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A technique for optimally designing fibre-reinforced laminated structures for minimum weight with manufacturing uncertainties accounted for

M. Walker†

Centre for Advanced Materials, Design & Manufacture Research, Durban University of Technology, South Africa
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Abstract. A methodology to design symmetrically laminated fibre-reinforced structures under transverse loads for minimum weight, with manufacturing uncertainty in the ply angle, is described. The ply angle and the ply thickness are the design variables, and the Tsai-Wu failure criteria is the design constraint implemented. It is assumed that the probability of any tolerance value occurring within the tolerance band, compared with any other, is equal, and thus the approach is a worst-case scenario approach. The finite element method, based on Mindlin plate and shell theory, is implemented, and thus effects like bending-twisting coupling are accounted for. The Golden Section method is used as the search algorithm, but the methodology is flexible enough to allow any appropriate finite element formulation, search algorithm and failure criterion to be substituted. In order to demonstrate the procedure, laminated plates with varying aspect ratios and boundary conditions are optimally designed and compared.

Keywords: design technique, manufacturing uncertainty, minimum weight, fibre-reinforced laminated structures.

1. Introduction

Design for minimum weight is becoming popular as the necessity for improved structural performance increases. This is particularly true of structures made from fibre reinforced composites, which offer superior stiffness to weight and strength to weight ratios when compared to conventional materials. An advantage of these materials over conventional ones is the possibility of tailoring their properties to the specific requirements of a given application. The tailoring is mostly achieved by maximizing the mechanical properties as a result of selecting the fibre angles and thicknesses of the layers optimally, and thus realizing the full potential of fibre-reinforced composites.

A number of studies concerning the minimum weight design of composite structures appear in the literature. Angle-ply laminates subjected to uncertain loads were studied by Adali *et al*. (1995) who used a convex modelling approach in their analysis. Adali *et al*. (1994) investigated the minimum weight and deflection design of thick sandwich laminates via symbolic computation. Optimal weight design of shells was tackled by Min and de Charanteney (1986), who investigated sandwich cylinders under combined loadings. A study by Ostwald (1990) considered the combined loading cases of

†E-mail: walker@dut.ac.za

external pressure and axial compression in the optimization of thin walled shells. The Bubnov-Galerkin method was used to solve the stability problem. A paper by Walker *et al.* (1997) focuses on the minimum deflection and weight designs of laminated composite plates. The finite element method using Mindlin plate theory was used in conjunction with optimization routines in order to obtain the optimal designs. Comparative results are presented for minimum weight priority design as an alternative to minimum deflection/minimum weight priority design to investigate the effect of priority on the deflection and weight.

Various researchers have proposed design methodologies or techniques for the optimization of composite structures. For example, Walker *et al.* (1997) describes a methodology that can be used to select the best material combinations and optimize the design of hybrid composite plates for minimum weight and cost. In two papers, Walker and Smith (2002, 2003) describe several different techniques for the design optimization of composite structures. In the first, the fibre orientations, laminae thicknesses and material combinations are used as design variables, and the technique is used to select the best combination for minimum cost for plates subject to compressive loads. The second details a simple-self design methodology that can be used to minimize the mass of composite structures. The procedure is based on the finite element method, and suitable elements are removed without affecting the overall structural integrity. Here, a failure criterion is implemented.

Tolerances of design variables due to variations in manufacturing processes and user environment may affect the quality and performance of a product (Messac and Sundararaj 2002). It is usually beneficial to account for such variances in the design process, and in fact, sometimes it may be crucial, particularly when the effect is of consequence. Robust Design Optimization (RDO) is intended to yield a system that performs with minimal variability in the face of input variations or uncertainties. RDO methods generally seek to minimize the variation of an Aggregate Objective Function (AOF) and to maintain design feasibility under input variations. The optimization outcome depends on (i) the acceptable level of variations in performance, and (ii) the level of input variations (Bates *et al.* 2002). Robust design, then, is an approach that explicitly recognizes the effects of these variations and seeks to minimize their consequences - without eliminating their sources.

