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Meshless formulation for shear-locking free bending elements

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Abstract. An improved version of the Element-free Galerkin method (EFGM) is presented here for addressing the problem of transverse shear locking in shear-deformable beams with a high length over thickness ratio. Based upon Timoshenko's theory of thick beams, it has been recognized that shear locking will be completely eliminated if the rotation field is constructed to match the field of slope, given by the first derivative of displacement. This criterion is applied directly to the most commonly implemented version of EFGM. However in the numerical process to integrate strain energy, the second derivative of the standard Moving Least Square (MLS) shape functions must be evaluated, thus requiring at least a C^1 continuity of MLS shape functions instead of C^0 continuity in the conventional EFGM. Yet this hindrance is overcome effortlessly by only using at least a C^1 weight function. One-dimensional quartic spline weight function with C^2 continuity is therefore adopted for this purpose. Various numerical results in this work indicate that the modified version of the EFGM does not exhibit transverse shear locking, reduces stress oscillations, produces fast convergence, and provides a surprisingly high degree of accuracy even with coarse domain discretizations.

Key words: meshless methods; element-free Galerkin method; shear locking; thick beam.

1. Introduction

In recent years, the Element-free Galerkin Method (Belytschko *et al.* 1994) has been extensively applied to problems in applied mechanics and related field. It has been recognized as one of the most viable alternatives to the standard Galerkin Finite Element Method (FEM). As its name implies, EFGM requires only nodal data and a domain boundary description for formulation of the discrete Galerkin equations. This implies the complete absence of predefined connectivity between nodes, which may eliminate natural limitations caused by element structure. The applicability of EFGM, therefore, appears to be far more flexible than that of the standard FEM, especially in problems requiring unrestrained mesh entanglements such as crack propagation and large displacement/large strain analyses. Despite many notable advantages stated previously, very few researchers have adopted EFGM to address the problem of numerical difficulty in bending elements, known as transverse shear locking (see e.g., Cook *et al.* 1989, Prathap 1993). In the framework of general FEM, it has long been known that the uniformly/selectively reduced integration approaches (see e.g., Hughes 1977) and assumed strain methods (see e.g., Ma and Kanok-Nukulchai 1989) can

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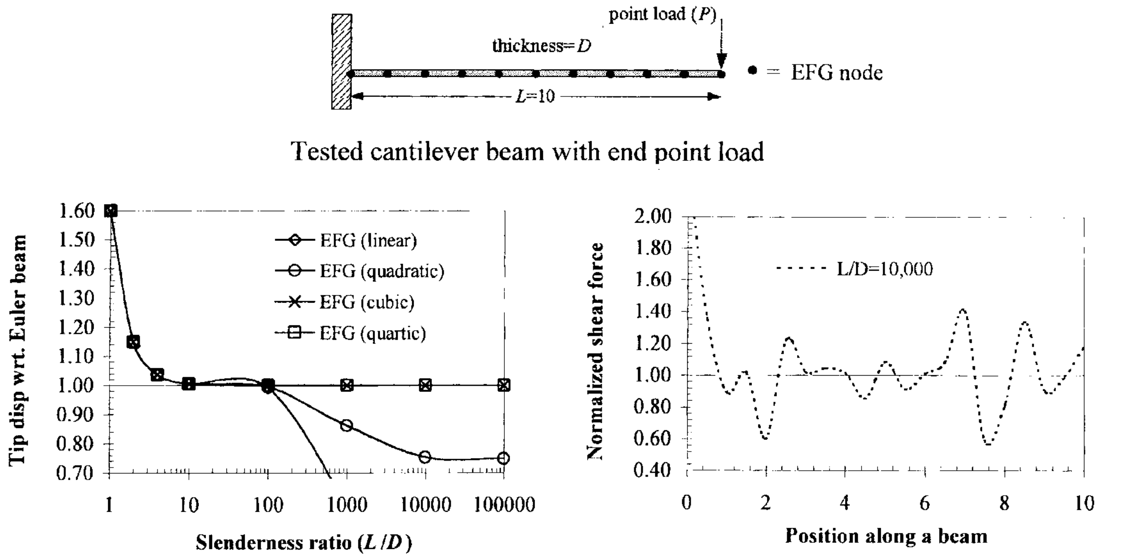


a) Tip displacement vs. slenderness ratio for various orders of basis functions

b) Oscillation in shear force along beam axis even quartic basis function is adopted

