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Effects of shear deformation on the effective length of tapered columns with I-section for steel portal frames

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Abstract. Based on the stiffness equation of the tapered beam element involving the effects of axial force and shear deformation, numerical investigations are carried out on elastic instability for web-linearly tapered columns with I-section of steel portal frames. Effects of shear deformation on the effective length of the tapered columns with I-section are studied. An efficient approach for determining the effective length of the tapered portal frame columns considering effects of shear deformation is proposed.

Key words: tapered column with I-section; effective length; finite element; shear deformation; steel portal frame.

1. Introduction

Tapered members are widely used in steel portal frames to make the stress more evenly distributed, so as that the consumption of steel can be reduced. For safety, stability check is required for the columns of steel portal frames. The requirements on tapered columns are specified in the codes of some countries for the design of steel structures. For example, an effective length factor K_γ is introduced for compressive strength check of web-tapered columns in the America code (AISC, LRFD 1994), although the calculation of K_γ is not specified. In the code of China (CECS 1998), the effective length factors of tapered columns for steel portal frames are presented, but effects of shear deformation are not taken into account.

It is found that shear deformation has significant influence on stability of prismatic steel columns with I-section under certain conditions (Li and Shen 1998). It should also be true for tapered members. For analysis of tapered members, a number of approaches have been proposed, which include normal beam elemental approach (Lindberg 1963 and Gallagher 1970), direct integral approach (Just 1977 and Karabalis 1983), Bessel function approach (Banerjee 1985 and 1986), principle of virtual forces (Kim, Lee and Chang 1995), and boundary integral approach (Al-Gahtani 1996), etc. The stability of tapered members was studied by Ermoupoulos and Kounadis (1985), Kim, Lee and Chang (1995) and Wang (1998) recently. On the other hand, Timoshenko beams including effects of shear deformation were widely employed in the study of natural frequencies for isotropic (Gupta and Rao 1978, To 1981 and Khulief and Bazoune 1992) and composite (Ramalinggeswara and Ganesan 1995 and Oral 1995) tapered beams. Vu-Quoc and Leger (1992) evaluated numerically the flexibility matrix of tapered I-beams accounting for shear deformation by

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principle of complementary virtual work, but not comprising the effect of axial forces. To the authors' knowledge, the investigation of Timoshenko-Euler tapered beams considering effects of shear deformation and axial force simultaneously or including the effect of shear deformation in the stability analysis of tapered members has not appeared in the reported literatures.

In this paper, a Timoshenko-Euler tapered beam element model is presented for analysis of structures consisting of tapered members by FEM involving effects of shear deformation. The elastic buckling of steel portal frames employing tapered columns with I-section is analyzed by using the proposed element.

2. Timoshenko-Euler tapered beam element

The cross-section of tapered members for steel portal frames is usually in symmetrical I-shape, welded by three plates. The height of the web is normally linearly varied, which the flanges keep uniform in width along the length of a member, as shown in Fig. 1. For the tapered member described above, the axis of the member remains straight and the applied forces as well as the corresponding deformations of the element representing this kind of member can thus be modelded as shown in Fig. 2. Following the same procedure given by the first author of this paper (Li and Shen 1998) for dealing with uniform Timoshenko-Euler beams, the equilibrium differential equation of the tapered Timoshenko-Euler beam element can be established as follows.

Under the simultaneous action of moment, shear force and axial force (positive for tension and negative for compression), the deflection of the element consists of two portions. One portion is induced by bending deformation and the other by shear deformation, that is

$$y=y_M+y_Q \tag{1}$$

The curvature of the element caused by bending is

$$y_M''=-\frac{M}{E\cdot I(z)} \tag{2}$$

where $I(z)$ is the inertia of second-moment of the cross-section at the location of distance z from left end of the element, E is the elastic modulus and M is the cross-sectional moment which can be expressed by

