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Effect of axial stretching on large amplitude free vibration of a suspended cable

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Abstract. This paper presents the effect of axial stretching on large amplitude free vibration of an extensible suspended cable supported at the same level. The model formulation developed in this study is based on the virtual work-energy functional of cables which involves strain energy due to axial stretching and work done by external forces. The difference in the Euler equations between equilibrium and motion states is considered. The resulting equations govern the horizontal and vertical motion of the cables, and are coupled and highly nonlinear. The solution for the nonlinear static equilibrium configuration is determined by the shooting method while the solution for the large amplitude free vibration is obtained by using the second-order central finite difference scheme with time integration. Numerical examples are given to demonstrate the vibration behaviour of extensible suspended cables.

Key words: cables; axial stretching; free vibration; large amplitude vibration; nonlinear vibration.

Introduction

Cable structures have been used extensively in many civil and ocean engineering applications. Practical considerations of cable behaviour may be limited to the case of small sag or small amplitude of vibration such as finding the natural frequencies and mode shapes. However, in some engineering applications such as cables in offshore engineering operations, large amplitudes of vibration are encountered. The large amplitude refers to the amplitude of vibration measured from the cable's equilibrium position which may be the same as or larger than the order of magnitude of the sag. The subject of large amplitude vibration of cables has been investigated by many researchers over the past several years. Early papers on the subject include those by Keller (1959) and Anand (1969). Rega *et al.* (1984) used a simple model to investigate non-linear free vibration of a suspended cable. Forced vibration of elastic suspended cables was investigated by Al-Noury and Ali (1985), and Benedettini and Rega (1987). Ali (1986) investigated the nonlinear response of sagged cables with movable supports. Cai and Chen (1994) investigated the nonlinear dynamic

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response of an inclined elastic cable under parametric and external resonances. Leissa and Saad (1994) studied the large amplitude free vibration of elastic strings in which the effects of longitudinal and transverse displacements are coupled.

In almost all of the aforementioned papers, the static equilibrium configuration of the cables is assumed to be straight line or having a small sag and the effect of axial stretching is not considered seriously. The present paper aims to develop a rigorous formulation of the cable problem that includes the effect of axial stretching on the large amplitude free vibration of a suspended cable with a large sag. Equations of motion in the horizontal and vertical directions are obtained by considering the difference in the virtual work-energy functional of the cable between two states, namely the equilibrium state and the motion state. A finite difference scheme with time integration is used to solve these equations. The effect of axial stretching on the large amplitude free vibration behaviour of cables is investigated by considering some numerical examples for the case of specified total unstrained arc-length of cable and the case of specified applied tension.

2. Development of equations of motion

Fig. 1 shows a typical cable problem considered in this study. The cable with a span length L is supported at the same level, with one end of the cable fixed and the other end free to slide over the support where the specified tension is applied to maintain the cable in equilibrium position. Three distinct states of the cable configuration are to be noted. The first state is the unstretched state, the second the equilibrium state, and the third the displaced state. In the unstretched state, the cable suspends by its own weight and its configuration takes on the catenary form. Due to axial stretching, the cable moves to the equilibrium position which is considered as an initial configuration for the cable. Owing to disturbances in loading, the cable is in the vibration or displaced state. The coordinate parameters in the three states are represented as follows:

i) Unstretched state: X, Y, S
ii) Equilibrium state: x_0, y_0, s_0
iii) Displaced state: x, y, s.

