Structural Engineering and Mechanics, Vol. 10, No. 2 (2000) 181-195

DOI: http://dx.doi.org/10.12989/sem.2000.10.2.181

Collapse behaviour of three-dimensional brick-block systems using non-linear programming

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Abstract. A two-step procedure for the application of non linear constrained programming to the limit analysis of rigid brick-block systems with no-tension and frictional interface is implemented and applied to various masonry structures. In the first step, a linear problem of programming, obtained by applying the upper bound theorem of limit analysis to systems of blocks interacting through no-tension and dilatant interfaces, is solved. The solution of this linear program is then employed as initial guess for a non linear and non convex problem of programming, obtained applying both the 'mechanism' and the 'equilibrium' approaches to the same block system with no-tension and frictional interfaces. The optimiser used is based on the sequential quadratic programming. The gradients of the constraints required are provided directly in symbolic form. In this way the program easily converges to the optimal solution even for systems with many degrees of freedom. Various numerical analyses showed that the procedure allows a reliable investigation of the ultimate behaviour of jointed structures, such as stone masonry structures, under statical load conditions.

Key words: limit analysis; mathematical programming; masonry.

1. Introduction

Although there have been considerable developments on modelling of masonry structures, the problem is still far from being solved in general. On one hand, most of the constitutive models formulated during the last decades, based on strong idealisation of masonry as a no-tension material (Heyman 1982, Giaquinta and Giusti 1985, Del Piero, 1989), led to the solution in closed form of a few class of problems having essentially a theoretical interest (Villaggio 1981, Bennati and Padovani 1997), on the other hand, many technical solutions proposed have been applied to specific problems only (Blasi 1994, Braga *et al.* 1998).

Some appreciable efforts in deriving continuous models for masonry materials properly taking into account not only the mechanical properties of the constituents but also the geometry of the bricks and their texture, have been made in the framework of homogeneization theories (Pande

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1989, Pietruszczak and Niu 1992, Lourenco *et al.* 1998). However, being classical continuum models, they do not account for the size of the units and consequently they cannot consider any scale effect. Moreover, it is known that the lack of a parameter of internal length scale can be related to ill-positioning in the differential equations of motion and consequently to mesh-sensitivity in the finite element formulations (Bertram and Sidoroff 1998). To take into account size effects and to avoid ill-conditionings, it is preferable to choose models of continua with microstructure and in particular the micropolar ones (Masiani *et al.* 1996, Iordache and Willam 1998, Trovalusci and Masiani 1999). Nevertheless, in many cases it could be useful to adopt a more refined modelling, resorting to numerical simulations based on the distinct description of bricks and mortar, to deal with some specific problems of masonry mechanics.

This happens for example in ancient masonry. Due to the extremely variable nature of the bricks, the mortar and the structural typologies, a unified description in terms of continuum can hardly be found. The strength of this kind of masonry, made of stones dry-assembled or of joints filled by scattered and poor mortar, strongly depends on parameters not always easily determinable: the mechanical properties of the constituents; the size of the blocks; the interlocking among the blocks; the lack of coherence along the joints and their frictional properties. In earlier works the influence of the brick texture on the whole mechanical behaviour of masonry has been investigated by experimental and numerical analyses performed for walls made of bricks of various sizes and arrangements (Baggio and Trovalusci 1993, 1995, 1998). The physical tests concerned walls made of bricks in dry-frictional contact acted upon by the self weight and by an increasing horizontal body force simulating statically the seismic action. The experimental results showed the predominance, at the collapse, of the rigid displacements of the blocks with respect to their deformability. This pointed out the necessity to take into account the geometry of the units and their disposition in the modelling as well as the inability to carry tension and the friction along the joints. The corresponding computations were performed by using different discrete approaches (Baggio and Trovalusci 1995). By accounting directly for the geometry of the stones and the interlocking among them, numerical results could be obtained consistent with the physical ones. For example, a surprising physical result was simulated: squat walls, made of bricks sufficiently small, collapse by detachment and rotation of a part of the wall while more slender walls, made of larger bricks, collapse through sliding mechanisms (Baggio and Trovalusci 1993, Figs. 1-6).

In this work the attention is still focused on the study of the collapse behaviour of brick-block masonry. The model selected to describe brick or stone masonry is a system of blocks supposed to have infinite strength, and considered rigid, interacting through no-tensional and frictional interfaces. The assumption that bricks cannot break is not so restrictive if masonry with bricks essentially dry assembled, like ancient masonries, is considered and if the influence of the geometry of the assembly on the ultimate behaviour of the masonry is investigated.

As widely acknowledged, the problem of the evaluation of the ultimate load for an assembly of rigid bodies with no-tension and frictional constraints subjected to proportional load can be studied as a problem of limit analysis of finite-dimensional rigid-plastic systems with non-associative flow rules. Though appealing from a theoretical point of view, limit analysis of masonry structures gives rise to a nearly prohibitive numerical task when friction has to be taken into account: this is why currently, it is not often employed. Moreover, the many degrees of freedom of the actual structures increase the computational complexity of the problem and, except for a few works (Livesley 1992, Melbourne and Gilbert 1995), plane structures without practical interest have generally been analysed.