Fig. 1 EFGM solution for CN-T beam: shear locking tests for a cantilever beam problem

be employed effectively to overcome shear-locking phenomenon. Nevertheless these techniques cannot be applied practically to the conventional meshless methods because of their nature of complex and non-rational shape functions. Due to this hindrance, Donning and Liu (1998) have recently introduced the modified spline shape functions to alleviate shear locking in both Mindlin beams and plates. However their method is still limited only to a uniform nodal discretization of the problem domain. When looking back to the standard EFGM, shear locking seems to be remedied by an increase in the order of the basis functions (Krysl and Belytschko 1995, 1996, Noguchi 1997, Noguchi *et al.* 2000). As shown in Fig. 1a, cubic and quartic basis functions used in the conventional EFGM are apparent to avoid shear locking in a cantilever beam under tip load (CN-T), as experienced by lower-order basis functions. Yet an oscillation is still evident in the result of the shear stress (Fig. 1b), while the rate of convergence with respect to the refinement of discretization would be rather poor because the zero shear constraint in thin beam limit is still not exactly satisfied.

To alleviate these shortcomings, a matching interpolation field derived from the derivative of the displacement shape functions is proposed for rotations. To illustrate the effectiveness of this enhanced version of the EFGM for bending elements, several thick beam examples are analyzed. The oscillations in stress are also investigated. Shear locking in thick beams is reviewed shortly in the next section. A detailed review of the EFGM and other meshless methods can be found in the earlier papers (Belytschko *et al.* 1994 and 1996).

2. Shear-locking phenomenon in Timoshenko beams

After loaded, a plane section of a Timoshenko beam remains plane but not necessarily normal to the neutral axis. The two independent variables are displacement, u, and rotation, θ, at the beam centerline. Transverse shear deformation, γ, can be calculated from the following relationship;

$$\gamma = \frac{du}{dx} - \theta \tag{1}$$

From Eq. (1), the total strain energy, U, comprised of contribution from bending strain energy, U_b, and shear strain energy, U_s, can be obtained as

$$U = U_b + U_s = \frac{1}{2}\int EI\left(\frac{d\theta}{dx}\right)^2 dx + \frac{1}{2}\int kGA\left(\frac{du}{dx} - \theta\right)^2 dx \tag{2}$$

or

$$U' = \frac{2U}{EI} = \int\left(\frac{d\theta}{dx}\right)^2 dx + \alpha\int\left(\frac{du}{dx} - \theta\right)^2 dx \tag{3}$$

in which $\alpha = (kG/E)\times(12/D^2)$ for a rectangular beam section, where D=beam thickness, E=Young's modulus, G=shear modulus, A=cross-sectional area, and k =shear correction factor. By minimizing the total potential energy, the stiffness matrix is obtained with bending and shear contributions, i.e., $[\boldsymbol{K}]=[\boldsymbol{K}_b]+\alpha[\boldsymbol{K}_s]$. The term α therefore acts as a penalizing parameter in the thin beam situation ($D \ll L$) essentially suppressing shear deformation. Thus, the strain energy is predominantly due to bending. Under this situation, shear locking occurs if the displacement derivative in the second term of Eq. (3) is represented by an interpolation function of lower order than that of the rotation field. This results in bending being locked up by the displacement derivative. A great variety of remedies for the problem of shear locking have been proposed. The matching of shape functions for the displacement derivative and the rotation is the most straightforward approach (Kanok-Nukulchai *et al.* 1981, Prathap 1993, Reddy *et al.* 1997). In the thin beam situation, the Kirchhoff constraint ($du/dx - \theta = 0$) implies that shape functions for rotation must be able to match those for the first derivative of the transverse displacement. In FEM, shape functions chosen for displacement must be of one order higher than those for rotations. In beam problems, this is easily accomplished. In plate bending and shell elements however, this requirement is not easily met due to the general requirements of the element geometry. However, EFGM is free from element structure, and therefore its Moving Least Square (MLS) shape functions (Lancaster and Salkauskas 1981) can be constructed to meet the previously described consistency requirement at the thin beam limit, i.e., $du/dx - \theta = 0$. In the proposed procedure, the shape functions for rotation are directly derived from the derivative of the shape functions used for the displacements. By this straightforward procedure, the slope and rotation fields are perfectly matched and shear locking would be completely eliminated.

