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Error propagation effects for explicit pseudodynamic algorithms

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Abstract. This paper discusses the error propagation characteristics of the Newmark explicit method, modified Newmark explicit method and α-function dissipative explicit method in pseudodynamic tests. The Newmark explicit method is non-dissipative while the α-function dissipative explicit method and the modified Newmark explicit method are dissipative and can eliminate the spurious participation of high frequency responses. In addition, error propagation analysis shows that the modified Newmark explicit method and the α-function dissipative explicit method possess much better error propagation properties when compared to the Newmark explicit method. The major disadvantages of the modified Newmark explicit method are the positive lower stability limit and undesired numerical dissipation. Thus, the α-function dissipative explicit method might be the most appropriate explicit pseudodynamic algorithm.

Keywords: error propagation analysis; explicit pseudodynamic algorithm.

1. The step-by-step pseudodynamic test procedure

The pseudodynamic test procedure (Takanashi *et al.* 1975) is basically the same as the step-by-step integration procedure to evaluate the time history response of the tested structure. At first, the test specimen is idealized as a discrete system and thus the equations of motion to govern the dynamic behaviors can be easily derived. Then, a step-by-step integration method must be used to solve the governing equations in performing a pseudodynamic test. It should be mentioned that the inertial and damping properties in the equations of motion are analytically prescribed while the restoring forces are no longer formulated as the product of the stiffness matrix and displacements as usually seen in a step-by-step integration procedure. In fact, in order to overcome the difficulty in mathematically modeling the stress-strain or load-displacement relations accurately for nonlinear structures, the restoring forces will be experimentally measured in performing a pseudodynamic test. This strongly indicates that pseudodynamic tests will provide more accurate test results than for the step-by-step time history analysis since the idealized mathematical models are not employed. The displacement responses in each step of the pseudodynamic test are computed and imposed upon the test structure through hydraulic actuators.

Unlike the quasi-static or cyclic loading tests, pseudodynamic errors will be propagated and accumulated (Chang *et al.* 1998, Shing *et al.* 1987, Shing *et al.* 1990). In the pseudodynamic testing procedure, the computed displacements may not be perfectly imposed upon the specimen due to

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control errors and thus lead to the restoring force errors. The experimentally measured restoring forces with errors will be returned to the on-line computer and be used to compute the next step displacements. This implies that once the pseudodynamic errors are introduced at any time step they will be carried over to the rest of the test. Therefore, it is very important to explore the error propagation characteristics of a step-by-step integration method before its application to perform pseudodynamic tests since the pseudodynamic test errors must be controlled within a certain limits in order to have reliable test results. In this paper, the error propagation effects for the Newmark explicit method (Newmark 1959), the modified Newmark explicit method (Shing and Mahin 1987) and the α-function dissipative explicit method (Chang 1997, Chang 2000) in pseudodynamic tests will be theoretically analyzed and cautiously compared. In addition, numerical illustrations will be provided.

2. Explit pseudodynamic algorithms

All the above mentioned three explicit pseudodynamic algorithms can be obtained from the following expressions:

$$\boldsymbol{M}\boldsymbol{a}_{i+1}+\boldsymbol{C}\boldsymbol{v}_{i+1}+\boldsymbol{r}_{i+1}+\alpha(\boldsymbol{r}_{i+1}-\boldsymbol{r}_i)=\boldsymbol{f}_{i+1}$$

$$\boldsymbol{d}_{i+1}=\boldsymbol{d}_i+(\Delta t)v_i+\frac{1}{2}(\Delta t)^2\boldsymbol{a}_i$$

$$\boldsymbol{v}_{i+1}=\boldsymbol{v}_i+\frac{1}{2}(\Delta t)(\boldsymbol{a}_i+\boldsymbol{a}_{i+1}) \tag{1}$$