$$M=M_1-Q_1\cdot z-N\cdot y \tag{3}$$

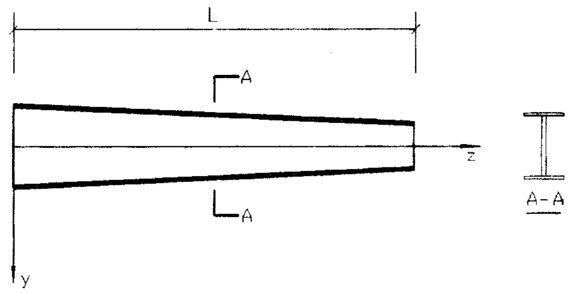

Fig. 1 A steel tapered member

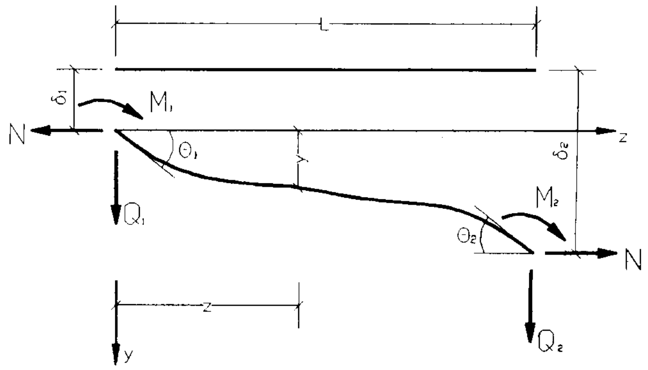

Fig. 2 Applied forces and deformations of an element

The slope of the element caused by shearing is

$$y_Q'=\frac{\mu\cdot Q}{G\cdot A(z)}=\frac{\mu}{G\cdot A(z)}\cdot\frac{dM}{dz} \tag{4}$$

where $A(z)$ is the area of the cross-section at location z, Q is the cross-sectional shear force, G is the shear modulus and μ is the factor varying with the shape of the cross-section. For I-shaped section, μ can be approximately calculated by

$$\mu=A(z)/A_w(z) \tag{5}$$

where $A_w(z)$ is the area of the web at the same cross-section for $A(z)$.

Substituting (5) into (4) gives

$$y_Q'=\frac{1}{G\cdot A_w(z)}\cdot\frac{dM}{dz} \tag{6}$$

By substituting (3) into (6), we have

$$y_Q'=\frac{1}{G\cdot A_w(z)}\cdot(-Q_1-N\cdot y') \tag{7}$$

Differentiating (7) gives

$$y_Q''=\frac{1}{G\cdot A_w(z)}\cdot\left[\frac{A_w'(z)}{A_w(z)}\cdot(Q_1+N\cdot y')-N\cdot y''\right] \tag{8}$$

Differentiating (1) twice and associating (2) and (8), we obtain

$$y''=y_M''+y_Q''=-\frac{M_1-Q_1\cdot z-N\cdot y}{E\cdot I(z)}+\frac{1}{G\cdot A_w(z)}\cdot\left[\frac{A_w'(z)}{A_w(z)}\cdot(Q_1+N\cdot y')-N\cdot y''\right] \tag{9}$$

Eq. (9) can be simplified as

$$\alpha(z)\cdot y''-\beta(z)\cdot N\cdot y'-N\cdot y=\beta(z)\cdot Q_1-(M_1-Q_1\cdot z) \tag{10}$$

in which

$$\alpha(z)=E\cdot I(z)\cdot\gamma(z)$$

$$\beta(z)=E\cdot I(z)\cdot\frac{A_w'(z)}{G\cdot A_w^2(z)}$$

$$\gamma(z)=1+\frac{N}{G\cdot A_w(z)}$$

Eq. (10) is the governing equation for the equilibrium of tapered Timoshenko-Euler beams. Let $\xi=\frac{z}{L}$, (10) is converted to non-dimensional form by

$$\alpha(\xi)\cdot y''-\beta(\xi)\cdot L\cdot N\cdot y'-L^2\cdot N\cdot y=\beta(\xi)\cdot L^2\cdot Q_1-L^2\cdot(M_1-Q_1\cdot L\cdot\xi) \tag{11}$$