In the above mentioned paper (Baggio and Trovalusci 1998) a computer procedure which provides the ultimate load and the collapse mechanism for two and three-dimensional masonries made of blocks with frictional interfaces has been implemented. From a mathematical point of view, this procedure is based on the solution of a problem of constrained programming which, due to presence of friction, is non-linear and non-convex. It is not guaranteed that the solution of this difficult problem corresponds to a feasible global minimum. In any case, its complex determination strongly depends on the choice of the initial estimate for the unknowns. In this work one of the strategies, outlined in (Baggio and Trovalusci 1995) to solve such limit analysis problems is adopted and improved. The computational aspects of the procedure selected are also specified and discussed. In particular, the solution of the problem of non linear programming is approached using as initial estimates the results of a linearised programming problem obtained by replacing friction with dilatancy. The advantages of using two steps of mathematical programming, the first one linear and the second one non-linear, are also pointed out. The effectiveness of the procedure is investigated by analysing non-trivial problems concerning the safety of two and three-dimensional structures of practical interest made of many blocks assembled together in various dispositions. The results of the analysis show the strong influence of the geometry and the texture of the bricks on the ultimate strength of the masonry for all the structures analysed.

2. Collapse load for rigid block systems with no-tension and frictional interfaces

The procedure we adopt in this work to investigate the ultimate behaviour of systems of rigid blocks interacting through non-linear and non-elastic actions is the one described in (Baggio and Trovalusci 1998) and is briefly summarised here below. This procedure corresponds to the combined ‘equilibrium’ and ‘mechanism’ approach for determining the minimum collapse load when non-associative flow rules are present. The collapse mechanisms considered involve the rotations of the blocks around the edges of the contact surfaces (hinging), the sliding along the joints and the relative rotations of the blocks around the axis normal to the plane of their contact surfaces (twisting). This problem is related to a non-linear and non-convex optimisation program with the load factor as objective function, with linear inequality constraints and, due to the presence of friction, with a non linear equality constraint.

It is known that the governing relations of a system of rigid bodies subjected to proportional loads and interacting through contact surfaces unable to carry tension and resistant to sliding by friction, formally correspond to those of a non standard rigid perfectly-plastic discretized body, where the nodes are replaced by the blocks and the elements by the joints. The blocks of parallelepiped form are subjected to the action of external forces and couples and interact through plane contact surfaces by a force and a couple. They can translate and rotate and the strain measures of the assembly are defined as the relative infinitesimal displacement and the relative infinitesimal rotation between a pair of adjacent blocks. As the joints cannot carry tension, and the tangential forces, as well as the torsional moments, are limited by the frictional strength, bounds on the components of the blocks interactions are posed. These delimitations define a piece-wise linear yield domain in the superposed space of stresses and strains. The governing relations of this problem are summarised as follows:

kinematic compatibility equations

$$\boldsymbol{u} = \boldsymbol{A}_0\boldsymbol{q}_1$$
$$\boldsymbol{q}_2 = \boldsymbol{A}\boldsymbol{q}_1, \qquad \text{with} \qquad \boldsymbol{A}_0 = \boldsymbol{B}_1^{-1}, \boldsymbol{A} = \boldsymbol{B}_2\boldsymbol{B}_1^{-1}; \tag{1}$$

balance equations

$$\boldsymbol{A}_0^T(\boldsymbol{f}_0 + \alpha \boldsymbol{f}_L) - \boldsymbol{A}^T\boldsymbol{r}_2 - \boldsymbol{r}_1 = 0 \tag{2}$$

yield domain

$$\boldsymbol{y} = \boldsymbol{N}^T\boldsymbol{r} = \boldsymbol{N}_1^T\boldsymbol{r}_1 + \boldsymbol{N}_2^T\boldsymbol{r}_2 \le 0 \tag{3}$$

flow rule

$$\boldsymbol{q} = \boldsymbol{M}\boldsymbol{\lambda} \quad \text{or} \quad \boldsymbol{q}_1 = \boldsymbol{M}_1\boldsymbol{\lambda}, \ \ \boldsymbol{q}_2 = \boldsymbol{M}_2\boldsymbol{\lambda} \tag{4}$$

positive work of the live loads

$$\boldsymbol{u}^T\boldsymbol{f}_L > 0 \tag{5}$$

complementarity condition

$$\boldsymbol{y}^T\boldsymbol{\lambda} = 0 \tag{6}$$

In the above relations: $\boldsymbol{u}$ is the vector of the generalised displacements; $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are respectively the vectors of the free and of the linearly dependent generalised relative displacements (inelastic strains) ($\boldsymbol{q}=\{\boldsymbol{q}_1, \boldsymbol{q}_2\}^T$); $\boldsymbol{B}_1$ is the kinematical submatrix of maximum rank and $\boldsymbol{B}_2$ the rest of the kinematical matrix; $\boldsymbol{f}_0$ and $\boldsymbol{f}_L$ are the vectors, dead and live respectively, of the generalised actions on the centres of the blocks; α is the positive load multiplier; $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the vectors of the generalised contact actions (stresses) ($\boldsymbol{r} = \{\boldsymbol{r}_1, \boldsymbol{r}_2\}^T$), with $\boldsymbol{r}_2$ the statically undetermined term; $\boldsymbol{y}$ is the vector of the yield functions defining the piecewise linear yield domain in the stress space; $\boldsymbol{N}$ is the transpose of the block-diagonal gradient matrix ($\boldsymbol{N} = [\boldsymbol{N}_1, \boldsymbol{N}_2]$); $\boldsymbol{\lambda}$ is the vector of the inelastic multipliers and $\boldsymbol{M}$ is the block-diagonal matrix of the modes of failures ($\boldsymbol{M} = [\boldsymbol{M}_1, \boldsymbol{M}_2]^T$).