Various researchers have used robust optimization techniques in the design optimization of structures. For example, Liou & Jang (1997) describe a procedure for considering stress distributions in forged products and use the finite element method together with a robust design approach. In order to extend the operating life of products and satisfy the quality of operation during the customer usage, it is necessary to monitor residual stresses during the forging operations. The finite element method and robust design methodology were utilized to identify the controlling process parameters which can monitor the residual stresses in forged products. Lee & Park (2002) describe a robust optimization strategy for dealing with discrete constrained design problems. A relatively simple method is proposed to select discrete and robust optimum. At first, the discrete design is achieved as the postprocess of conventional optimization. An orthogonal array is established around a conventional optimum, and the characteristic functions are evaluated. The characteristic function is defined by considering the robustness of the objective and constraints. The parameter design of the Taguchi method is introduced to obtain the robust solution in discrete space. The method is insensitive to variations of the design variables within the selected discrete values enhancing the feasibility of constraints. Several structural problems are solved to demonstrate the technique.

Very few researchers have dealt with the robust design of composite structures. Chiang (1996) used a robust design approach to improve the accuracy of the Iosipescu shear test specimen. The statistical design of experiments based on the finite element method was employed, and was able to identify the influential design variables. Kristinsdottir (1995, 1996) describe a methodology which uses a random global search algorithm to explore near optimal designs with different tolerances. In two papers by Walker and Hamilton (2005, 2005) a technique for optimally designing laminated plates with manufacturing tolerances present in the design variable (which is the fibre orientation) is described. The objective is to maximize the buckling load carrying capacity and in the first, a closed form solution for plates is implemented, whilst in the second, the FEM is used. The techniques are aimed at optimally designing for the worst-case scenario, and the results presented (as a means of illustrating the methodology) demonstrate the importance of accounting for manufacturing uncertainties.

This paper describes a procedure to design symmetrically laminated structures under transverse loads for minimum mass with manufacturing uncertainty in the ply angle. The ply angle and the ply thickness are the design variables, and the Tsai-Wu failure criteria is the design constraint implemented. As in Walker and Hamilton (2005, 2005), it is assumed that the probability of any tolerance value occurring within the tolerance band, compared with any other, is equal, and thus the approach is a worst-case scenario approach. The finite element method, based on Mindlin plate and shell theory, is implemented, and thus effects like bending-twisting coupling are accounted for. The Golden Section method is used as the search algorithm, but the methodology is flexible enough to allow any appropriate finite element formulation, search algorithm and failure criterion to be substituted. In order to demonstrate the procedure, laminated plates with varying aspect ratio and boundary conditions are optimally designed and compared. The results show that if the manufacturing uncertainty in the fibre orientation is neglected, for the tolerance scenario implemented, the plate thickness may be as much as 12% underspecified, and consequently would fail in practice.

2. Bending of rectangular laminates

Consider a symmetrically laminated rectangular plate of length a, width b and total thickness H under a transverse bending load $q(x, y)$, as shown in Fig. 1. The plate is located in the x, y, z plane and