3. Consistent EFG approximation

As stated previously, shear locking at the small thickness limit is associated with the inability of a bending element to satisfy the Kirchhoff hypothesis exactly. This numerical difficulty occurs when the field of bending rotation appears to be restrained by the first derivative of the displacement field, as they are represented by incompatible functions. In the present version of EFGM, as MLS shape functions are no longer tied to the element nodal structure, it is therefore possible to choose shape functions for rotation to match perfectly with that for the corresponding slope. By using this strategy, shape functions for rotation are adopted from the first derivative of the shape functions for

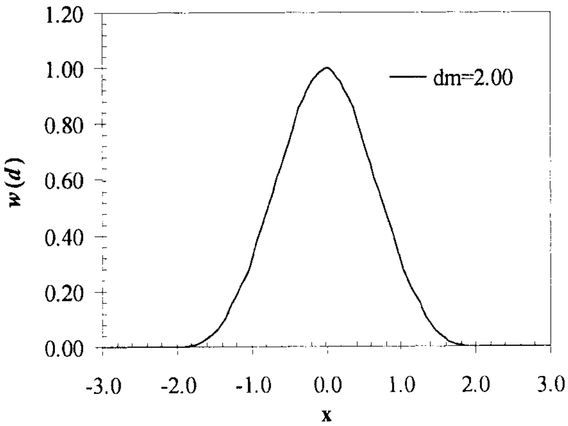


Fig. 2 Quartic spline weight function of node I (at x=0.0)

the transverse displacement. In Timoshenko beam problem, if

$$u^h(x) = \sum_{I=1}^{n} \phi_I(x) u_I \tag{4}$$

$$\theta^h(x) = \sum_{I=1}^{n} \eta_I(x) \theta_I \tag{5}$$

in which $u^h(x)$ is approximated displacement field; u_1 is nodal displacement parameters; $\theta^h(x)$ is approximated rotation filed; θ_I is nodal rotation parameters; $\phi_I(x)$ is the standard MLS shape function of node I for displacement field; and $\eta_I(x)$ is the modified shape function of node I for rotation field, it follows that

$$\eta_I(x) = \partial\phi_I(x)/\partial x \tag{6}$$

However, this criterion implies a C^1 continuity requirement for the displacement field to form the global stiffness matrix. In the procedure of MLS approximation, such a condition can be ensured if the weighting function possesses at least C^1 continuity. For this purpose, the one-dimensional quartic spline weight function, $w_I(d)$ in Eq. (7) with C^2 continuity over the entire problem domain is adopted, resulting in C^2 continuous displacement fields and C^1 continuous stress fields.

$$w_I(d) = \begin{cases} 1 - 6\left(\dfrac{d}{dm_I}\right)^2 + 8\left(\dfrac{d}{dm_I}\right)^3 - 3\left(\dfrac{d}{dm_I}\right)^4 & 0 \le d/dm_I \le 1 \\ 0 & d/dm_I > 1 \end{cases} \tag{7}$$

where $d = |x - x_I|$ and dm_I is the domain of influence for node I. Based upon quartic spline weight functions with domain of influence of 2.00 (Fig. 2), the current MLS shape functions using linear basis function for displacement and rotation fields are plotted in Fig. 3a and 3b respectively.

On the other hand, if the problem domain is discretized by eleven uniformly spaced nodes, quadratic basis functions are adopted, and the size of support radius is chosen to be 5.4. The set of current MLS shape functions for displacement and rotation fields are illustrated in Fig. 4.