Let $\{\boldsymbol{e}_i^c\}$ be an ortho-normal basis in the three-dimensional space for each c-th contact surface between bricks, where i=1, 2, or i=4, 5, indicates respectively the two directions tangential to the contact surface and i=3, or i=6, the direction normal to this surface (Baggio and Trovalusci 1998, Fig. 1). The vector $\boldsymbol{r}$ is made of subvectors $\boldsymbol{r}^c$ of three components of force, r_1^c, r_2^c, r_3^c, and three components of couple r_4^c, r_5^c, r_6^c. The matrix $\boldsymbol{N}$ is made of diagonal blocks of submatrices

$$\boldsymbol{N}^c = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -l_2^c & -l_2^c & -l_1^c & -l_1^c & -tg\phi^c & -tg\phi^c & -tg\phi^c & -tg\phi^c & -tg\phi^c\beta l^c & -tg\phi^c\beta l^c \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

where ϕ^c is the angle of friction, l_1^c and l_2^c the half lengths of the edges of the c-th contact surface, l^c a characteristic length of the surface and β^c an undimensional factor depending on the shape of the surface. Supposing a uniform distribution of pressure on plane circular contact surfaces of radius $l^c = l_1^c + l_2^c$, $\beta^c = (2/3)$.

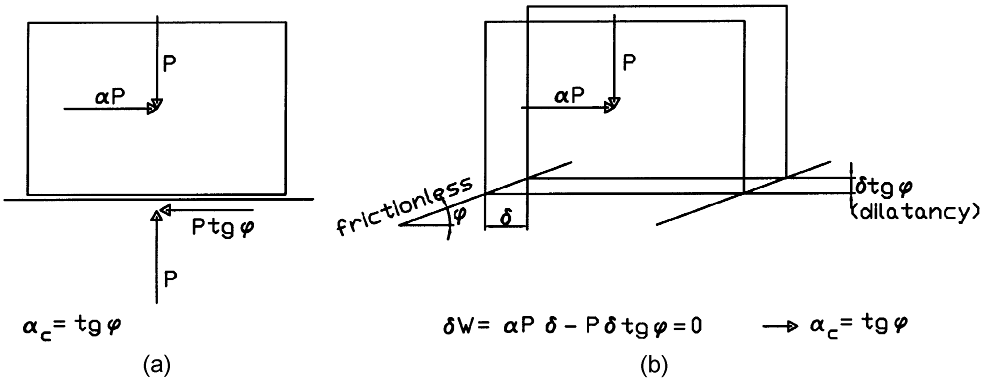


Fig. 1 Collapse mechanism and collapse multiplier of a block with frictional (a) or dilatant (b) interface

Therefore, the yield domain (3) is made of ten faces corresponding, first, to the activation of the rotational mechanisms about the edges of the parallelepiped blocks (four faces), then to the slipping along the directions of the two edges (four faces) and finally to the twisting (two faces). As first attempt, sliding was only allowed in two orthogonal directions, but there are no theoretical limits to consider more possible directions of sliding by increasing the number of the faces of the yield domain. Finally, the block-diagonal matrix $\boldsymbol{M}$ is made of submatrices

$$[\boldsymbol{M}^c]=\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -l_2^c & -l_2^c & -l_1^c & -l_1^c & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

where the j-th column (j=1, 10) amplified by the multiplier λ_j represents the contribution to the relative displacement of the interface due to the activation of the j-th face of the yield domain at a given interface c. It is easy to verify that the normality rule holds only in case of relative displacements normal to the plane of the joints. To determine the collapse load multiplier α_c for the described assemblage of blocks we can resort to the methods of the theory of optimisation under constraints. Mathematical programming has been considered for some time as an effective tool to deal with problems involving inequality boundary conditions on the contact interfaces between solid bodies, such as constraints against interpenetration of adjacent elements, no-tension constraints and friction conditions (Stavroulakis 1991, Simo and Laursen 1992). All of these problems, belonging to the class of the so-called free-boundary problems, can be formulated in variational inequality form and solved using constrained minimisation routines. To this purpose, several improvements in the methods of non-linear programming have been provided (Li *et al.* 1997). Moreover, it is recognised that in the presence of load governed by a single parameter, the inequality constrained minimum formulations, in terms of potential or complementary energy, lead to the statements of the kinematic and the static theorems of limit analysis, extended even to the case of non-conservative systems (Maier and Nappi 1990, Boothby and Brown 1992, Boothby 1994). Non-linear optimisation under constraints becomes the natural tool to identify the safety load factor for a system of bodies subjected to unilateral and frictional constraints.

Due to the presence of non-associative flow-rules (4) (in this case $\boldsymbol{N} \neq \boldsymbol{M}$) the Drucker's stability postulate is no longer valid, the power of dissipation cannot be uniquely defined and the solution in terms of load contact actions and collapse factor also loses its uniqueness. Coulomb, in studying the strength of masonry vaults (Coubomb 1776), recognised that the collapse load is not unique when friction has to be taken into account. However, delimitations can be found for it. More recently Drucker, by computing the dissipation due to friction, states safe and unsafe conditions to provide some delimitations for the collapse load (Drucker 1954). The class of the load multiplier is therefore bounded, and in order to evaluate the structural safety we search within this class the minimum (safe) load factor corresponding to both admissible kinematical mechanisms and statical stress states. That is, the minimum load multipliers such that the relations (1-6) are verified. This corresponds, after some algebra, to solve the following problem of non linear programming (NLP) with constraints