where $\boldsymbol{M}$ and $\boldsymbol{C}$ are the mass and damping matrices; $\boldsymbol{d}_{i+1}$, $\boldsymbol{v}_{i+1}$ and $\boldsymbol{a}_{i+1}$ are the vectors of displacements, velocities, and accelerations; $\boldsymbol{f}_{i+1}$ is the external force vector; and $\boldsymbol{r}_{i+1}$ is the restoring force vector. The subscript $(i+1)$ indicates the time step at $t=(i+1)(\Delta t)$. For linear elastic systems, the restoring force vector can be further expressed as $\boldsymbol{r}_{i+1}=\boldsymbol{K}\boldsymbol{d}_{i+1}$ where $\boldsymbol{K}$ is the structural stiffness matrix. In this general formulation, the symbol α can be used to represent a constant or a matrix for multi-degree-of-freedom systems and is a constant for single-degree-of-freedom systems. Various forms of α can lead to the following three different algorithms:

$$\begin{aligned}
&\alpha=0 && \text{Newmark explicit method} \\
&\alpha=a\boldsymbol{I}+\left[\frac{b}{(\Delta t)^2}\right]\boldsymbol{M}\boldsymbol{K}^{-1} && \text{modified Newmark explicit method} \\
&\alpha=c(\Delta t)^2\boldsymbol{K}\boldsymbol{M}^{-1} && \alpha\text{-function dissipative explicit method}
\end{aligned} \tag{2}$$

where $\boldsymbol{I}$ represents an identity matrix. In addition, α denotes a scalar for the Newmark explicit method (Newmark 1959) while it is a matrix for the other two methods (Chang 1997, Shing and Mahin 1987) where the coefficients a, b and c are some appropriate constants. For the modified Newmark explicit method and the α-function dissipative explicit method, α is computed based on the initial stiffness matrix $\boldsymbol{K}$ and kept constant for a complete test since the stiffness matrix is not determined during a pseudodynamic test.

3. Results of error propagation analysis

Employing the same procedure developed by Shing and Mahin (1990) to complete the error propagation analysis of a single-degree-of-freedom system if using the step-by-step integration procedure of Eq. (1), the cumulative displacement error is found to be:

$$e_{n+1}^d = \sum_{i=0}^{n} \{D_i \cos[(n-i)\bar{\Omega} + \theta]\} e_i^d + \sum_{i=0}^{n} \{F_i \sin[(n-i)\bar{\Omega}]\}[\alpha e_i^{rd} - (1+\alpha) e_{i+1}^{rd}] \tag{3}$$

In this equation, $\bar{\Omega} = \bar{\omega}(\Delta t)$, $\bar{\omega}$ is used to represent the computed natural frequency. The symbols e_i^d and e_i^r denote the displacement error and force error introduced at step i, respectively. In addition, e_i^{rd} is introduced to denote the amount of displacement error corresponding to e_i^r and has the relationship of $e_i^r = k e_i^{rd}$. It will be more convenient to compare the propagation effects of the displacement and force errors and to derive the cumulative Eq. (3) if the new error term e_i^{rd} is defined as such. The first and second terms on the right hand side are the cumulative errors due to displacement feedback errors and force feedback errors, respectively. The coefficients D_i and F_i are the amplification factors for the displacement feedback errors and the force feedback errors and are:

$$D_i = \frac{\left(\sqrt{1-\alpha\Omega^2}\right)^{(n-i)}}{\sqrt{1-\frac{1}{4}(1+\alpha)^2\Omega^2}}, \quad F_i = \frac{\Omega\left(\sqrt{1-\alpha\Omega^2}\right)^{(n-i)}}{\sqrt{1-\frac{1}{4}(1+\alpha)^2\Omega^2}}, \quad \theta = \tan^{-1}\left[\frac{\frac{1}{2}(1+\alpha)\Omega}{\sqrt{1-\frac{1}{4}(1+\alpha)^2\Omega^2}}\right] \tag{4}$$

where $\Omega = \omega(\Delta t)$, and $\omega = \sqrt{k/m}$ is the natural frequency of the single-degree-of-freedom system. The symbols k and m are used to represent the stiffness and mass of the system. It is clear that D_i and F_i depend on the values of α and $(n-i)$, where n is the number of total time steps and i is the specific i-th time step. In addition, $F_i = \Omega D_i$ can be easily obtained. It is worth noting that for the case of $\alpha = 0$, Eq. (4) becomes:

$$D_i = \frac{1}{\sqrt{1-\frac{1}{4}\Omega^2}}, \quad F_i = \frac{\Omega}{\sqrt{1-\frac{1}{4}\Omega^2}}, \quad \theta = \tan^{-1}\left[\frac{\frac{1}{2}\Omega}{\sqrt{1-\frac{1}{4}\Omega^2}}\right] \tag{5}$$

This clearly reveals that the amplification factors for the Newmark explicit method are independent of $(n-i)$.