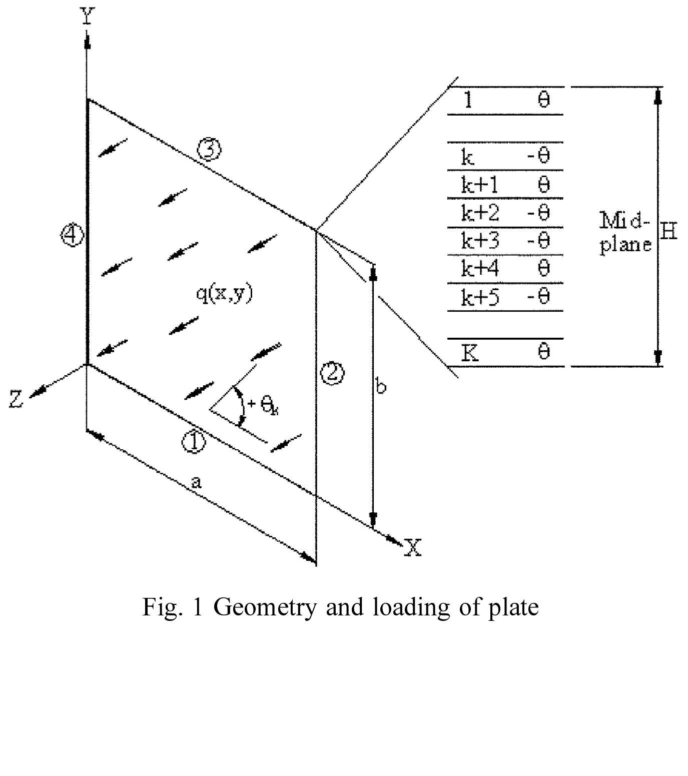

Fig. 1 Geometry and loading of plate

constructed of an arbitrary number K of orthotropic layers of thickness t_k and fibre orientation θ_k where $k=1, 2, ..., K$. The displacement of a point (x^0, y^0, z^0) on the reference surface is denoted by (u^0, v^0, w^0). Different combinations of free (F), simply supported (S) and clamped (C) boundary conditions are implemented at the four edges of the plate. In particular, four different combinations are studied, namely, (C, S, F, S), (S, S, S, S), (C, S, C, S) and (C, C, C, C), where the first letter refers to the first plate edge, and the others follow in the anti-clockwise direction as shown in Fig. 1.

3. Finite element formulation

We now consider a finite element formulation of the problem (based on Mindlin type theory, although any suitable formulation can be substituted). Let the region S of the plate be divided into n sub-regions S_r ($S_r \in S$; $r = 1, 2, ..., n$) such that

$$\Pi(u) = \sum_{r=1}^{n} \Pi^{Sr}(u) \tag{1}$$

where Π and Π^{Sr} are potential energies of the plate and the element, respectively, and u is the displacement vector. Using the same shape functions associated with node j ($j = 1, 2, ..., n$), $S_j(x, y)$, for interpolating the variables in each element, we can write

$$u = \sum_{r=1}^{n} S_j(x, y) u_j \tag{2}$$

where u_j is the value of the displacement vector corresponding to node j, and is given by

$$u = \{u^{(j)}, v^{(j)}, w^{(j)}, \phi_1^{(j)}, \phi_2^{(j)}\}^T \tag{3}$$

The displacements $\{u, v, w, \phi_1, \phi_2\}$ are approximated as

$$u = \sum_{j=1}^{n} u_j \Psi_j(x, y), \quad v = \sum_{j=1}^{n} v_j \Psi_j(x, y), \quad w = \sum_{j=1}^{n} w_j \Psi_j(x, y)$$

$$\phi_1 = \sum_{j=1}^{n} S_j^1 \Psi_j(x, y), \quad \phi_2 = \sum_{j=1}^{n} S_j^2 \Psi_j(x, y) \tag{4}$$

where ψ_j are Lagrange family of interpolation functions. From the equilibrium equations of the first order theory, and Eq. (6), we obtain the finite element model of the first-order theory,

$$\sum_{\beta=1}^{5} \sum_{j=1}^{n} K_{ij}^{\alpha\beta} \Delta_j^{\beta} - F_i^{\alpha} = 0, (\alpha = 1, 2, \ldots, 5) \tag{5}$$

or

$$[K]\{\Delta\} - \{F\} = \{0\} \tag{6}$$

where K and F are the stiffness and force coefficients respectively, and the variable Δ denotes the nodal values of w and its derivatives.

4. The Tsai-Wu failure criteria

The Tsai-Wu failure criteria stipulates that the condition for non-failure for any particular ply is

$$F(\theta) = F_{11}\sigma_{11}^{(k)}\sigma_{11}^{(k)} + F_{22}\sigma_{22}^{(k)}\sigma_{22}^{(k)} + F_{66}\tau_{12}^{(k)}\tau_{12}^{(k)} + 2F_{12}\sigma_{1}^{(k)}\sigma_{2}^{(k)} + F_{1}\sigma_{1}^{(k)} + F_{2}\sigma_{2}^{(k)} \leq 1 \tag{7}$$

where the strength parameters F_{11}, F_{22}, F_{66}, F_1, F_2 and F_{12} are given by

$$F_{11} = 1/(X_tX_c);\ F_{22} = 1/(Y_tY_c);\ F_{66} = 1/G^2$$

$$F_1 = 1/X_t - 1/X_c;\ F_2 = 1/Y_t - 1/Y_c;\ F_{12} = -\frac{1}{2}\sqrt{F_{11}F_{22}} \tag{8}$$

and X_t, X_c, Y_t, Y_c are the tensile and compressive strengths of the composite material in the fibre and transverse directions, and G is the in-plane shear strength.