$\alpha_c = \min\{\alpha\}$ subject to

$$(\boldsymbol{A}\boldsymbol{M}_1 - \boldsymbol{M}_2)\boldsymbol{\lambda} = \boldsymbol{0} \tag{7}$$

$$(\boldsymbol{A}_0\boldsymbol{N}_1)^T(\boldsymbol{f}_0 + \alpha\boldsymbol{f}_L) + [\boldsymbol{N}_2^T - (\boldsymbol{A}\boldsymbol{N}_1)^T]\boldsymbol{r}_2 \leq \boldsymbol{0} \tag{8}$$

$$\boldsymbol{\lambda}^T(\boldsymbol{A}_0\boldsymbol{M}_1)^T\boldsymbol{f}_L - 1 = \boldsymbol{0} \tag{9}$$

$$\boldsymbol{\lambda}^T\{(\boldsymbol{A}_0\boldsymbol{M}_1)^T(\boldsymbol{f}_0 + \alpha\boldsymbol{f}_L) + [\boldsymbol{N}_2^T - (\boldsymbol{A}\boldsymbol{N}_1)^T]\boldsymbol{r}_2\} = 0 \tag{10}$$

with the unknowns α, $\boldsymbol{r}_2$, $\boldsymbol{\lambda}$, and the bounds on the unknowns $\alpha \geq 0$ and $\boldsymbol{\lambda} \geq \boldsymbol{0}$. The linear constraints (7-9) represent respectively the kinematic compatibility condition, the static admissibility condition and the normalized positive work of the live loads; the non-linear constraint (10) represents the complementarity condition.

3. Non-linear programming: Computational aspects

Due to the non-linear equality constraint (10), the constrained programming problem of the previous section is non-linear and non-convex and its solution is mathematically and numerically difficult to be found for several reasons. It is known that the solution of a convex problem of programming corresponds to a Kuhn-Tucker point and that this point is certainly a global minimum. In non-convex programming problems, however, this is not guaranteed because the Kuhn-Tucker conditions (KT) are necessary but not sufficient conditions for a feasible point to be a minimum (Kirsh1993). Moreover, it is not guaranteed that a feasible minimiser can define a global minimum. Further computational problems are related to the numerical evaluation of the gradients of the objective function and of the constraints performed by many algorithms of programming. Either because the correct evaluation of the gradients guarantees the convergence to a Kuhn-Tucker point and because it enhances the velocity of convergence to the optimal solution (Best *et al.* 1979, pgs. 521-522).

Therefore, we here proposed some devices to cope with this difficult numerical task. First, we used a minimisation routine based on the use of a Lagrange multiplier method that directly implies the KT optimality conditions, which otherwise should be verified a posteriori. This routine exploits the iterative solution of convex programming subproblems with linear constraints. The solution of a subproblem allows finding a search direction which is used to minimize the constrained objective function through the minimization of a free constraint function (the augmented Lagrangian function) (Schittkowski 1986). Secondly, we provided the routine with the symbolic evaluation of the gradients of the objective function and of the constraints to avoid inaccuracy in their numerical evaluation and the termination at a wrong point. Finally, in order to avoid the termination at a local minimum, we performed sensitivity analyses by varying the initial guesses to check the optimal point generated by the algorithm. Nevertheless, the procedure hardly performed as expected: the solutions without a good initial estimate of the collapse mechanism was found to result in spurious local minimum solutions. Thus it became necessary to define a method which performed well in solving the NLP problem.

The convergence to the optimal solution strongly depends on the selection of the initial guess for the variables of the problem. Therefore, by finding an initial estimate of the unknowns of the NLP problem close to their actual values, and then by starting the program from this estimate it is easier to obtain the convergence to the optimal solution. In the paper (Baggio and Trovalusci 1998) two ways to select the initial estimate for the NLP problem were indicated. The first one, based on the reduction of the statical undetermined unknowns, is such that the non-linear constraint becomes quasi linear and also the NLP problem becomes quasi linear. This approach requires the approximate knowledge 'a priori' of the collapse mechanism which often it is not easy to identify. The second way outlined is based on the solution of a linear problem of programming and does not require any user-decision. This approach was adopted and its effectiveness was verified here below by analysing complex structures of practical interest with many degrees of freedom.

A suitable initial estimate for the variables α, $\boldsymbol{\lambda}$ and $\boldsymbol{r}_2$ is then obtained by solving a linearised programming problem corresponding to the limit analysis in presence of associative flow-rules. This program can provide lower or upper bounds to the safe collapse load. In particular, by reinterpreting Drucker through the theorem of Radenkovic (Sacchi and Save 1968), it can be stated that the collapse load for a block system with frictional interfaces is bounded from below by the collapse load of the same assemblage in which the surfaces between the blocks are supposed perfectly polished and cemented together by cohesion; that is, by the collapse load of a system with the same flow rule but with the plastic potential as yield function. Otherwise, it is bounded from above by the collapse load of the same assemblage in which the interfaces are supposed cemented together by a dilatant material rather than by friction; that is, by the collapse load of a system with the same yield surface but associative flow-rule. Here, due to lack of cohesion along the contact surfaces, the lower bound corresponds to the trivial solution, α_c=0, while the solution obtained through the latter approach can be used to initialise the NLP problem.

In presence of dilatancy, the sliding along the joints is allowed only if the blocks move both in the direction tangential and normal to the joints. The rate of dilatancy, defined as the ratio between the normal and the tangential component of the relative displacement between the blocks, is calculated in such a way that the direction of that displacement is normal to the corresponding yield face. In this circumstance the yield domain (3) remains unchanged while, since the normality rule holds, the flow rule becomes associative

$$\boldsymbol{q}=\boldsymbol{N}\lambda, \tag{11}$$

where $\boldsymbol{N}$ is the transpose of the block-diagonal gradient matrix (made of the submatrices in Baggio and Trovalusci 1998, p. 298).