By using Chebyshev Polynomial, the function $y(\xi)$, $\alpha(\xi)$, $\beta(\xi)$ can be approached by

$$y(\xi)=\sum_{n=0}^{M}y_n\cdot\xi^n \tag{12a}$$

$$\alpha(\xi)=\sum_{n=0}^{M}\alpha_n\cdot\xi^n \tag{12b}$$

$$\beta(\xi)=\sum_{n=0}^{M}\beta_n\cdot\xi^n \tag{12c}$$

Substituting (12) into (11) leads

$$\sum_{n=0}^{M}\left[\sum_{i=0}^{n}\alpha_i(n+2-i)(n+1-i)y_{n+2-i}\right]\cdot\xi^n-$$

$$L\cdot N\cdot\sum_{n=0}^{M}\left[\sum_{i=0}^{n}\beta_i(n+1-i)y_{n+1-i}\right]\cdot\xi^n-L^2\cdot N\cdot\sum_{n=0}^{M}y_n\cdot\xi^n=L^2\cdot Q_1\cdot\sum_{n=0}^{M}\beta_n\cdot\xi^n-L^2\cdot M_1+L^3\cdot Q_1\cdot\xi \tag{13}$$

According to the principle that the factors at two sides of (13) for the same exponent of ξ should be equal, then
for $n=0$:

$$2\alpha_0\cdot y_2-L\cdot N\cdot\beta_0\cdot y_1-L^2\cdot N\cdot y_0=L^2\cdot Q_1\cdot\beta_0-L^2\cdot M_1 \tag{14}$$

for $n=1$:

$$6\alpha_0\cdot y_3+2\alpha_1\cdot y_2-L\cdot N(2\beta_0\cdot y_2+\beta_1\cdot y_1)-L^2\cdot N\cdot y_1=L^2\cdot Q_1\cdot\beta_1+L^3\cdot Q_1 \tag{15}$$

and for $n\geq 2$:

$$\sum_{i=0}^{n}\alpha_i(n+2-i)(n+1-i)y_{n+2-i}-L\cdot N\cdot\sum_{i=0}^{M}\beta_i(n+1-i)y_{n+1-i}-L^2\cdot N\cdot y_n=L^2\cdot Q_1\cdot\beta_n \tag{16a}$$

Rewrite (16a) as

$$y_{n+2}=\left[L\cdot N\cdot\sum_{i=0}^{n}\beta_i(n+1-i)y_{n+1-i}+L^2\cdot N\cdot y_n+L^2\cdot Q_1\cdot\beta_n-\sum_{i=0}^{n}\alpha_i(n+2-i)(n+1-i)y_{n+2-i}\right]$$
$$/\alpha_0(n+2)(n+1) \tag{16b}$$

Since $\alpha(z)$, $\beta(z)$ are functions known already, the series $\{\alpha_n\}$, $\{\beta_n\}$ are determinate. Hence, it can be conducted from (16b) that $\{y_n\}$ (for any $n\geq 4$) can be expressed as the linear combination of y_0, y_1, y_2, y_3 and Q_1 with known coefficients or the series $\{y_n\}$ may be determined when the values of y_0, y_1, y_2, y_3 and Q_1 are obtained.

Considering the following boundary condition
for $\xi=0$:

$$y(0)=y_0=0 \tag{17a}$$

$$y'(0)=y_1=\frac{L}{\gamma(0)}\left[\theta_1-\frac{Q_1}{G\cdot A_w(0)}\right] \tag{17b}$$

for $\xi=1$:

$$y(1)=\sum_{n=0}^{M}y_n=c_1y_1+c_2y_2+c_3y_3+c_4Q_1=\delta_2-\delta_1 \tag{18a}$$

$$y'(1)=\sum_{n=0}^{M}n\cdot y_n=c_5y_1+c_6y_2+c_7y_3+c_8Q_1=\frac{L}{\gamma(1)}\left[\theta_2-\frac{Q_1}{G\cdot A_w(1)}\right] \tag{18b}$$

The reason that $y(1)$ and $y'(1)$ in (18) are expressed as the linear combination of y_0, y_1, y_2, y_3 and Q_1 is the conclusion obtained from (16b) and $y_0=0$ from (17a).

