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A general method of analysis of composite beams with partial interaction

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Abstract. This paper presents a generic modelling of composite steel-concrete beams with elastic shear connection. It builds on the well-known seminal technique of Newmark, Siess and Viest, in order to formulate the partial interaction formulation for solution under a variety of end conditions, and lends itself well for modification to enable direct quantification of effects such as shrinkage, creep, and limited shear connection slip capacity. This application is possible because the governing differential equations are set up and solved in a fashion whereby inclusion of the kinematic and static end conditions merely requires a statement of the appropriate constants of integration that are generated in the solution of the linear differential equations. The method is applied in the paper for the solution of the well-studied behaviour of simply supported beams with partial interaction, as well as to provide solutions for a beam encastré at its ends, and for a propped cantilever.

Keywords: composite beams; differential equations; elasticity; indeterminate; interface slip; partial interaction.

1. Introduction

Composite steel-concrete beams are used extensively in contemporary engineering structures, since the attributes that best suit both the concrete (its relatively high compressive strength) and the steel (its high tensile strength) are utilised, whilst the ramifications of the undesirable attributes of the concrete (its low tensile strength owing to cracking) and the steel (its low compressive strength due to buckling) are minimised. In many nations, the use of composite construction is necessary on the basis of economic considerations, since a structure built in steel alone, or in concrete alone, may be undesirable on a cost basis compared with the alternate composite steel-concrete design solution that provides the best optimisation of cost. The fundamental characteristic of composite beams that enables composite action to be mobilised is the shear connection between the concrete slab and steel joist, and this paper is concerned with the stiffness aspects of the shear connection, that is partial interaction at the interface.

In order to provide a load path that enables a composite beam to be stressed in the benign way that utilises the compressive strength of the concrete slab and the tensile strength of the steel joist,

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mechanical shear connection between the slab and joist is essential. This is most often achieved by the use of headed stud connectors (Oehlers and Bradford 1995, 1999), whose load-slip characteristics can be determined from standard push-out tests. Although this load-slip characteristic is usually nonlinear, up to quite moderate load levels it can be considered to be sensibly linear (Oehlers and Bradford 1995), as can the stress-strain response of the concrete slab in compression and the steel joist in tension. At service load levels, the three components of the composite beam, viz. the concrete slab, shear connection and the steel joist, can therefore be considered to behave in a fashion that is characterised by linear material response. The elastic (rather than rigid) response of the shear connection produces partial interaction, and the structural mechanics of a composite beam with partial interaction makes the analysis of composite members much more difficult than would appear at face value. The influence of the elastic shear connection was addressed over half a century ago in the seminal and highly quoted work of Newmark, Siess and Viest (1951), which established that when even elementary assumptions in structural mechanics (such as elastic behaviour, Euler-Bernoulli bending theory and a constant curvature at a cross-section) overarch the analysis, the solution is far more complex than the familiar midspan deflection of $(5/384)wL^4/EI$ for an elastic beam under a uniformly distributed load.

Many investigators have utilised or extended the work of Newmark, Siess and Viest for specific applications. A mixed formulation has fairly recently been used in several applications to extend the analysis of partial interaction to investigate the influence of the limited slip capacity of headed stud connectors (Oehlers and Sved 1995). This technique assumed that the slab and the joist were elastic, but that the shear connector behaved as if rigid and then with a plastic plateau until its ultimate slip was attained, after which fracture occurred and its strength was zero. When calibrated against tests, the mixed approach produced accurate solutions that can quantify clearly the significance of the relevant parameters on beam strength. Another formulation of a mixed approach is that of Nguyen, Oehlers and Bradford (1998), Ahmed, Oehlers and Bradford (2000) and Oehlers *et al.* (2000), in which the elastic formulation for the concrete was combined with elastic assumptions for steel plate, in which steel plating is bolted to the sides of a concrete beam in a retrofit process, and this was combined with plastic-fracture assumptions for the shear connectors. This analysis is different from that for the composite T-beam considered by Newmark, Siess and Viest, as the curvature in the concrete beam and in the steel side plate cannot be considered to be equal. Nevertheless, the mixed approach leads to analytical solutions, which when calibrated with test data, allow for an accurate quantification of the parameters that may cause the shear connection to be lost due to fracture of the bolt connectors.