In this way the first member in Eq. (6) corresponds to the work of the interactions in the inelastic strains and, under the hypothesis of perfect plasticity, the non-linear constraint (10) is identically fulfilled.

In this formulation, both the static and the kinematical limit theorems can be used to determine the sole collapse multiplier. The use for example of the upper bound theorem requires only the kinematic compatibility and the positiveness of the work of the live loads. Thence, the following linear programming problem (LP) must be solved

$$\alpha^c = \min \; \{-\boldsymbol{\lambda}^T(\boldsymbol{A}_0\boldsymbol{N}_1)^T\boldsymbol{f}_0\}$$

subjected to

$$(\boldsymbol{A}\boldsymbol{N}_1-\boldsymbol{N}_2)\boldsymbol{\lambda}=\boldsymbol{0} \tag{12}$$

$$\boldsymbol{\lambda}(\boldsymbol{A}_0\boldsymbol{N}_1)^T\boldsymbol{f}_L-1=\boldsymbol{0} \tag{13}$$

with the bound on the unknown $\boldsymbol{\lambda} \geq \boldsymbol{0}$. It can be noticed that the lower bound approach, sometimes used to avoid the normality rule (Livesley 1978), corresponds to a problem solved by taking into account dilatancy instead of friction.

The procedure adopted to solve the NLP problem is based on two steps. In the first step the LP problem is solved. At the end of the process of minimizing, the algorithm produces the vector of inelastic multipliers $\boldsymbol{\lambda}$, the vector of the inelastic strains $\boldsymbol{q}$, and, as a by-product, the vector of the dual statical unknowns $\boldsymbol{r}_2$. In the second step, the NLP problem is solved using the solutions of the first step as initial guesses for the unknowns. In this way the analysis easily converges to the optimal point. This procedure is completely automatic and can be easily adopted by anyone by introducing the geometry of the structure and the coefficient of friction as sole input data.

It is worth noting that the LP solution, obtained by replacing friction with dilatancy, can be considered sufficiently close to the actual solution because of the real importance of the phenomenon of dilatancy. Due to the asperity of the contact surfaces between bricks, tangential sliding is generally accompanied by normal displacement, with a high rate of dilatancy in the presence of low compressive stresses and small cumulative tangential displacements (Lofti and Shing 1991). Moreover, several numerical analyses performed for various block systems verify that the solutions of the LP and the NLP codes are often comparable, both in terms of collapse loads and in terms of collapse mechanisms. In particular, when sliding mechanisms are predominant the two solutions coincide. This result can be clarified by focusing the attention on a single block system resting on a horizontal plane and subjected to an inclined body force (Fig. 1a). If the block moves horizontally and friction is present the collapse multiplier is the coefficient of friction. The linearised solution, obtained accounting for the presence of dilatancy such that the normality rule is satisfied, corresponds to the block considered moving on a plane inclined of the angle of friction (Fig. 1b). With frictionless planes, the virtual work is limited to the work of active forces and the collapse multiplier is still the coefficient of friction. Therefore, in some cases the approximate solution obtained at the first step of the procedure by the LP program can be accepted avoiding the burden of non-linear programming.

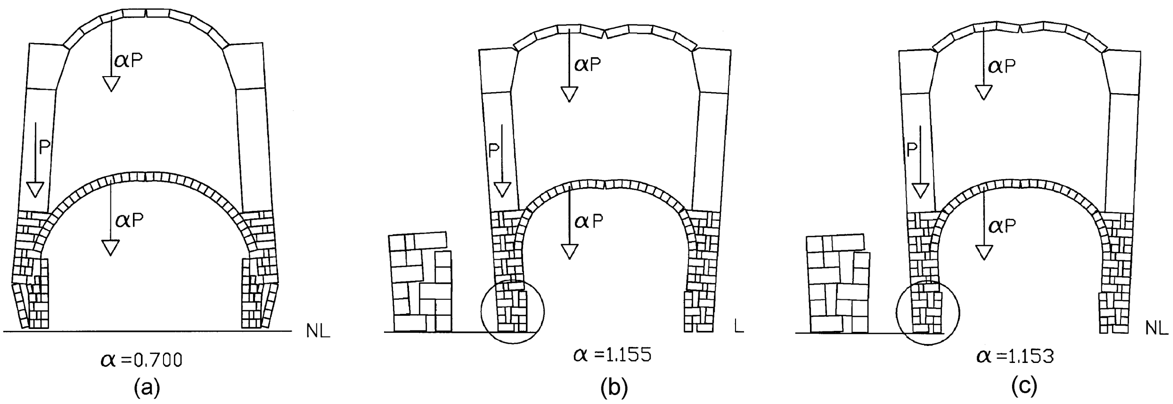


Fig. 2 Collapse mechanism and collapse multiplier of vaulted structures under the dead self weight and a live extra-charge

4. Non standard limit analysis of masonry structures

Thanks to the various block assemblages considered here below, we could test the effectiveness of the proposed two-step procedure in evaluating the ultimate behaviour of real masonry structures. Moreover, the different results obtained at each step, the solutions of the linear and the non-linear programming problem, are discussed pointing out in some cases the suitability of the sole linearized analysis.