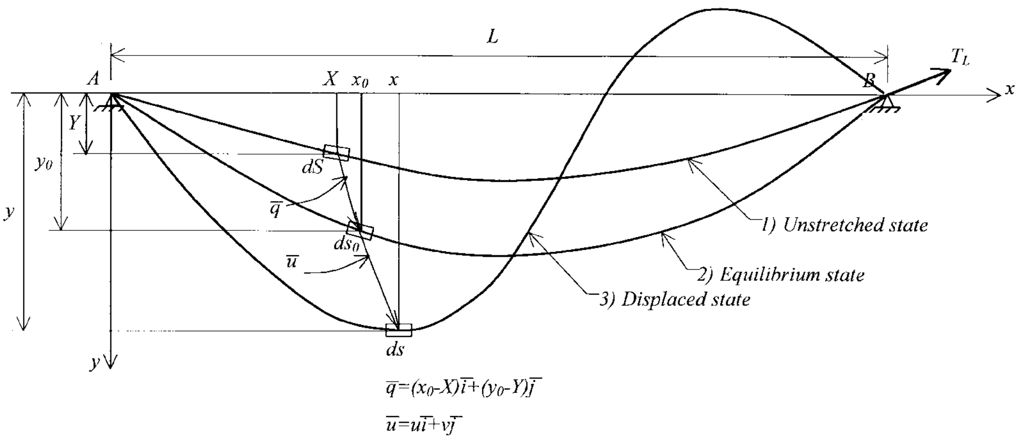

Fig. 1 Cable configurations at various states

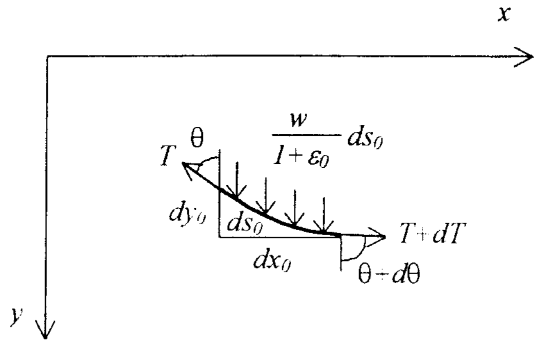

Fig. 2 Free body of infinitesimal cable segment

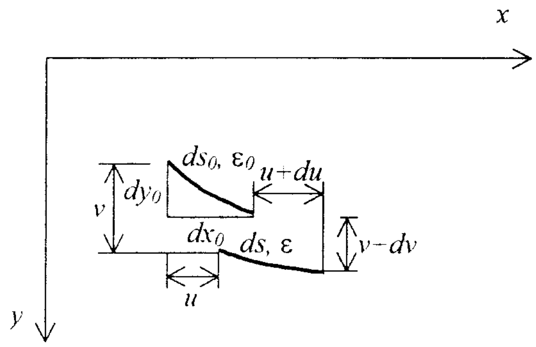

Fig. 3 Equilibrium and displaced positions of cable segment

Fig. 2 shows the components of the forces acting on an infinitesimal segment in the equilibrium position. Fig. 3 shows the same segment at the displaced position in which u and v are the displacement components in the x and y directions, respectively, and their magnitudes can be large. Consider an element having an unstretched length dS. At the equilibrium position, its length ds_0 is given by

$$ds_0 = \sqrt{1 + {y_0'}^2}\, dx_0 \tag{1}$$

in which a prime denotes differentiation with respect to x_0.

The strain ε_0 at the equilibrium position is

$$\varepsilon_0 = \frac{ds_0 - dS}{dS} \quad \text{and} \quad ds_0 = (1 + \varepsilon_0)dS \tag{2a, b}$$

Using Eqs. (1) and (2b), one gets

$$dS = \frac{\sqrt{1 + {y_0'}^2}}{(1 + \varepsilon_0)} dx_0 \tag{3}$$

The arc length ds in the displaced position (see Fig. 3) is given by

$$ds = \sqrt{(1 + u')^2 + (y_0' + v')^2}\, dx_0 \tag{4}$$

The strain at the displaced position is

$$\varepsilon = \frac{ds - dS}{dS} \tag{5}$$

In view of Eqs. (3) and (4), Eq. (5) can be written as

$$\varepsilon = \frac{1 + \varepsilon_0}{\sqrt{1 + y_0'^2}} \sqrt{(1 + u')^2 + (y_0' + v')^2} - 1 \tag{6}$$

The variation of strain ε furnishes

$$\delta\varepsilon = \frac{1 + \varepsilon_0}{\sqrt{1 + y_0'^2}} \frac{(1 + u')\delta u' + (y_0' + v')\delta v'}{\sqrt{(1 + u')^2 + (y_0' + v')^2}} \tag{7}$$

3. Virtual strain energy due to axial stretching

The virtual strain energy of the cable at the displaced position is given by

$$\delta U = \int EA\varepsilon\delta\varepsilon dS \tag{8}$$

where E is the elastic modulus of the cable, and A the undeformed cross-sectional area of the cable. Using Eqs. (3), (6), and (7), Eq. (8) becomes

$$\delta U = \int \left\{ \begin{aligned} &\left[\frac{EA(1 + \varepsilon_0)}{\sqrt{1 + y_0'^2}}(1 + u') - \frac{EA(1 + u')}{\sqrt{(1 + u')^2 + (y_0' + v_0')^2}}\right]\delta u' \\ &+ \left[\frac{EA(1 + \varepsilon_0)}{\sqrt{1 + y_0'^2}}(y_0' + v') - \frac{EA(y_0' + v')}{\sqrt{(1 + u')^2 + (y_0' + v')^2}}\right]\delta v' \end{aligned} \right\} dx_0 \tag{9}$$

4. Virtual work due to self-weight and inertia force

For free vibration, the virtual work due to the self-weight and the inertia force are given by

$$\delta W = \int \left\{ \left[-\frac{w\sqrt{1 + y_0'^2}}{g(1 + \varepsilon_0)}\ddot{u}\right]\delta u + \left[\frac{w\sqrt{1 + y_0'^2}}{(1 + \varepsilon_0)} - \frac{w\sqrt{1 + y_0'^2}}{g(1 + \varepsilon_0)}\ddot{v}\right] \delta v \right\} dx_0 \tag{10}$$

in which w is the weight of the cable per unit unstretched length, and the dot denotes differentiation with respect to time.