Variations of D_i and F_i versus Ω for all the three algorithms are shown in Fig. 1 and Fig. 2, respectively. In this study, $a=0.25$, $b=-0.012$ and $c=0.10$ are considered and the number of total time steps is assumed to be $n=500$. For the Newmark explicit method, D_i and F_i increase with increasing Ω. These curves for D_i and F_i are increased starting from 1 and 0 each very slowly for the small value of Ω and they grow up very rapidly as Ω tends to stability limit 2, which is the maximum value of $\Omega = \omega(\Delta t)$ to have stable computations for the Newmark explicit method. It should be mentioned that the stability limit is shortened for the modified Newmark explicit method and the α-function dissipative explicit method due to the presence of numerical dissipation. Unlike the Newmark explicit method, the curves for D_i and F_i of the modified Newmark explicit method and the α-function dissipative explicit method are varied with i for a given value of n. In fact, for the two methods, D_i and F_i curves which correspond to a specified time step i move upward as the

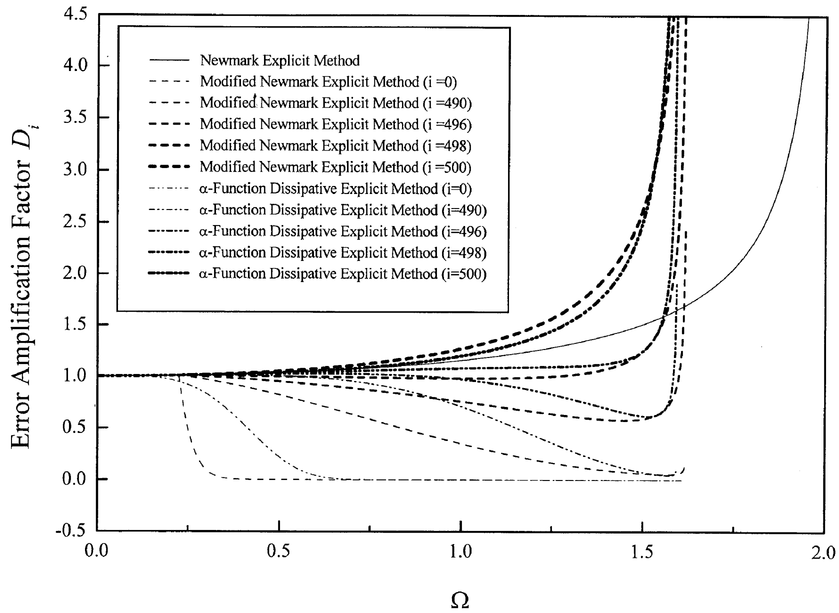

Fig. 1 Error amplification factor for displacement feedback error

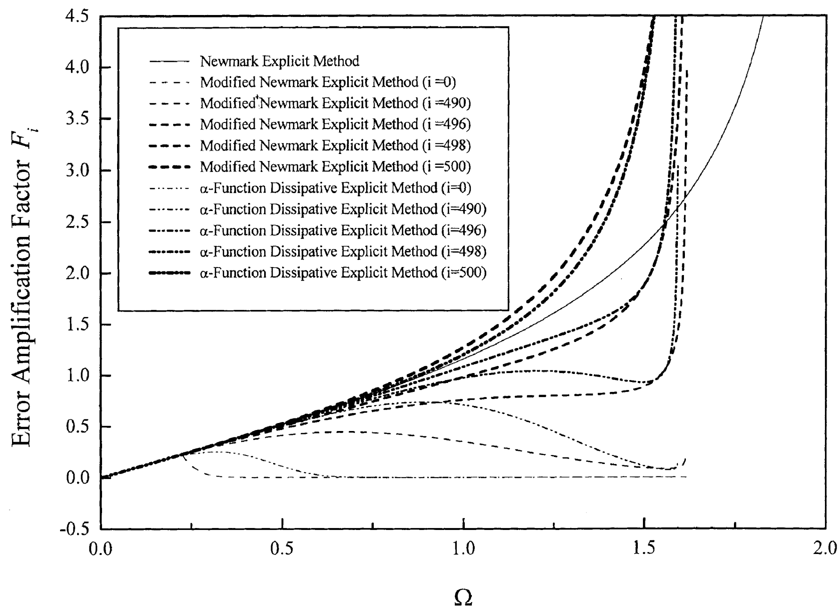

Fig. 2 Error amplification factor for force feedback error

step i increases from 0 to 500. In this example, for the most time steps (say i between 0 and 490), all the D_i curves drop from 1 to 0 as Ω increases from 0 to 2 and they are all under the D_i curve for the Newmark explicit method. For the last few steps (say $490 < i < 500$), the Newmark explicit method still has larger amplification factor D_i than those of the other two methods and only at