The double level arches of Fig. 2 belong to an old dwelling site in South Italy, "I Sassi", in Matera. As often happens to vaulted structures, the number of interfaces (74 in the case of Fig. 2a) does not greatly exceed the number of units (55). This gives rise to a relatively simple optimization problem, easily solved by the two-step procedure, owing to the low level of indeterminacy. The two models are subjected to dead loads, the self weight of each stone of the piers, and to live vertical loads, applied only to the arch voussoirs, proportional to the self weight through the multiplier α. Due to the unequal texture of the pier units, the two collapse mechanisms, obtained using the NLP code, are very different: the first assemblage is unable to carry the full weight of the arch (Fig. 2a), whereas a suitable arrangement of stones not only prevents the breaking down of the second arch under its self weight but also allows an additional load of 15 percent of the self weight (Fig. 2b). By comparing Figs. 2b and 2c, where the results of the linearised and the non linear analyses of the same double arch are showed, we can find a very small numeric difference in the collapse load. In the two details the effect of dilatancy can be observed (Fig. 2b) compared with the absence of dilatancy (Fig. 2c). The small difference in the geometry of the collapse mechanisms, due to the presence of many little stones affected by small displacements because of dilatancy, explains the small difference in the collapse loads. Since the non linear analysis run lasts for some time and requires the selection of an acceptable estimate of the starting parameters, then the linearised procedure, which ends safely in few seconds without these starting parameters, could be preferred.

The samples of Fig. 3 are two-dimensional assemblages made of polygonal stones modelling the masonry walls of three different historical centres in Italy. These are part of a research work on the evaluation of the seismic resistance of ancient masonry to out of plane horizontal forces. These assemblages are characterised by the presence of many contact surfaces for each stone and by their scattered orientation (Fig. 3a). The load condition for each wall is the self weight as dead load and

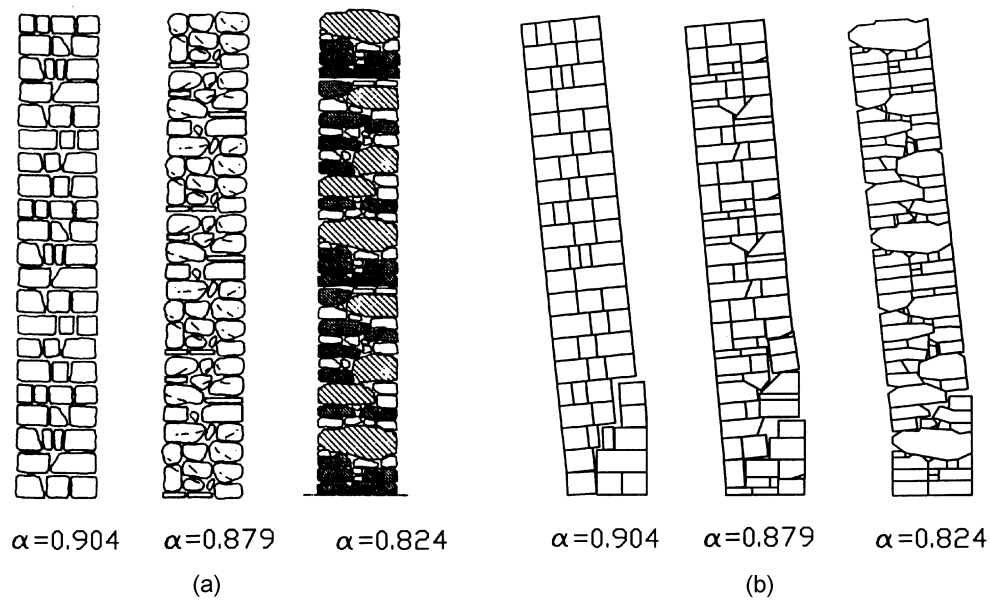


Fig. 3 Masonry textures (a), collapse mechanism and collapse multiplier (b) of walls in 'opus poligonale' subjected to inclined body force

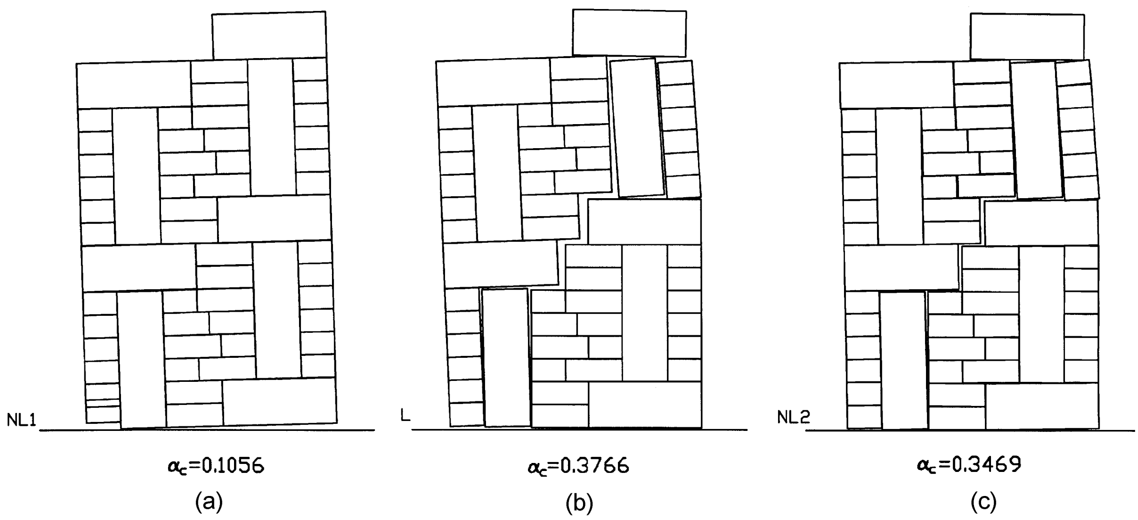
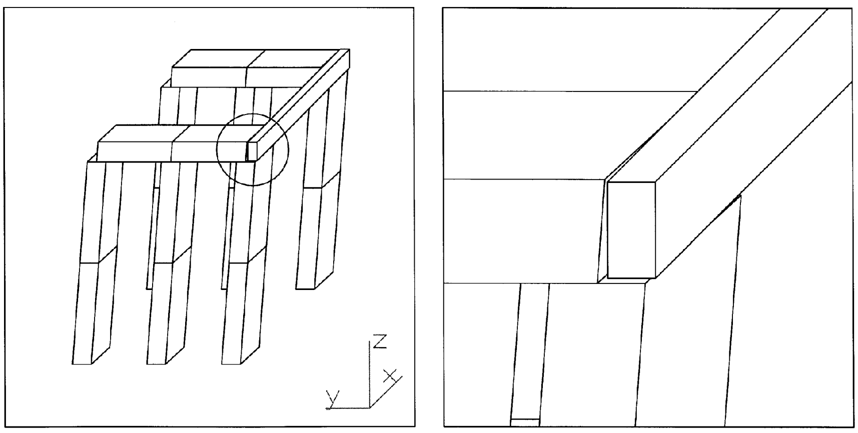