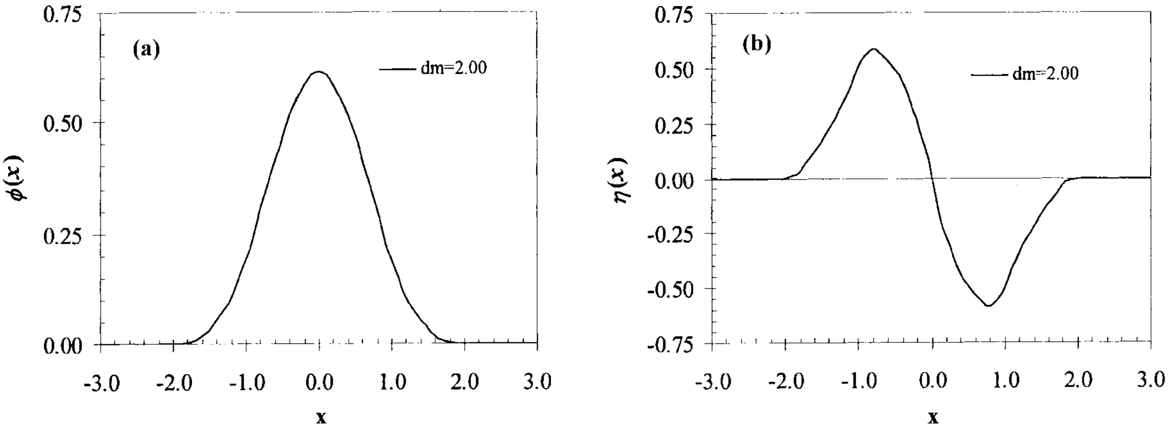


Fig. 3 Modified shape functions of node I (at x=0.0) for (a) displacement, $\phi_I(x)$, and (b) rotation, $\eta_I(x)$, fields in thick beam

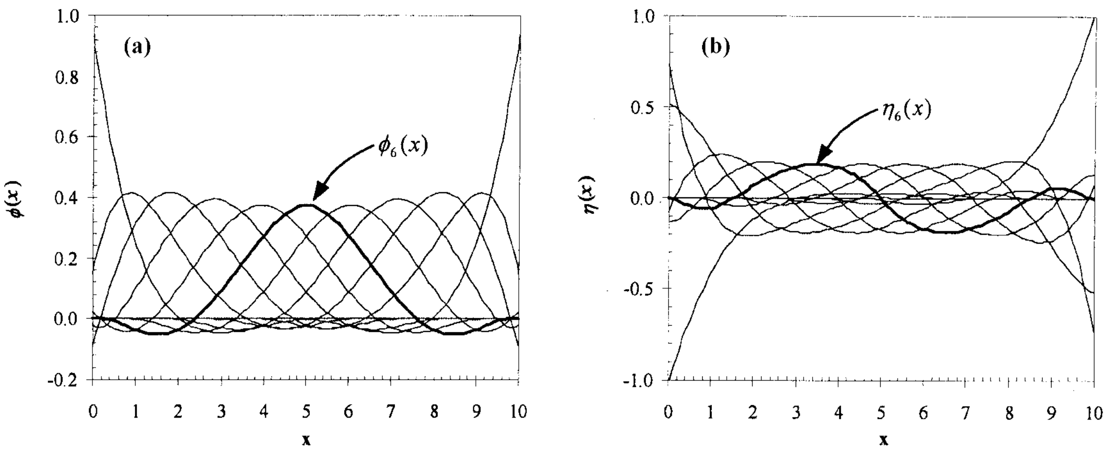


Fig. 4 Sets of modified shape functions for (a) displacement, $\phi(x)$, and (b) rotation, $\eta(x)$, fields (quadratic bases and domain of influence=5.4 are utilized)

4. Numerical tests

Several numerical examples were tested in this part to evaluate the effectiveness of the present EFGM application to shear-deformable beams. In our study, essential boundary conditions are imposed by the use of Lagrange multiplier technique even though this approach may increase the size of system matrix and computational-time cost. To minimize numerical integration errors, an integration rule of high order up to six is intentionally employed in each background cell for the calculation of the global stiffness matrix. In the standard EFGM with arbitrary nodal positioning, the procedure to evaluate the MLS shape functions requires an inversion of a moment matrix $\boldsymbol{A}(x)$ (see detail in Belytschko *et al.* 1994 and Lancaster and Salkauskas 1981), which can be quite ill-conditioned in particular cases. To overcome the near singularity value of the matrix, the conditioning of this matrix is enhanced by normalizing the nodal coordinates within each domain of influence into a local coordinate varying from –1 to +1, i.e.,

$$x' = 2\frac{(x - x_{\min})}{(x_{\max} - x_{\min})} - 1 \tag{8}$$

where x', x are positions of points expressed in the local and global coordinates; $x_{\max}$, $x_{\min}$ are global coordinates of the left and the right boundary nodes influenced by x_I respectively.

For all examples, the following beam properties are adopted: Young's modulus E=2×10^6, beam length L=10, and beam width b=1. If not stated otherwise, the domain of influence for all nodes is set equally at 5.4 times the nodal spacing.