i=500 it shows a smaller value. Very similar results are also found for the amplification factor F_i. Generally speaking, the modified Newmark explicit method and α-function dissipative explicit method possess much less error propagation effect for both the displacement feedback errors and the force feedback errors when compared to the Newmark explicit method.

4. Numerical simulation

In order to illustrate that the α-function dissipative explicit method can effectively eliminate the spurious growth of high frequency response due to the presence of errors and thus has improved error propagation characteristics, a computer simulation of a pseudo dynamic test is investigated. In this simulation, the imposed displacement increment Δd at each step is assumed to be a random variable X with a truncated normal distribution (Chang 1992, Chang and Mahin 1993), since abnormally large displacement errors do not occur in a properly adjusted system. In fact, the probability density function of the random variable X with truncated normal distribution is taken to be:

$$f(x)=\frac{1.00135}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \mu-3\sigma < x < \mu+3\sigma \tag{6}$$

where σ is the standard deviation and μ is the mean value of the random variable X. Fig. 3 shows the probability density function of the truncated normal distribution, in which the curve with μ=Δd stands for the probability density functions of a properly adjusted system. In the following simulations, the case of σ=0.0001m and μ=Δd for each degree of freedom is further assumed.

In this example, a four-story shear-beam type structure is studied. The mass and stiffness for each story are assumed to be 1 kg and 1000 N/m. The structure is subjected to the El Centro 1940 (NS) ground acceleration whose peak ground acceleration is scaled to 0.35 g. Numerical results are