Fig. 4 Collapse mechanism and collapse multiplier of a wall in 'opus africanum' subjected to inclined body force: (a) NLP solution with arbitrary initial guess; (b) LP solution (step 1); (c) NLP solution starting from the LP solution (step 2)

an increasing horizontal body action, the self weight factorised by a real multiplier α, as live load. As in the previous examples, the results of the linearised limit analysis can be accepted. These show a quasi-monolithic behaviour of all the three specimens in spite of the scatter size, texture and orientation of stone in the masonry. The attempt to model the actual masonry texture, also shown in the figure, appears to be successful and the procedure is able to give account of a sound collapse mechanism (Fig. 3b).

The structure in Fig. 4 deserves a more accurate description: it is a model of a masonry typology called ‘opus africanum’, sometimes but not extensively used during the Roman Empire, as in Pompei or in the radial walls of the Coliseum. The main feature of this wall is the presence of large vertical stones, as pilasters or framing of the other stones of common size. In this example the number of stones (57) and the number of joints (138) are well within the possibility of the described programming algorithm; but the fact that the vertical stones are in contact with dozen of little stones makes the problem more complicated. External forces in the analysis are the self-weight, input as dead load, and horizontal mass actions factorised by α, input as live load. Fig. 4a plots the result of the non-linear limit analysis started with a casual initial guess for the unknowns. The result is poor in terms of geometry of collapse mechanism as well as in terms of collapse load: the value α_c=0.106 is far from the actual one and the collapse mechanism is badly mistaken, based on the interpenetration of two stones at the foot of the wall. The linearised solution in Fig. 4b is far more accurate: there is no interpenetration and the collapse load value is acceptable. In the right upper part of the figure we can observe the effect of dilatancy: the slipping of the pilaster with respect to the stone stack in the right enforces the separation along all the slipped joints. Using the results of the linearised solution to evaluate the initial guesses for the NLP problem, the solution of Fig. 4c can be obtained at the end of the second step of the procedure. As in this problem the effects of dilatancy are not negligible, due to the presence of large stones, the solutions of the two steps of programming are quite different.

If we focus our attention on real structures, using either the linear or the non-linear approach, we need to deal with three-dimensional models. Particular importance in studying the collapse behaviour of buildings must be attributed to the problem of connections, for example those between intersecting walls or between column rows in trilithic systems. From an operating point of view, the introduction of spatial problems highly complicates the numerical task. In systems of blocks connected together in three-dimensional arrangements, the number of contact interfaces per block generally increases, the yield domain of each interface is at least composed of ten faces and the contact actions introduce in the problem six unknowns per each interface. If we hold to the linear



(a) (b)

Fig. 5 3D failure behaviour of a trilithic structure under inclined body force

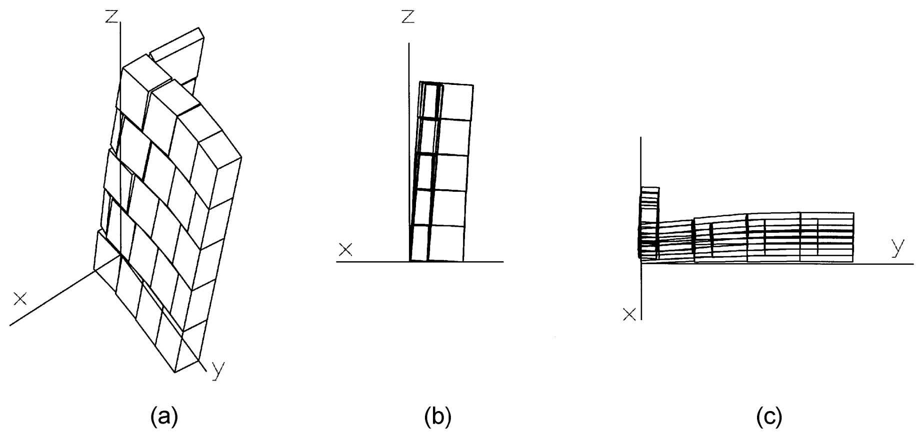


Fig. 6 3D failure behaviour of intersecting walls under inclined body force: (a) Axonometric view; (b) Section; (c) Plan

approach the problem is simplified by the vanishing of these last unknowns. Thus, the cases studied below pertain to the restricted approach only.