4.1 Patch tests

The patch test is specifically aimed at determining the convergence behavior of non-conforming elements. Under particular loading patterns, patch test will be considered "pass" if the constant strain is numerically attained in machine precision. In Timoshenko beams, total strain energy comprises contributions from bending and shear terms (see Eq. 2). Both constant bending and shearing strains should then be achieved to ensure convergence performance in this newly developed version of EFG. For this purpose two loading cases, tip moment and point end load applied to cantilever beam, are considered. The beam is discretized by five non-uniformly spaced nodes. The size of support radius for all nodes is set at 10.00. EFG solutions from both loading patterns are demonstrated in Figs. 5 and 6 respectively. Fig. 5 shows linearly varied rotation field implying constant bending strain while constant shear force along the beam in Fig. 6 verifies constant shearing strain. These results confirm the satisfaction of patch test, and thus guarantee convergence characteristic of the proposed version of EFG.

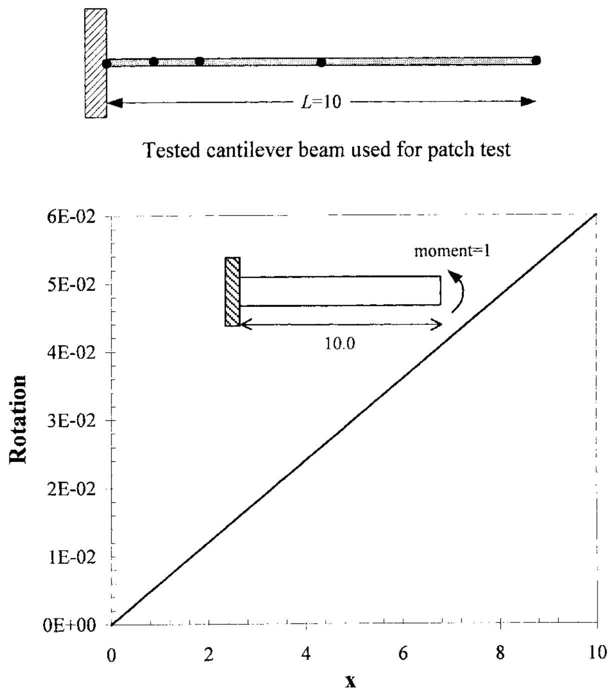


Fig. 5 Linearly varied rotation field verifying constant bending strain

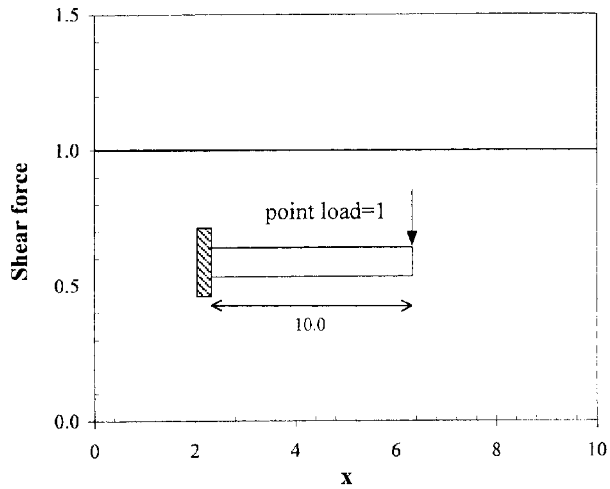


Fig. 6 Constant shear force verifying constant shear strain

4.2 Investigation of transverse shear locking

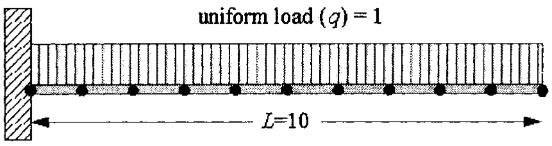
To ensure a complete absence of transverse shear locking at thin beam limit, cantilever beams under uniformly distributed load (CN-U) with varying length-to-thickness aspect ratios (*L/D*) are analyzed by both the standard version of EFGM (S-EFG) and the modified version proposed in this study (M-EFG). The beams are discretized by eleven nodes uniformly spaced. Normalized displacement at position x=4.00 from the fixed support is plotted versus the beam slenderness ratio as shown in Fig. 7a. In this figure, the results of S-EFG using quadratic basis functions show deterioration when the beam slenderness ratio is greater than 100. This is an evidence of transverse shear locking. Although shear-locking effects seem to decrease with the increase of the order of the basis function, they cannot be completely suppressed. On the contrary, using perfectly matched shape functions with built-in capacity to satisfy Kirchhoff's hypothesis, M-EFG consistently produces highly accurate results over the full range of beam slenderness ratios, even with quadratic basis functions. To further demonstrate the accuracy of the M-EFG model, the correct presentation of overall bending strain energy is assessed. The normalized bending energy, e, of this beam model is plotted against its slenderness ratio in Fig. 7b, where