Letting $y_1=1$, $y_2=y_3=Q_1=0$, c_1 and c_5 can be determined from (18) by

$$c_1=1+\sum_{i=4}^{M}y_n^{(1)} \tag{19a}$$

$$c_5=1+\sum_{i=4}^{M}n\cdot y_n^{(1)} \tag{19b}$$

in which $\{y_n^{(1)}\}$ is the series of $\{y_n\}$ determined under the condition $y_1=1$, $y_2=y_3=Q_1=0$.

If we let $y_2=1$, $y_1=y_3=Q_1=0$, c_2 and c_6 can be determined in the same way. So do c_3 and c_7, c_4 and c_8.

Up to here, y_1, y_2, y_3 and Q_1, M_1 are unknown variables. If the boundary deformations of the element δ_1, δ_2, θ_1, θ_2 are treated as known variables, the five equations numbered (14), (15), (17b), (18a) and (18b) can be combined for solving Q_1 and M_1. The applied forces at the other end of the element, Q_2 and M_2, may expressed as the function of Q_1 and M_1 by considering the following equilibrium conditions

$$Q_2+Q_1=0 \tag{20}$$

$$M_2+M_1-Q_1\cdot L-N\cdot(\delta_2-\delta_1)=0 \tag{21}$$

Then, the stiffness equation of the element is obtained as

$$[k]\cdot\{\delta\}=\{f\} \tag{22}$$

in which

$$\{\delta\}=[\delta_1,\ \theta_1,\ \delta_2,\ \theta_2]^T$$

$$\{f\}=[Q_1,\ M_1,\ Q_2,\ M_2]^T$$

$$[k]=\begin{bmatrix} -\phi_1 & \phi_2 & \phi_1 & \phi_3 \\ -\phi_4 & \phi_5 & \phi_4 & \phi_6 \\ \phi_1 & -\phi_2 & -\phi_1 & -\phi_3 \\ -\phi_7 & \phi_8 & \phi_7 & \phi_9 \end{bmatrix} \tag{23}$$

The expressions of ϕ_i (i=1, 2, ..., 9) are given in Appendix.

The stiffness matrix expressed in (23) seems unsymmetrical. But it could be verified by numerical studies that the values of elements in that matrix are in a symmetric pattern.

In theory, the approach described above is accurate. The only error comes from the representation of the real deflection y by Chebyshev Polynomial with definite terms. So long as the number of terms for y, i.e. M, is suitably chosen to make the coefficients c_1~c_8 accurate enough, the satisfactory accuracy of stiffness matrix of the element can be achieved. According to the actual numerical experience, it may be concluded that M around 13 can produce satisfactory results in stability analysis of structures.

3. Application

3.1. The effective length factors of tapered columns for steel portal frames excluding effects of shear deformation

The steel portal frame with tapered columns as shown in Fig. 3 is widely used for industry buildings. Effective length factors for the tapered columns are given in the China Specification (CECS 1998) for simplifying safety check on stability.

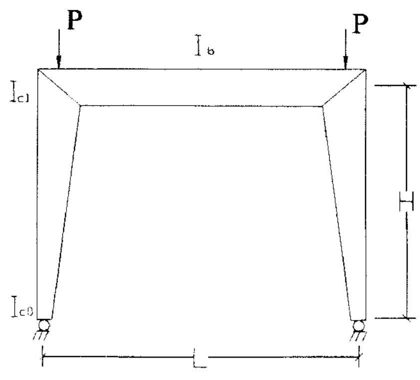

Fig. 3 Diagram of a steel portal frames

Define the effective length factor of a tapered column in the portal frame shown in Fig. 3 as

$$\mu=\sqrt{\frac{\pi^2 EI_{c0}}{P_{cr}H^2}} \tag{24}$$

where P_{cr} is the elastic critical load of the steel portal frame, I_{c0} is the inertia moment of the smaller end of the tapered column as illustrated in Fig. 3 and E is elastic modulus.