5. Euler equations

The virtual work of the cable is written as

$$\delta\pi = \delta U - \delta W \tag{11}$$

Using Eqs. (9) and (10), Eq. (11) becomes

$$\delta\pi = \int\left\{\left[\frac{EA\sqrt{1+\varepsilon_0}}{\sqrt{1+y'^2_0}}(1+u') - \frac{EA(1+u')}{\sqrt{(1+u')^2+(y_0'+v')^2}}\right]\delta u' + \frac{w\sqrt{1+y_0'^2}}{g(1+\varepsilon_0)}\ddot{u}\delta u\right\}dx_0$$
$$+\int\left\{\left[\frac{EA(1+\varepsilon_o)}{\sqrt{1+y'^2_0}}(y_0'+v') - \frac{EA(y_0'+v')}{\sqrt{(1+u')^2+(y_0'+v')^2}}\right]\delta v' + \left[-\frac{w\sqrt{1+y_0'^2}}{(1+\varepsilon_0)} + \frac{w\sqrt{1+y_0'^2}}{g(1+\varepsilon_0)}\ddot{v}\right]\delta v\right\}dx_0 \tag{12}$$

For static equilibrium, $\delta\pi$=0 and $u = v = u' = v' = \ddot{u} = \ddot{v} = 0$. Thus, the integration of Eq. (12) by parts yields two Euler equations with respect to the virtual displacements δu and δv.

$$\left[\frac{EA\varepsilon_0}{\sqrt{1+y'^2_0}}\right]' = 0\ , \tag{13}$$

$$\left[\frac{EA\varepsilon_0 y_0'}{\sqrt{1+y'^2_0}}\right]' + \frac{w\sqrt{1+y_0'^2}}{(1+\varepsilon_0)} = 0 \tag{14}$$

These two equations can also be obtained by considering the equilibrium of forces on the cable segment in the x and y directions, respectively.

For the cable in motion, $u \neq v \neq u' \neq v' \neq \ddot{u} \neq \ddot{v} \neq 0$, and the integration by parts gives two Euler equations associated with the virtual cable motion δu and δv.

$$\left[\frac{EA(1+\varepsilon_0)}{\sqrt{1+y'^2_0}}(1+u')\right]' - \left[\frac{EA(1+u')}{\sqrt{(1+u'^2)+(y_0'+v'^2)}}\right]' - \frac{w\sqrt{1+y'^2_0}}{g(1+\varepsilon_0)}\ddot{u} = 0 \tag{15}$$

$$\left[\frac{EA(1+\varepsilon_0)}{\sqrt{1+y'^2_0}}(y_0'+v')\right]' - \left[\frac{EA(y_0'+v')}{\sqrt{(1+u')^2+(y_0'+v')^2}}\right]' + \frac{w\sqrt{1+y'^2_0}}{(1+\varepsilon_0)} - \frac{w\sqrt{1+y'^2_0}}{g(1+\varepsilon_0)}\ddot{v} = 0 \tag{16}$$

By subtracting Eq. (13) from Eq. (15), and Eq. (14) from Eq. (16), one obtains the equations of motion in the u and v directions, respectively, as

$$\frac{EA\left[(1+\varepsilon_0)(1+y'^2_0)u'' - (1+u'+\varepsilon_0 u'')y_0'y_0''\right]}{\left(1+y'^2_0\right)^{3/2}} - \frac{EA(y_0'+v')\left[(y_0'+v')u'' - (1+u')v''\right]}{\left[\left(1+u'\right)^2+\left(y_0'+v'\right)^2\right]^{3/2}}$$
$$= \frac{w\sqrt{1+y'^2_0}}{g(1+\varepsilon_0)}\ddot{u} \tag{17}$$

$$\frac{EA\left[(1+\varepsilon_0)(1+y'^2_0)v'' + (1 - y_0'u' - \varepsilon_0 y_0'u')y_0''\right]}{\left(1+y'^2_0\right)^{3/2}} - \frac{EA(1+u')\left[(1+u')(y_0''+v'') - (y_0'+v')u''\right]}{\left[\left(1+u'\right)^2+\left(y_0'+v'\right)^2\right]^{3/2}}$$
$$= \frac{w\sqrt{1+y'^2_0}}{g(1+\varepsilon_0)}\ddot{v} \tag{18}$$

Eqs. (17) and (18) are the equations of motion for large amplitude free vibration of extensible suspended cables. If the terms y_0, y_0', and y_0'' are set to zero (implying the equilibrium configuration of the cable is a straight line), then Eqs. (17) and (18) are reduced to the same form given by Leissa and Saad (1994).

