\frac{\rho_2 g}{2}(2y-h-h_1), \tag{A2c}$$

$$\sigma_{2yp}(y)=\rho_2 g(y-h), \qquad (h_1\le y\le h), \tag{A2d}$$

$$\tau_{1xyp}(x,y)=\tau_{2xyp}(x,y)=0, \tag{A2e}$$

where g, ρ_1 and ρ_2 are gravity acceleration, mass density of the strip 1 and 2, respectively.
Kernels of integral Eqs. (7), (9), (16) and (27) are expressed as follows:

$$k_1(x,t)=\int_0^\infty\left\{\frac{1}{\Delta^*(\omega)}[K1(\omega).K2(\omega)+\beta K3(\omega).K4(\omega)]-1\right\}\sin(t-x)\frac{\omega}{h}d\omega, \tag{A3}$$

$$k_2(x)=\int_0^\infty \frac{4\beta e^{-\omega}}{\Delta^*(\omega)}\left\{(1-e^{2\omega r}).K5(\omega)+\omega r[(1+e^{2\omega r}).K6(\omega)+2(e^{-2\omega}e^{2\omega r}-e^{-2\omega r})]\right\}$$
$$*\left[\sin(b+x)\frac{\omega}{h}-\sin(b-x)\frac{\omega}{h}\right]d\omega, \tag{A4}$$

$$k_3(x,t)=\int_0^\infty 2\frac{e^{-\omega}e^{-\omega r}}{\Delta^*(\omega)}K2(\omega)[e^{-2\omega r}-e^{-2\omega}-K6(\omega)]\cos(t-x)\frac{\omega}{h}d\omega, \tag{A5}$$

$$k_4(x)=\int_0^\infty 2\beta\frac{e^{\omega r}}{\Delta^*(\omega)}K7(\omega).K8(\omega).\left[\cos(b+x)\frac{\omega}{h}+\cos(b-x)\frac{\omega}{h}\right]d\omega, \tag{A6}$$

$$k_1^*(x,t)=\int_0^\infty\left\{\frac{1}{\Delta^*(\omega)}[K1(\omega).K2(\omega)+\beta K3(\omega).K4(\omega)]-1\right\}*\left[\sin(t+x)\frac{\omega}{h}-\sin(t-x)\frac{\omega}{h}\right]d\omega, \tag{A7}$$

$$k_5(x,t)=\int_0^\infty 2\frac{e^{-\omega}e^{-\omega r}}{\Delta^*(\omega)}K2(\omega).K9(\omega).\left[\cos(t-x)\frac{\omega}{h}-\cos(t+x)\frac{\omega}{h}\right]d\omega, \tag{A8}$$

$$k_6(x,t)=\int_0^\infty 2\frac{e^{-\omega}e^{-\omega r}}{\Delta^*(\omega)}K2(\omega).K9(\omega)\left[\cos(t-x)\frac{\omega}{h}+\cos(t+x)\frac{\omega}{h}\right]d\omega, \tag{A9}$$

$$k_7(x,t)=\int_0^\infty\left[\frac{(1+\beta)}{\Delta^*(\omega)}K2(\omega).K7(\omega)-1\right]\left[\sin(t+x)\frac{\omega}{h}+\sin(t-x)\frac{\omega}{h}\right]d\omega, \tag{A10}$$

where,

$$\Delta^*(\omega)=-K2(\omega).K3(\omega)-\beta.K4(\omega).K7(\omega),$$
$$K1(\omega)=e^{-4\omega r}+e^{-4\omega}-2e^{-2\omega}e^{-2\omega r},$$
$$K2(\omega)=1+e^{2\omega r}(-2-4\omega^2 r^2+e^{2\omega r}),$$
$$K3(\omega)=e^{-4\omega}-e^{-4\omega r}-2e^{-2\omega}e^{-2\omega r}(2\omega-2\omega r),$$
$$K4(\omega)=1-e^{2\omega r}(4\omega r+e^{2\omega r}),$$
$$K5(\omega)=(1+\omega)e^{-2\omega r}+(-1+\omega)e^{-2\omega},$$

$$K6(\omega)=(-\omega+\omega r)(e^{-2\omega r}+e^{-2\omega}),$$
$$K7(\omega)=e^{-4\omega r}+e^{-4\omega}-e^{-2\omega}e^{-2\omega r}(2+4\omega^2+4\omega^2 r^2-8\omega^2 r),$$
$$K8(\omega)=-1+\omega r+e^{2\omega r}(1+\omega r),$$
$$K9(\omega)=e^{-2\omega}-e^{-2\omega r}+K6(\omega),$$

and,

$$\beta=\frac{1+\kappa_1}{1+\kappa_2}\frac{\mu_2}{\mu_1}, \qquad \omega=\alpha h, \qquad r=h_1/h.$$