Two parameters are introduced for expressing μ. They are

$$n=I_{c1}/I_{c0} \tag{25}$$

$$K=\frac{I_b H}{I_{c1} L} \tag{26}$$

where I_{c1} is the inertia moment of the larger end of the tapered column, I_b is the inertia moment of the girder for the portal frame, H and L are heights and span of the portal frame respectively, as shown in Fig. 3.

By using finite element method, the elastic critical load of portal frames can be obtained. Then, the effective length factors of the tapered portal frame columns can be calculated by Eq. (24). Comparison is made on values of μ in Table 1 obtained respectively by CECS(1998) using Euler beam element and this paper using Timoshenko-Euler beam element and assuming $G=\infty$ for excluding effects of shear deformation.

3.2. *Effects of shear deformation on the effective length factors of tapered columns for steel portal frames*

In order to discuss effects of shear deformation on the effective length factor of tapered columns for steel portal frames, a slenderness parameter is defined as

$$\lambda_c=H/\sqrt{I_{c0}/A_{c0}} \tag{27}$$

Table 1 Values of μ excluding effects of shear deformation

n \ K		0.1	0.2	0.5	1.0	2.0	10.0
50	CECS	0.706	0.591	0.518	0.494	0.484	0.475
	This paper	0.704	0.588	0.516	0.493	0.481	0.473
20	CECS	1.095	0.889	0.758	0.713	0.693	0.682
	This paper	1.077	0.881	0.751	0.708	0.687	0.670
10	CECS	1.473	1.208	1.008	0.942	0.929	0.869
	This paper	1.495	1.201	1.002	0.933	0.918	0.871
5	CECS	2.053	1.641	1.341	1.229	1.176	1.140
	This paper	2.065	1.644	1.337	1.228	1.173	1.129
1	CECS	No value	3.420	2.630	2.330	2.170	2.000
	This paper	4.405	3.404	2.627	2.327	2.164	2.033

where A_{c0} is the area of the smaller end of the tapered columns.

It is found that the effective length factor of tapered columns, μ, varies only with n and K if effects of shear deformation are neglected. When effects of shear deformation considered, however, the slenderness parameter, λ_c, has significant influence on values of μ. In general, the larger the slenderness parameter, the smaller the effects of shear deformation on the effective length factor of tapered columns.

For convenience of practical application, a magnification factor is introduced to consider the effects of shear deformation on the effective length factor of tapered columns, i.e.

$$\beta=\mu_s/\mu_0 \tag{28}$$

where μ_s and μ_0 are effective length factors of tapered portal frame columns including and excluding effects of shear deformation respectively. The values of μ employed by CECS (1998), as given in Table 1, are just values of μ_0. The values of μ_s can be determined by implementing the elastic buckling analysis of the portal frame as shown in Fig.3 using the Timoshenko-Euler beam element presented in this paper. It is clear that the magnification factor, β, indicates the severity of shear deformation effects. If β is obtained, the effective length factor of tapered columns involving shear deformation effects may be easily determined, with previously known values of μ_0.

Table 2 gives the values of the magnification factor varying with n and K, when λ_c=23.67. It can be found from Table 2 that the magnification factor reduces with n when K keeps constant. However, when n keeps constant, the magnification factor varies slightly with K. For simplification, a constant β_E may be adopted to represent all the values of β for a constant n with K varying from 0.1 to 10.0. The even value of the maximum and the minimum value of β for each n can be used for β_E, as listed in Table 2.

By using β_E, the magnification factor becomes the function of λ_c and n. Fig. 4 gives a group of curves for β_E varying with the slenderness parameter, λ_c, under the condition of a number of constant values for n, which can be used to quickly estimate the magnification factor in practice.

Taking a careful observation of Fig. 4, we can find that when

$$\lambda_c=36\sqrt{0.02n}+26 \tag{29}$$

the magnification factor becomes less than 1.05. In this case, the effects of shear deformation on the effective length factors of tapered portal frame columns may be approximately neglected.