6. Method of solution

The shooting method is used to solve the nonlinear equilibrium Eqs. (13) and (14). The results of the static solution are the strain at the equilibrium state ε_o at any position and the equilibrium position y_0 at any position x_0 along the span length. For the case of specified end tension, it is more convenient to use an expression for the tension at any position in the calculations. This expression can be obtained by considering the equilibrium equation in the tangential direction which is

$$dT = \frac{w}{1+\varepsilon_0} dy_0 \tag{19}$$

Integrating from x_0 to L and replacing $T(x_0)$ by $EA\varepsilon_0$, one obtains

$$T(x_0) = EA\varepsilon_0 = T_L - \frac{w}{1+\varepsilon_0} y(x_0) \tag{20}$$

Eqs. (13) and (20) are used for solving the case of specified end tension.

For free vibration analysis, Eqs. (17) and (18) are solved using the finite difference approach. The derivatives of u and v appearing in the equations of motion are replaced by the following finite difference approximations:

$$u' = \frac{u_{i+1}^t - u_{i-1}^t}{2h}, \; v' = \frac{v_{i+1}^t - v_{i-1}^t}{2h} \tag{21a, b}$$

$$u'' = \frac{u_{i+1}^i - 2u_i^t + u_{i-1}^t}{h^2}, \; v'' = \frac{v_{i+1}^t - 2v_i^t + v_{i-1}^t}{h^2} \tag{22a, b}$$

$$\ddot{u} = \frac{u_i^{t+1} - 2u_i^t + u_i^{t-1}}{k^2}, \; \ddot{v} = \frac{v_i^{t+1} - 2v_i^t + v_i^{t-1}}{k^2} \tag{23a, b}$$

in which h is the grid size and k is the time step.

7. Numerical examples and results

Numerical examples involving two practical cases are given to demonstrate the vibration behaviour of u and v. The first case deals with a cable with a fixed value of unstrained total arc-length and the elastic modulus is varied. The input data for this case are as follows: S=800 m, L=854 m, w=9.478 kN/m, A=0.1159 m^2, and the values of the elastic modulus E are 1.294×10^6 kN/m^2, 1.794×10^6 kN/m^2, and 2.294×10^6 kN/m^2. The second case deals with a cable with a fixed value of applied tension and the elastic modulus is varied. The input parameters, other than A, L and w as given in the first case, are taken as: T_L=17,000 kN, the assigned values of E are 1.794×10^5 kN/m^2, 1.794×10^6 kN/m^2, and 1.794×10^7 kN/m^2.

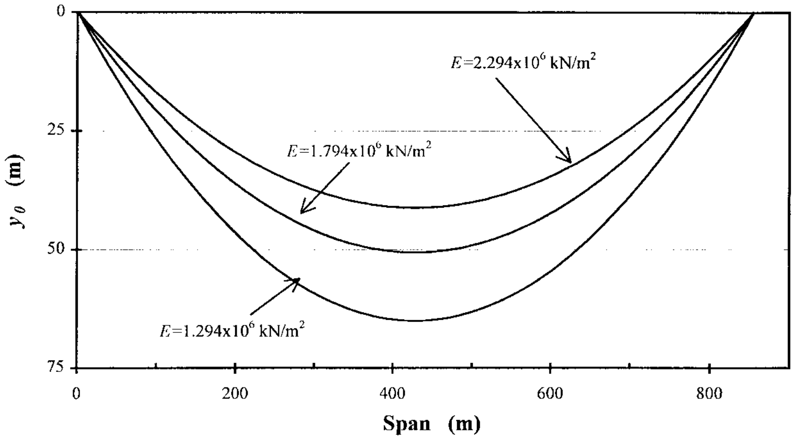

Fig. 4 Equilibrium profiles of cables for the specified unstretched total arc-length with different values of elastic modulus