5. Optimal design problem and solution procedure

The objective of the design problem is to minimize the mass of the plate, with manufacturing uncertainty in the layup angle θ accounted for. The problem can thus be stated as

$$W_{\min} \triangleq \min_{\theta, H}[W(\theta, H)],\ \ 0^\circ \leq \theta \leq 90^\circ,\ \ H_{\min} \leq H \leq H_{\max} \tag{9}$$

where the mass of a plate is given by

$$W = Hab\rho \tag{10}$$

and where H is the total thickness of the plate and ρ the density.

In this case the minimum mass is found by determining

$$\min_{H} |F(\theta) - 1| \tag{11}$$

at each value of θ until $H_{\min}$ (and thus θ_{opt}) is obtained.

When composite laminates are manufactured, the desired fibre orientation in different plies may deviate from their intended design values by a few degrees. These deviations are referred to as manufacturing tolerances. Assume that for the interval $0^\circ \leq \theta \leq 90^\circ$, a manufacturing tolerance in the layup angle θ is incurred, and must be accounted for during the design stage, if optimal performance is required. Furthermore, assume that the probability of any tolerance value occurring within the tolerance band, compared with any other, is equal. For example, there may be a maximum variation band of $+g$ or $-h$, with $0^\circ \leq g, h \leq 90^\circ$. In addition, when accounted for, it is assumed that $\theta + g \leq 90^\circ$ and $0^\circ \leq \theta + h$. In order to illustrate the problem, consider the following scenario: assume a manufacturer incurs the following maximum tolerances, $\theta + 13^\circ$ and $\theta - 7^\circ$; viz. when trying to achieve the value θ, the actual value that results is $\theta - 7^\circ \leq \theta \leq \theta + 13^\circ$. Fig. 2 shows the effect of the tolerance on the minimum

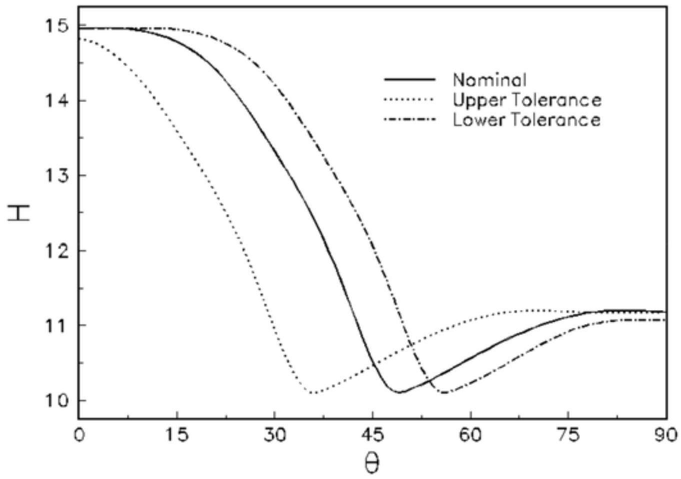

Fig. 2 Effect of manufacturing tolerance in θ on the minimum plate thickness with $a/b = 1.25$ and (CCCC) boundary condition

required plate thicknesses for a (*CCCC*) laminated plate with four symmetric layers of equal thickness and with $\theta_1 = -\theta_2 = -\theta_3 = \theta_4 = \theta$. The plate has dimensions a = 1.25 m, b = 1m and is subjected to a uniform transverse bending pressure of 100,000 Pa. The material properties are those for a typical T300/5208 graphite/epoxy material with E_1 = 181 GPa, E_2 = 10.3 GPa, G_{12} = 7.17 GPa and γ_{12} = 0.28. There are three trendlines given, and these represent the nominal layer thickness (viz. the value at θ), along with the upper and lower bounds (viz. the values at $\theta + g$ and $\theta + h$; thus the plate thickness required is $H_{lower} \leq H \leq H_{upper}$ due to the presence of tolerance in the layup angle). Note that each value in the trendlines has been determined by using Eq. (11). It is evident that the effect of the upper and lower tolerances is to shift the nominal trend right and left. The design problem becomes one of determining the value of θ at which the layer thickness is minimized thus reducing the weight of the