β_E in Fig. 4 could further be fitted by the second-order exponent-decay function as

Table 2 Values of the magnification factor when λ_c = 23.67

n \ K	0.1	0.2	0.5	1.0	2.0	10.0	β_E
50	1.278	1.330	1.365	1.372	1.360	1.358	1.325
20	1.177	1.215	1.227	1.226	1.208	1.182	1.202
10	1.161	1.159	1.168	1.158	1.119	1.142	1.144
5	1.124	1.126	1.128	1.125	1.115	1.091	1.110
1	1.064	1.064	1.071	1.067	1.059	1.066	1.065

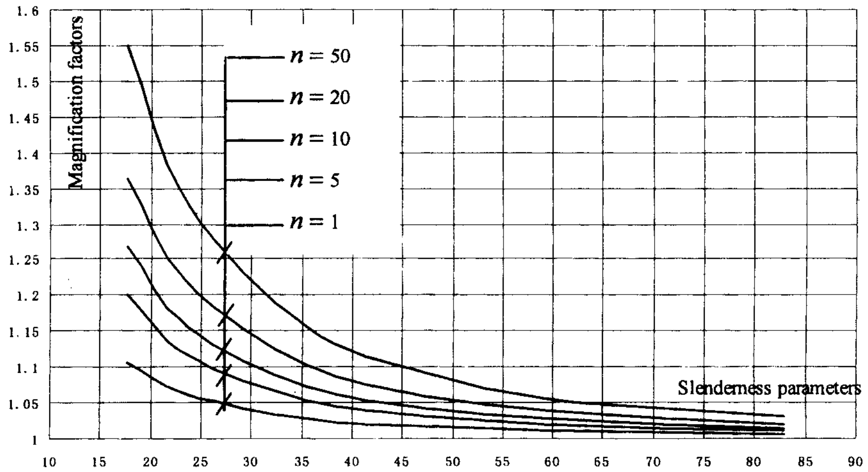

Fig. 4 Curves of magnification factor vs slenderness parameter

$$\beta_E=\beta_E(n,\lambda_c)=1.0+f_1(n)\cdot e^{-(\lambda_c-17.75)/4}+f_2(n)\cdot e^{-(\lambda_c-17.75)/22} \tag{30}$$

where

$$f_1(n)=0.02903+0.0099n-0.0003416n^2+0.000004155n^3 \tag{31a}$$

$$f_2(n)=0.06523+0.01120n-0.0001056n^2 \tag{31b}$$

The fitted values by Eq. (30) are compared with the original values of β_E in Table 3.

Table 3 Comparison between the calculated and fitted values of β_E

n / λ_c		50	20	10	5	1
17.750	Calculated	1.550	1.364	1.267	1.200	1.105
	Fitted	1.550	1.371	1.265	1.189	1.115
23.670	Calculated	1.328	1.218	1.157	1.117	1.061
	Fitted	1.319	1.217	1.150	1.107	1.067
35.505	Calculated	1.157	1.102	1.072	1.054	1.028
	Fitted	1.163	1.112	1.076	1.054	1.035
47.340	Calculated	1.092	1.059	1.042	1.031	1.016
	Fitted	1.094	1.064	1.044	1.031	1.020
59.175	Calculated	1.056	1.038	1.027	1.020	1.010
	Fitted	1.055	1.038	1.025	1.018	1.012
71.010	Calculated	1.041	1.027	1.019	1.014	1.007
	Fitted	1.032	1.022	1.015	1.011	1.007
82.845	Calculated	1.031	1.020	1.015	1.010	1.005
	Fitted	1.019	1.013	1.009	1.006	1.004

3.3. Example

To explain the modification of effective length factors for tapered columns due to effects of shear deformation, take an example from the case shown in Table 2 where λ_c=23.67. When n=50 and K=0.1, we know from Table 2 that β=1.278 and β_E=1.325, and from Table 1 that μ_0=0.706. Hence, the accurate value of the effective length factor considering effects of shear deformation is $\mu_s=\beta\mu_0$=1.278×0.706=0.902, while the corresponding approximate value of the effective length factor is $\mu_s'=\beta_E\mu_0$=1.325×0.706=0.935. The relative error between μ_s' and μ_s is 3.7%. This demonstrates that the substitution of β with β_E can obtain satisfying estimation for μ_s. Fig. 4 or Eq. (30) can be employed by practicing engineers to determine β_E.