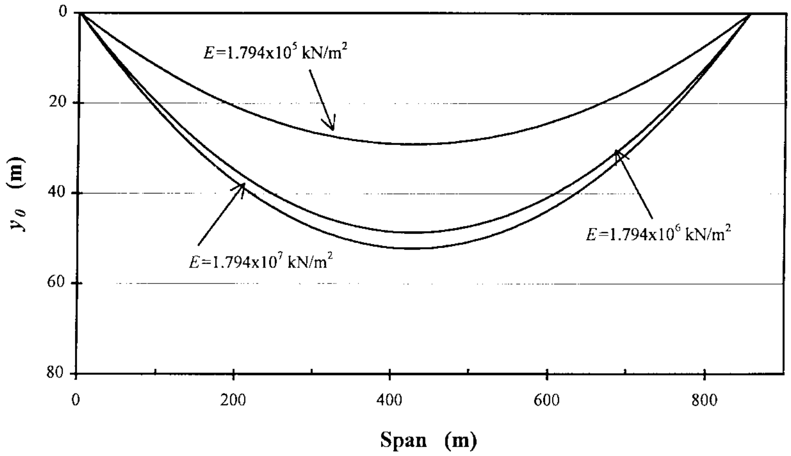

Fig. 5 Equilibrium profiles of cables for the specified end tension with different values of elastic modulus

Fig. 4 shows the static equilibrium positions of cables with a fixed value of unstrained total arc-length but different values of the elastic modulus. It is seen that the sag of the cable decreases when the elastic modulus is increased. The result is expected as the cable experiences a greater sag as EA decreases while T_L remains the same. On the other hand, if EA remains unchanged but T_L increases, the sag decreases. Fig. 5 shows the equilibrium positions for cables with a fixed value of the applied tension. Unlike the first case, it is seen that the sag increases with an increase in the elastic modulus. Figs. 6 to 11 show the motion of u and v plotted at the quarter and midspan length for the first case. The computed values of the maximum strain at the equilibrium position are also given. It can be seen from these figures that the motion of v is more periodic and has a much larger amplitude than the motion of u. The amplitude of v at midspan is slightly higher than the one at quarter span while the amplitude of u at quarter span has a much larger amplitude than the one at the midspan where it shows no oscillation. Fig. 12 shows the combined amplitude of vibration of u and v for the different values of E. The results show that

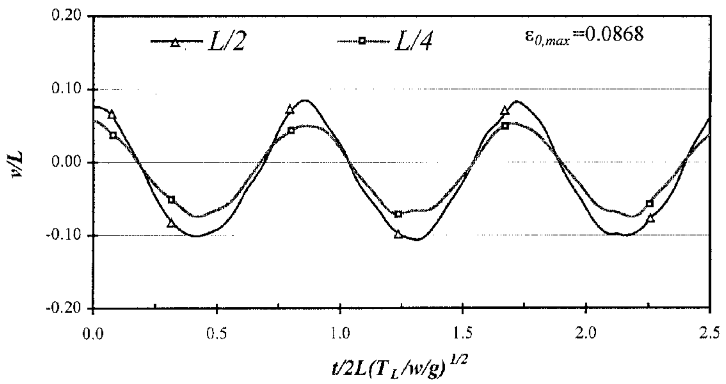

Fig. 6 Vibration amplitude of v at mid span and quarter span for S=800 m and E=1.294×10^6 kN/m^2

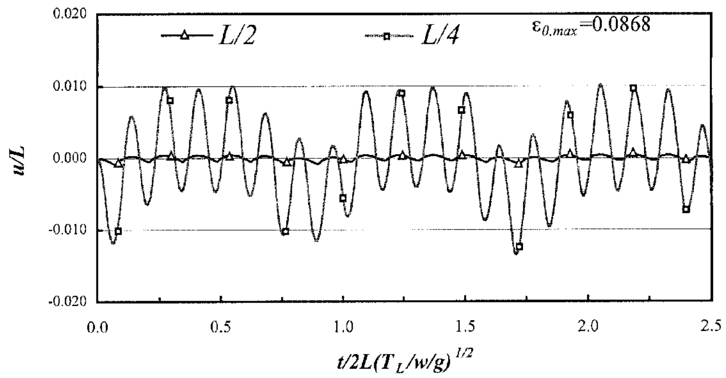

Fig. 7 Vibration amplitude of u at mid span and quarter span for S=800 m and E=1.294×10^6 kN/m^2

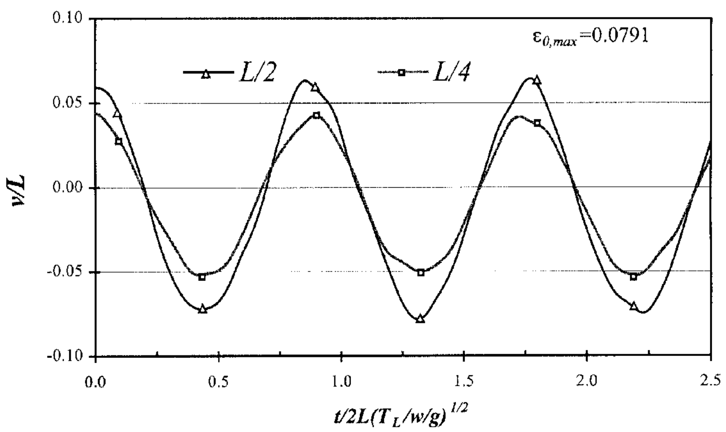

Fig. 8 Vibration amplitude of v at mid span and quarter span for S=800 m and E=1.794×10^6 kN/m^2

the amplitude of vibration decreases as the elastic modulus increases. These results have the same trends as those found in the static case.