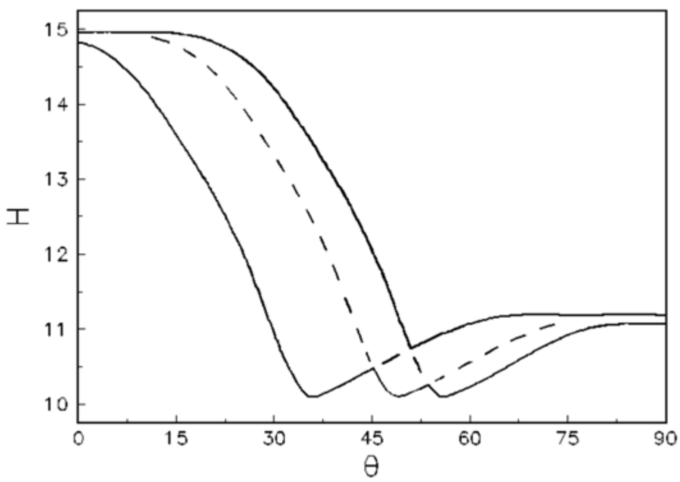

Fig. 3

laminate, with the tolerances accounted for, which effectively becomes one of determining the value of θ for which the trend described by the upper solid line in Fig. 3 (which needs no explanation) is minimized (viz. designing for the worst case scenario). In addition, in the event that two or more values of θ correspond to equal minimum thickness values, the one that gives the best value contained within the lower solid line should logically be selected. In this case, the optimal value is 51.43° which corresponds to a plate thickness of 10.89 mm. The optimal values for the nominal case are 48.84° and 10.10 mm. It is apparent from this example that the values of the actual optimal results are different from those of the nominal, and that if we were to ignore the manufacturing tolerances and choose 48.84° as the optimal fibre orientation, the corresponding minimum thickness required could be as high as 11.32 mm (viz. the value at 48.84°–7°), which is 12% more than the optimal value. Alternatively, if the nominal design values are selected, the plate would fail in practice. This fact emphasizes the importance of carrying out optimization in design work with the effects of manufacturing tolerances included.

The optimization procedure thus involves the stages of determining the minimum layer thickness required for a given θ, $\theta + g$ and $\theta + h$ to satisfy Eq. (11), selecting the greatest of the three values, and improving the fibre orientation to minimize the greatest value. Thus, the computational solution consists of successive stages of analysis and optimization until convergence is obtained and the optimal angle θ_{opt} and layer thickness t_{min} is determined within specified accuracy. In the optimization stage here, the Golden Section method is employed to determine both θ_{opt} and t_{min}, and θ_{opt} is determined to within 0.01°, whilst t_{min} is determined to within 0.01 mm.

6. Sample results and discussion

In order to further illustrate the methodology described above, plates with four equal thickness symmetric layers and with aspect ratios ranging from $a/b = 0.5$ to 2 are studied. In addition, the effect of the boundary condition is also considered. The plates have the same material properties and loading as that used for Fig. 2.

The effect of the aspect ratio is illustrated in Tables 1 and 2, for (*CCCC*) and (*SSSS*) plates (respectively). Also, the nominal optimal fibre angle and corresponding H_{min} are reported, for comparison purposes.

For Table 1, the trend in both the nominal thickness and actual thickness values is generally as expected, viz. as the plate gets longer, so the thickness increases. In addition, the actual values are greater than the nominal values (as expected), with the difference for the square plate being the largest (9.7%). The optimal fibre orientation values show the same (increasing) trend although the actual values are less than the nominal values for $0.5 \le a/b \le 1$. When $a/b>1$, the values are greater. The greatest difference occurs for the plate with $a/b = 0.5$.