$$e = \frac{U_b^{\mathrm{comp}}}{U_b^{\mathrm{exact}}} \tag{9}$$

in which U_b^{comp} denotes the bending strain energy of the discrete model, while the exact bending strain energy, U_b^{exact} can be evaluated from

$$U_b^{\mathrm{exact}} = \frac{1}{2}\int EI\left(\frac{d\theta}{dx}\right)^2 dx = \frac{q^2L^5}{40EI} \tag{10}$$

Results for both S-EFG in Fig. 7b confirm the deterioration in bending strain energy due to shear locking, where as the result for M-EFG does not exhibit any deterioration.



Tested cantilever beam with
uniformly distributed load (CN-U)

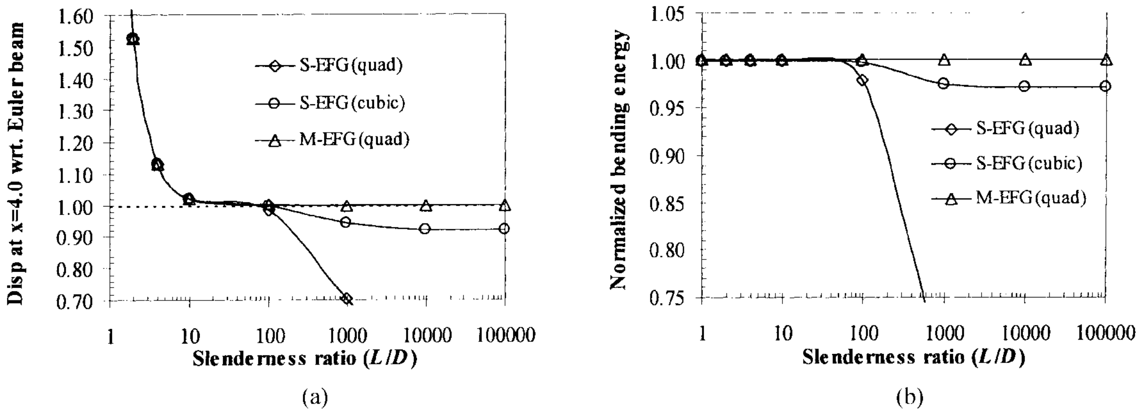


Fig. 7 Observation of shear locking in CN-U: (a) normalized displacement with respect to Euler beam solution; and (b) normalized bending energy (e) vs. slenderness ratio

4.3 Accuracy of results

To illustrate the accuracy computed by the proposed method, a very thin clamped beam (L/D= 10,000) loaded by a unit center point load (CL-C) is tested. For this case the beam is discretized evenly by thirty-one nodes. Solutions by standard version of EFGM (S-EFG) with high order of basis function up to quartic order is compared with those by modified EFGM (M-EFG) with only quadratic polynomial base. Profiles of displacement, rotation, bending moment, and shear force are demonstrated in Fig. 8 and show that inferior accuracy is observed for the case of S-EFG. An oscillation of shear stress is inherent due to the failure to satisfy Kirchhoff's constraint completely even though high order of basis function up to quartic is adopted. These shortcomings are not experienced at all for the case of M-EFG because of the perfect consistency between the slope and the rotation.