Figs. 13 to 18 demonstrate the effect of stretching on the amplitude of vibration for cables with a specified value of applied tension. The motion of u and v show the same vibration behaviour as in

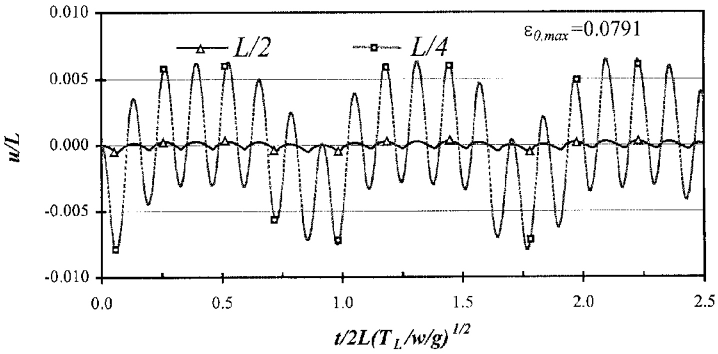

Fig. 9 Vibration amplitude of u at mid span and quarter span for S=800 m and E=1.794×10^6 kN/m^2

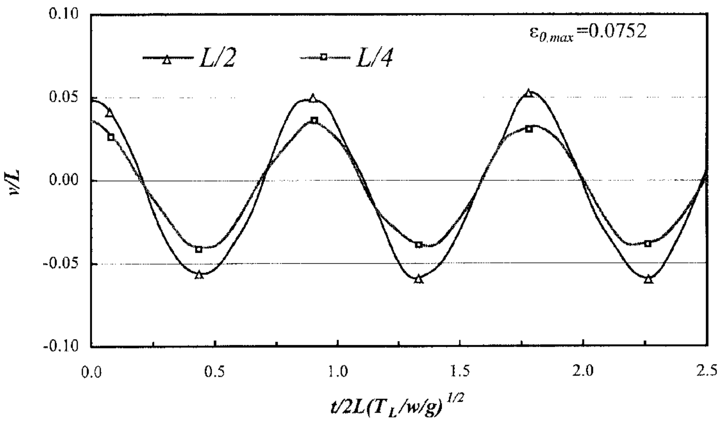

Fig. 10 Vibration amplitude of v at mid span and quarter span for S=800 m and E=2.294×10^6 kN/m^2

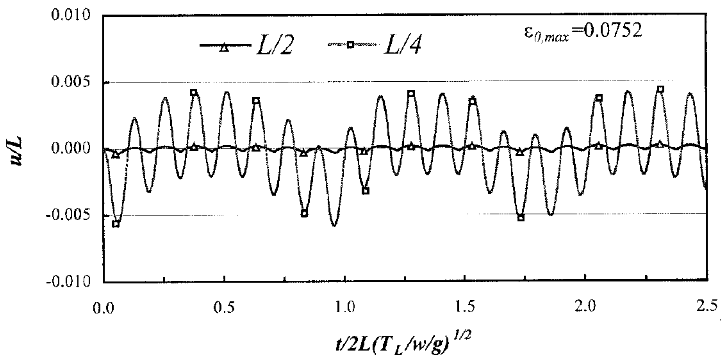

Fig. 11 Vibration amplitude of u at mid span and quarter span for S=800 m and E=2.294×10^6 kN/m^2

the first case. However, it is found that the amplitudes of u and v increase as the elastic modulus increases. The maximum combined amplitude of vibration of u and v versus elastic modulus is plotted in Fig. 19. It is noted that as the elastic modulus is increased beyond the values used earlier, the resulting cable motion is unstable and the amplitude of vibration cannot be determined.