4. Conclusions

The following concluding remarks can be summarized through studies of this paper.

(1) The equilibrium differential equation for the Timoshenko-Euler tapered element is established in this paper, and it is successfully solved by utilizing Chebyshev Polynomial approach technique.

(2) Effects of shear deformation on effective length factors of tapered columns with I-section for steel portal frames are discussed on the basis of the element developed in this paper. It is found that under certain circumstances shear deformation has significant influence on the effective length of tapered portal frame columns. The condition that effects of shear deformation need not considered is investigated.

(3) A magnification factor is defined for efficiently determining the effective length factor of tapered columns with I-section considering effects of shear deformation based on the values of the corresponding effective length factor excluding shear effects. Curves and fitted equation for determining the magnification factor are obtained and proposed for practical use.

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Appendix

Expression of elements in Eq. (23)

$$\phi_1=\frac{\psi_{15}}{\psi_{11}\psi_{15}-\psi_{14}\psi_{12}},\quad \phi_2=\frac{\psi_{13}\psi_{15}-\psi_{16}\psi_{12}}{\psi_{11}\psi_{15}-\psi_{14}\psi_{12}},\quad \phi_3=\frac{\psi_{12}\psi_{14}}{\psi_{11}\psi_{15}-\psi_{14}\psi_{12}},\quad \phi_4=\frac{1-\psi_{11}\phi_1}{\psi_{12}},$$

$$\phi_5=\frac{\psi_{13}-\psi_{11}\phi_2}{\psi_{12}},\quad \phi_6=\frac{\psi_{11}\phi_3}{\psi_{12}},\quad \phi_7=\phi_1 L+N-\phi_4,\quad \phi_8=\phi_2 L-\phi_5,\quad \phi_9=\phi_3 L-\phi_6,$$

$$\psi_1=-\frac{L}{G\cdot A_w(0)\cdot\gamma(0)},\quad \psi_2=\frac{L}{\gamma(0)},\quad \psi_3=-\frac{L}{G\cdot A_w(1)\cdot\gamma(1)},\quad \psi_4=\frac{L}{\gamma(1)},$$

$$\psi_5=\frac{L\cdot\beta_0(N\cdot\psi_1+L)}{2\alpha_0},\quad \psi_6=-\frac{L^2}{2\alpha_0},\quad \psi_7=\frac{L\cdot N\cdot\beta_0\cdot\psi_2}{2\alpha_0},$$

$$\psi_8=\frac{2(L\cdot N\cdot\beta_0-\alpha_1)\psi_5+L\cdot(\beta_1+L)(N\cdot\psi_1+L)}{6\alpha_0},\quad \psi_9=\frac{2(L\cdot N\cdot\beta_0-\alpha_1)\psi_6}{6\alpha_0},$$

$$\psi_{10}=\frac{2(L\cdot N\cdot\beta_0-\alpha_1)\psi_7+L\cdot N(\beta_1+L)\psi_2}{6\alpha_0},\quad \psi_{11}=c_1\psi_1+c_2\psi_5+c_3\psi_8+c_4,\quad \psi_{12}=c_2\psi_6+c_3\psi_9,$$

$$\psi_{13}=-(c_1\psi_2+c_2\psi_7+c_3\psi_{10}),\quad \psi_{14}=c_5\psi_1+c_6\psi_5+c_7\psi_8+c_8-\psi_3,\quad \psi_{15}=c_6\psi_6+c_7\psi_9,$$

$$\psi_{16}=-(c_5\psi_2+c_6\psi_7+c_7\psi_{10})$$