Table 1 Effect of varying the plate aspect ratio for (*CCCC*) plates

a/b	θ_{opt}	H_{min}	Optimal fibre angle (nominal)	H_{min} (nominal)
0.5	7.59°	4.46	35.46°	4.37
0.75	29.54°	7.65	39.64°	7.26
1	41.97°	10.20	45°	9.30
1.25	51.43°	10.89	48.84°	10.10
2	56.71°	11.09	52.15°	10.36

Table 2 Effect of varying the plate aspect ratio for (*SSSS*) plates

a/b	θ_{opt}	H_{min}	Optimal fibre angle (nominal)	H_{min} (nominal)
0.5	0°	6.06	0°	5.73
0.75	0°	9.79	0°	8.69
1	90°	12.50	0 or 90°	11.78
1.25	90°	12.46	90°	11.57
2	90°	12.43	86.81°	11.62

Table 3 Effect of boundary condition for plates with a/b = 1.25

Boundary condition	θ_{opt}	H_{min}	Optimal fibre angle (nominal)	H_{min} (nominal)
(*SSSS*)	90°	12.46	90°	11.57
(*CSFS*)	38.73°	21.23	39.98°	19.84
(*CSCS*)	84.14°	7.88	79.69°	7.82
(*CCCC*)	51.43°	10.89	48.84°	10.10

For Table 2, the trend in the thickness values is the same as that observed in Table 1, except when a/b >1, at which point the values start reducing. Nonetheless, the actual values are always greater than the nominal values due to the uncertainty in θ, and the greatest difference is 12.7% (in the case of the plate with a/b = 0.75). The values of the optimal fibre orientations are not particularly interesting, except in the case of the longest plate, which has a nominal θ_{opt} which is neither 0° or 90° (although close to 90°).

The effect of the boundary condition on the results for plates with a/b = 1.25 is illustrated in Table 3. There are no apparent trends in either the thickness or fibre orientation values (although, as usual, the actual thickness values are greater than the nominal values). The greatest difference in the thickness values is 7.6% in the case of the (*SSSS*) plate. Interestingly, the plate with the free edge, viz. the (*FSCS*) plate, is almost double the thickness of the thinnest (for both actual and nominal values).

7. Conclusions

A technique for designing symmetrically laminated structures under transverse loads for minimum mass with manufacturing uncertainty in the ply angle has been presented. The ply angle and the ply thickness are the design variables, and the Tsai-Wu failure criteria is the design constraint implemented. It is assumed that for a specified value of the fibre angle θ (for $0^\circ \leq \theta \leq 90^\circ$) there may be a maximum variation of $+g$ or $-h$, with $0^\circ \leq g, h \leq 90^\circ$, and also that the probability of any tolerance value occurring within the tolerance band, compared with any other, is equal. In addition, when accounted for, it is assumed that $\theta + g \leq 90^\circ$ and $0^\circ \leq \theta + h$. It is obvious then that for a specified value of θ, the actual outcome could vary between the values $\theta + g$ and $\theta + h$, and thus also the corresponding minimum plate thickness could vary between the values $H(\theta + g)$ and $H(\theta + h)$. The technique that has been described is designed to determine the value of θ at which the greatest corresponding thickness is minimized. Thus the worst case scenario is always accounted for. The search routine implemented is the Golden Section method, although any suitable one can be substituted. Similarly, the Tsai-Wu failure criterion is implemented, but any suitable alternative could also be substituted.

In order to illustrate the technique, the designs of symmetrically laminated plates are optimized. The plates are subjected to similar uniformly disctributed loads, and the effect of changing the aspect ratio and boundary condition is studied. In addition, the value of θ_{opt} is determined to within 0.01^o whilst H is determined to within 0.01 mm, and the optimized designs are compared to those for plates not subjected to manufacturing variations (termed here 'nominal'). The results demonstrate that if the manufacturing tolerances are neglected in the optimal design stage, for the tolerance scenario implemented, the plate can be as much as 12% heavier if it were to carry the load without failure, or would be too thin, and fail in practice. Finally, it should be noted that different finite element formulations can also be substituted into the technique when more appropriate.

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