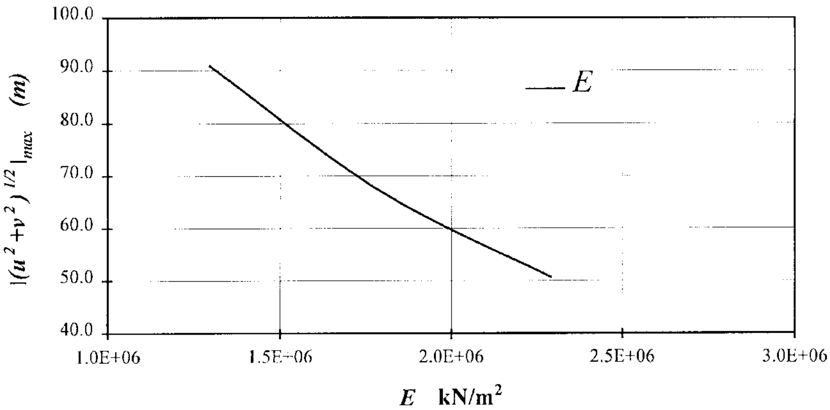

Fig. 12 Variation of the maximum combined amplitude versus elastic modulus for the specified total unstrained arc-length of 800 m

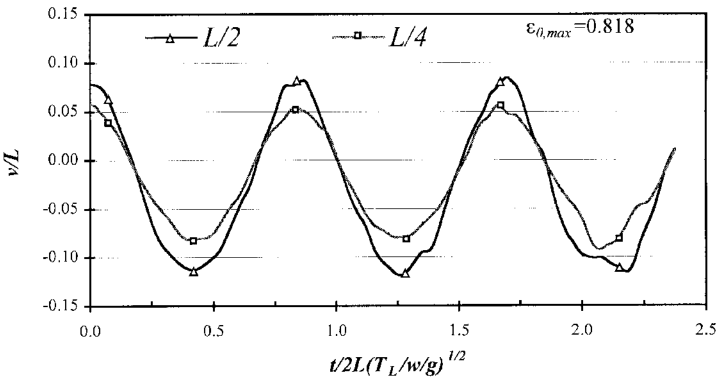

Fig. 13 Vibration amplitude of v at mid span and quarter span for T_L=17,000 kN and E=1.794×10^5 kN/m^2

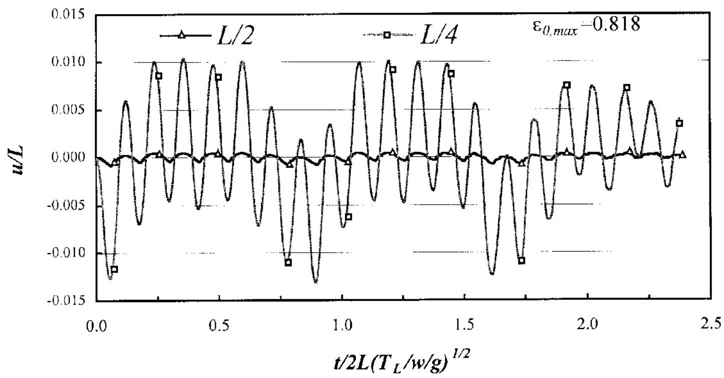

Fig. 14 Vibration amplitude of u at mid span and quarter span for T_L=17,000 kN and E=1.794×10^5 kN/m^2

8. Conclusions

The equations of motion of in-plane large amplitude free vibration of a suspended cable have

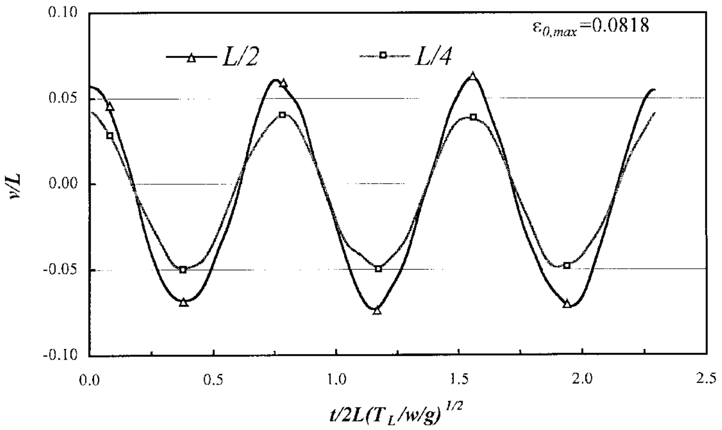

Fig. 15 Vibration amplitude of *v* at mid span and quarter span for T_L=17,000 kN and E=1.794×10^6 kN/m^2