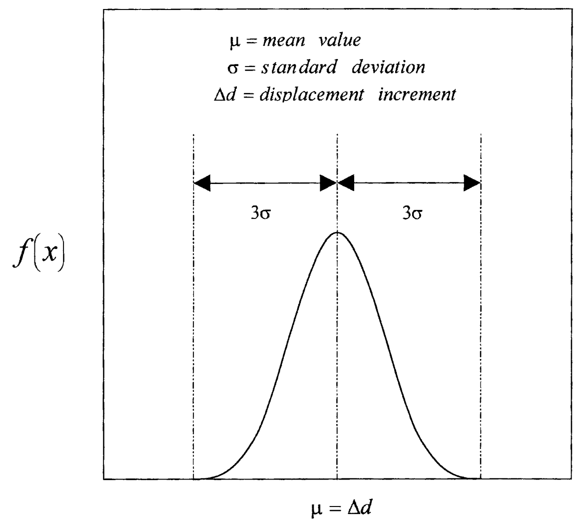

Fig. 3 Probability density function of truncated normal distribution

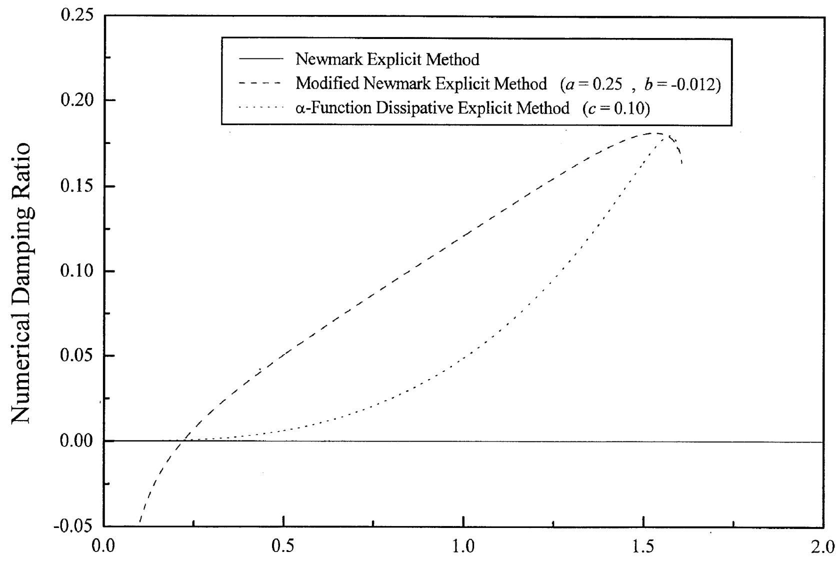

Fig. 4 Comparisons of numerical damping ratio

obtained by employing all the three algorithms using $\Delta t = 0.02$ second. For the modified Newmark explicit method, a=0.25 and $b = -0.012$ are chosen and for the α-function dissipative explicit method, c=0.10 is used. This will lead to roughly the same maximum numerical dissipation. For these specified coefficients, the numerical dissipation properties for all the three algorithms are depicted in Fig. 4. It is manifested from this figure that the Newmark explicit method does not exhibit any numerical dissipation while the numerical damping ratio for the modified Newmark explicit method is almost linearly proportional to the natural frequency and the α-function dissipative explicit method can have desired numerical dissipation. The numerical damping ratio for the α-function dissipative explicit method has a zero slope at the origin and then turns upward gradually. This indicates that the high frequency responses can be eliminated while lower modes are integrated accurately. For the convenience of subsequent discussions, the natural frequencies of the four-story structure and numerical damping ratios $\bar{\xi}$ of the four modes for all the three algorithms are summarized in Table 1.

The stability range for the modified Newmark explicit method is $0.219 \leq \Omega \leq 1.606$ for the given values of a=0.25 and b=–0.012 (Shing and Mahin 1987). In this numerical experiment, in order to have very small numerical damping for the first mode the step size of 0.02 second is used. Thus, it

Table 1 Numerical damping ratio

ω (rad/sec)	10.98	31.62	48.45	59.43
$\Omega = \omega(\Delta t)$	0.220	0.632	0.969	1.189
$\bar{\xi}$ Newmark explicit method	0.000	0.000	0.000	0.000
$\bar{\xi}$ modified Newmark explicit method	0.000	0.070	0.117	0.147
$\bar{\xi}$ α–function dissipative explicit method	0.000	0.012	0.045	0.082