5. Conclusions

An enhanced version of EFGM is proposed to address the problem of transverse shear locking in bending elements. Based upon this pilot study of Timoshenko beams, it is shown that to avoid shear locking, modified shape function for the rotation field may be derived consistently from the first derivative of the standard shape function for the displacement field. This direct application provides a perfect fit between the slope and rotation fields, thus automatically and completely satisfying the Kirchhoff's hypothesis at the thin beam limit. The uniformly/selectively reduced integration scheme, allowing the constraint to be *satisfied discretely* at only some points as in the standard FEM, is

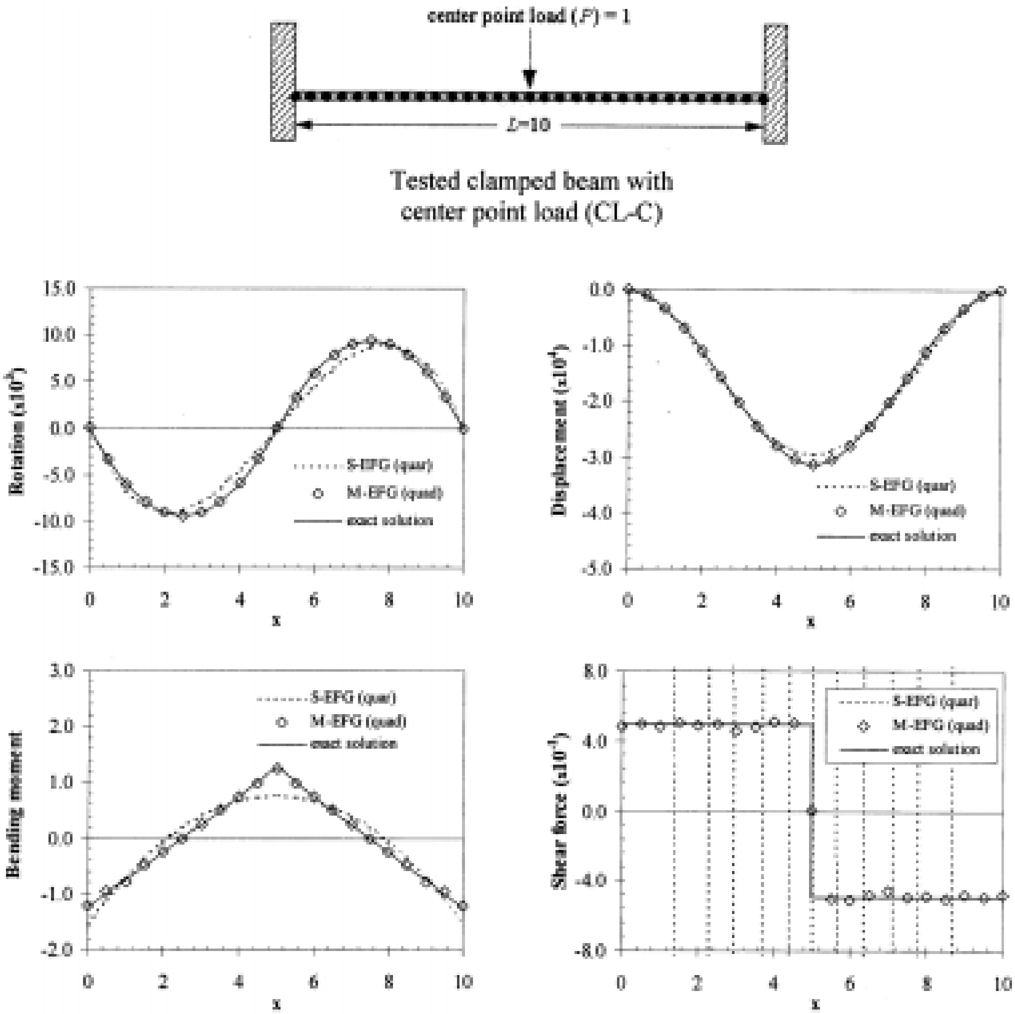


Fig. 8 Profile plots of displacement, rotation, moment, and shear force in a thin beam (L/D=10,000) using quadratic and quartic bases for S-EFG and M-EFG respectively

therefore not necessary any longer.

By the use of consistent MLS shape functions, numerical tests on various beam problems all reveal the absence of shear-locking phenomenon. The proposed model gives exceptionally accurate results for both the displacement and stress resultants over a full range of aspect ratios. Convergence can also be achieved earlier with coarse domain discretization. No oscillation in stress fields is evident at all. With little effort, the same approach should be extensible to overcome shear-locking pitfall in Mindlin plate and shell in a straightforward manner. In our opinion, the only disadvantage of the proposed EFGM is the increase in computational time needed to evaluate the second derivative of shape functions, particularly in two or three-dimensional problems.

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