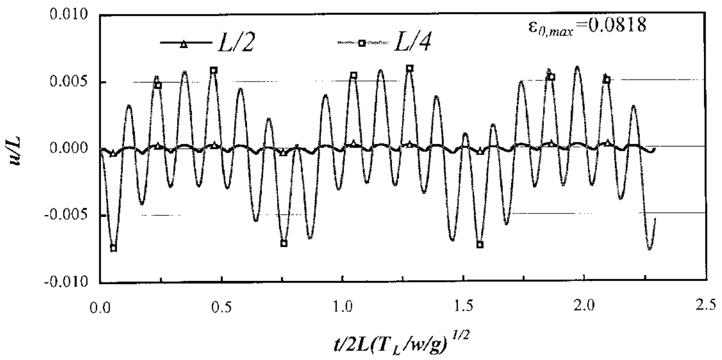

Fig. 16 Vibration amplitude of *u* at mid span and quarter span for T_L=17,000 kN and E=1.794×10^6 kN/m^2

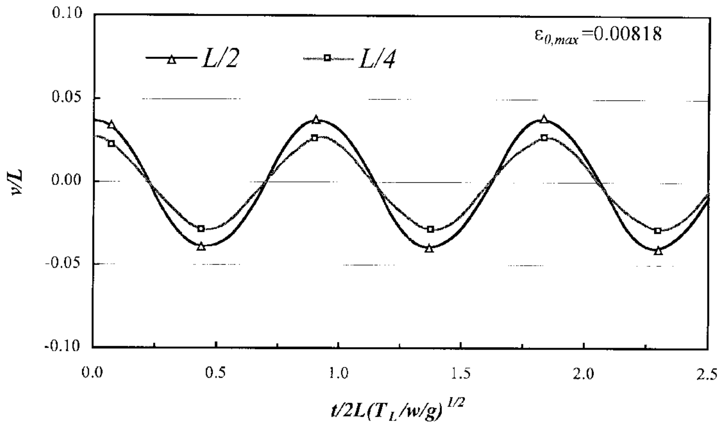

Fig. 17 Vibration amplitude of *v* at mid span and quarter span for T_L=17,000 kN and E=1.794×10^7 kN/m^2

been developed. The formulation is applicable to cables with large sag and large axial deformation as well as a variable tension force. The method can also be conveniently applied to cables with a specified end tension. Numerical examples for the cases of fixed unstrained total arc-length and

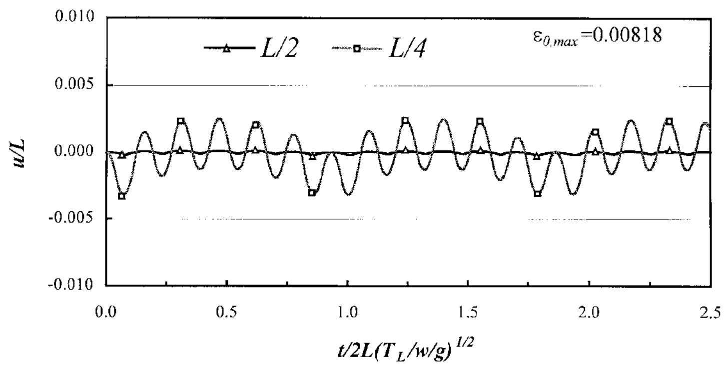

Fig. 18 Vibration amplitude of u at mid span and quarter span for T_L=17,000 kN and E=1.794×10^7 kN/m^2

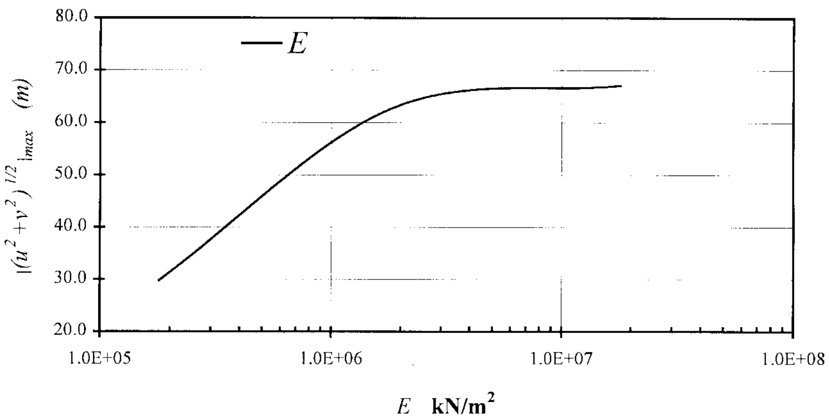

Fig. 19 Variation of the maximum combined amplitude versus elastic modulus for the specified end tension 17,000 kN

specified applied tension have been demonstrated. It can be concluded that for cables with a given total arc-length, the amplitude of vibration decreases when the elastic modulus is increased. However, for cables with a specified value of the applied tension, the results show that the amplitude of vibration increases with increasing values of elastic modulus.

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