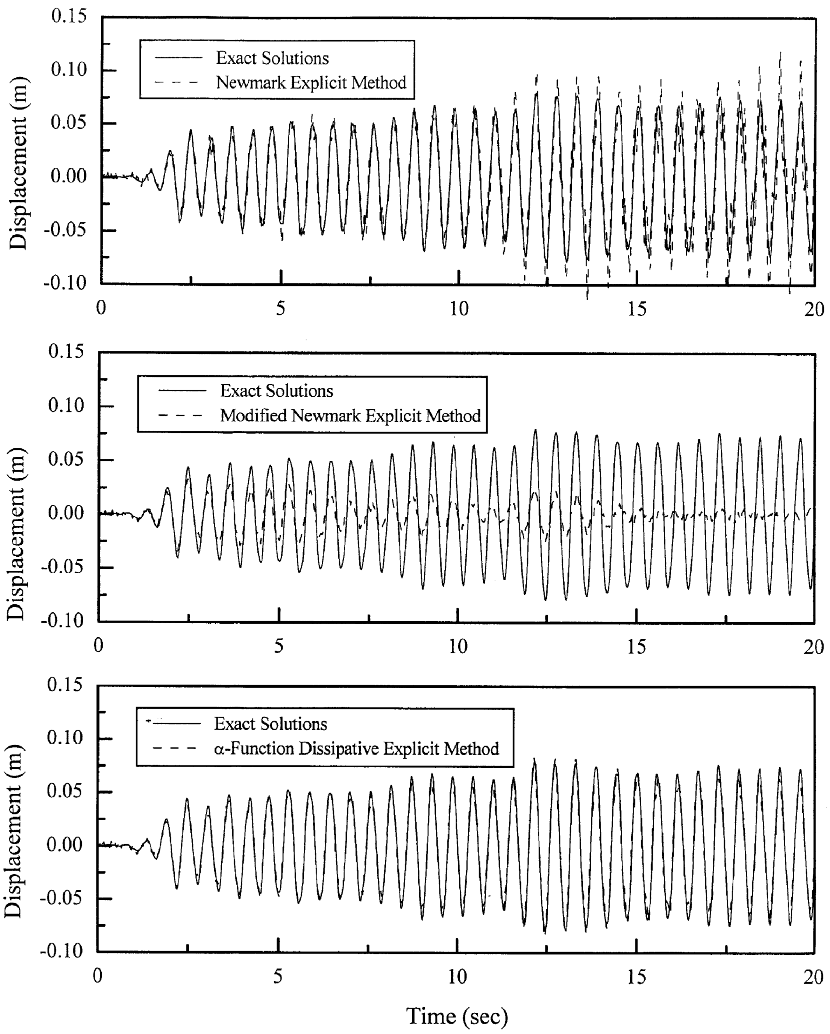

Fig. 5 Bottom story response of a 4-story structure

will result in $\Omega_1=\omega_1(\Delta t)=0.220$ which is slightly larger than the lower stability limit. It is clear that the value of Ω_1 may become smaller than the lower stability limit 0.219 as the structure yields. Thus, instability or negative damping might occur during a nonlinear oscillation (Shing and Mahin 1987).

Numerical results of the computer simulations are plotted in Fig. 5. The displacement responses obtained from the Newmark explicit method significantly deviate from the correct solutions due to the errors introduced since it does not possess any numerical dissipation to suppress the spurious growth of high frequency responses. Excessive numerical dissipation is manifested by the results of the modified Newmark explicit method. This is because that the modal responses of the second, third and fourth modes are quickly eliminated by large numerical dissipation as listed in Table 1. The results for the α-function dissipative explicit method indicates that this method can appropriately eliminate the spurious growth of higher modes while the lower modes are integrated very accurately. For this explicit method, numerical damping ratios for the first two modes are very small as indicated in Table 1. Thus, responses for these two modes can be integrated very accurately. The third mode is appropriately suppressed by the numerical damping ratio of 4.5% and the fourth mode can be entirely filtered out by the numerical damping ratio of 8.2%. In addition to desired numerical dissipation, the improved error propagation characteristics for the α-function dissipative explicit method lead to less error propagation effect for the higher modes. Thus, it gives pretty good results.

5. Conclusions

For all the three explicit pseudodynamic algorithms, the Newmark explicit method is shown to have the worst error propagation characteristics. In addition, it can not provide any numerical damping to suppress or eliminate the spurious growth of high frequency responses. Even though the modified Newmark explicit method is dissipative, the lower modes except the fundamental mode might be damped too strongly for a structure with uniform spread frequencies since its numerical dissipation is frequency-proportional. Furthermore, the positive lower stability limit indicates that the softening of the specimen might lead to an instability problem. The α-function dissipative explicit method possesses the desired numerical dissipation to damp out the spurious growth of higher modes while the lower modes can be obtained accurately. Furthermore, its applications in pseudodynamic tests are significantly enhanced by the improved error propagation characteristics. Thus, the use of α-function dissipative explicit method in explicit pseudodynamic tests is recommended.

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