The first three-dimensional sample shown is the assemblage of Fig. 5a, made of 17 blocks and 26 joints. This represents a trilithic structure (columns connected by architraves) acted upon by the self weight as dead load and by a live horizontal body action, the self weight factorised by α, parallel to the x-axis. The interconnection level is low: only 26/17=1.5 joints per block, so the linearized numerical problem is quite simple: 260+1 unknowns to deal with, using the simplex method. The collapse factor comes out to be α_c=0.086. The plot clearly shows a minor slippage of the transversal lintel beam (see also Fig. 5b). Due to the architraves connection, the rotations around the y-axis of all the piers are equal to each other. The two equal column rows undergo equal horizontal force, in such a way that the connecting beam, leaving aside its self-weight, should not exert any effect on the collapse mechanism or load, but it is forced to rotate with respect to the architraves and, due to dilatancy, the interposed joint opens.

Since the walls resistance to out of plane actions strongly depends on the quality of the interlocking with the orthogonal walls, the study of the problem of angular connections between intersecting walls is the first step of a research on the behaviour of masonry viewed as a spatial structure. The load condition is the same of the structure above: the dead self weight and a live body action parallel to the x-axis. Figs. 6a (axonometric view), 6b (section) and 6c (plan) represent a block wall restrained by the interlocking with a pilaster or intersecting panel. The interconnection level is higher than the preceding sample: 61/25=2.4 joints per block; a non-linear treatment would give rise to 610 kinematical unknowns plus $(61\text{-}25)\times 6$=216 unknowns (undetermined statical terms due to contact actions); the 826+1 unknowns would create a severe numerical non linear optimization problem, but, due to linearisation, the whole computer run of the procedure lasts few seconds. It can be observed, mainly in plan and section, that the horizontal body force along the x-axis enforces the rotation of the entire structure but the stones near the angle, more constrained, show minor displacement and rotate about the vertical z-axis.

Figs. 7a (axonometric view) and 7b (plan) show a partial model of a gothic fountain: the upper basin of the “Fontana Maggiore” in Perugia, under restoration in 1998. The analysis is aimed at suggesting the restoration techniques for the monument. Horizontal body forces along the x-axis

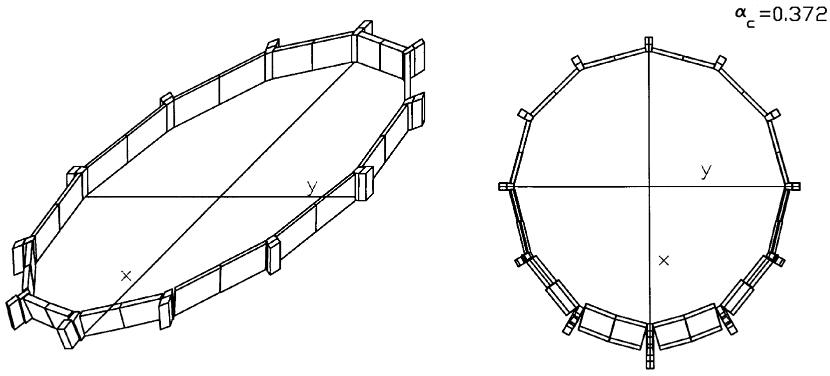


Fig. 7 Collapse mechanism of the 'Fontana Maggiore' in Perugia: (a) Axonometric view; (b) Plan

simulate a seismic (static) action with the scope of accepting or rejecting the presence of simple contact actions between pilasters and marble entablatures. The full model involves 48 blocks and 96 joints but symmetry is directly accounted for in the program, reducing the data size. The problem is thus limited to about 500 kinematical unknowns. The good behaviour of this space model brought to a "soft" restoration of the monument.

5. Conclusions

This work aims to contribute to the study of the collapse behaviour of brick-block structures with particular reference to ancient masonries made of stones dry assembled together or with joints filled with scattered and poor mortar. Since the mechanical behaviour of such assemblies strongly depends on the geometry of the units and on their texture, besides the mechanical properties of the constituents, the discrete approach seemed to be an appropriate tool to model these structures.

The study of the collapse behaviour of masonry, described as Lagrangian systems of blocks in unilateral and frictional contact, has been approached using the theorems of limit analysis in the presence of non-associative rules. This led to the formulation of a problem of non-convex and non-linear mathematical programming under constraints whose solution gives the collapse load and the collapse mechanism.

The analyses performed show how limit analysis of such structures through non-linear programming algorithms can only be approached by resorting to special devices. In this framework, remarkable results are obtained by requiring the preliminary solution of a linearised problem of programming achieved by replacing friction with dilatancy.

The effectiveness of the proposed procedure based on two different steps of mathematical programming-one linear and the other nonlinear- has been investigated by analysing non trivial, two and three-dimensional, problems with many degrees of freedom (with about a thousand unknowns). Real structures in fact are made of many bricks or stones with a large number of contact surfaces, and their study using discrete models could not be proposed without a numerical code able to deal with many unknowns. Moreover, the effectiveness in solving real problems is related to the possibility of dealing with three-dimensional structures.

In short, the results of the samples analysed suggest the following remarks. First, the suitability of the discrete approach and in particular of the limit analysis approach, especially when compared to other discrete approaches (Baggio and Trovalusci 1995). Secondly, the NLP problem starting from a random estimate does not always converges or terminates at a wrong point; if the initial estimate is the output of the LP problem the NLP program converges at an optimal point. It is worth noting that without resorting to two-steps programming, the linearised approach should be preferred to the non-linear one. Finally, the two-step procedure is able to deal with practical problems characterised by many degrees of freedom. Moreover, apart from the standard or non-standard approach, the three-dimensional code proposed allows to study real block structures.

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