This paper formulates an analysis of a composite steel-concrete composite T-beam with elastic shear connection in a more generic fashion to that usually developed in application-specific treatments. The formulation produces analytical results for beams with a number of support conditions, by solving the linear differential equation that is established in the paper in terms of constants of integration whose prescription for the support conditions is routine by invoking the kinematic and static boundary conditions. The formulation is demonstrated for the well-known case of a simply supported beam with a uniformly distributed load, and also for a beam that is encastré at both ends and for a propped cantilever beam. The solution is formulated in a fashion that lends itself to extension for beams with shrinkage and creep deformations, and to more general mixed analysis to investigate beam strength that is governed by limited slip capacity of the mechanical shear connectors.

2. Partial interaction analysis

2.1 General

The modelling of partial interaction developed in this section is based on the composite cross-section shown in Fig. 1. For simplicity, a single span beam is considered (as shown in Fig. 2) which is subjected to a pattern of loading that produces a variation of bending moment $M(z)$ whose variation is not necessarily known initially if the beam is statically indeterminate. Again for simplicity, it is assumed that both the concrete and steel behave elastically in both compression and tension (so that slab cracking is ignored), but modifying the procedure to include this is not difficult as will be explained later. It is assumed further that the shear connection between the concrete slab and steel is elastic, with a modulus k (force per length^2) that defines the relationship between the shear flow force q per unit length and the slip s at the interface by the relationship $q = k \times s$, as shown in Fig. 3.

The top fibre of the cross-section is selected as a convenient invariant reference position from which cross-sectional properties are defined, and it is assumed that the curvature ρ is the same in both the

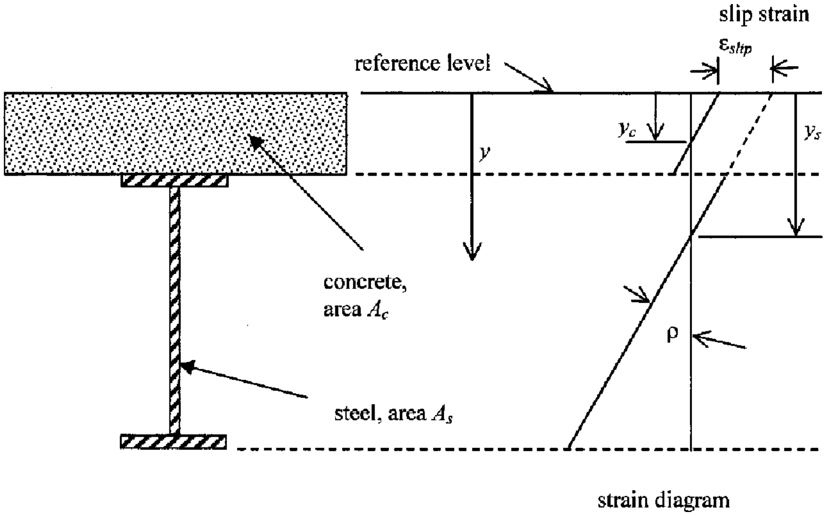


Fig. 1 Composite cross-section

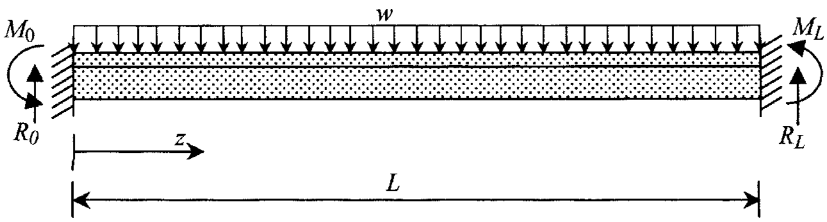


Fig. 2 General single span beam

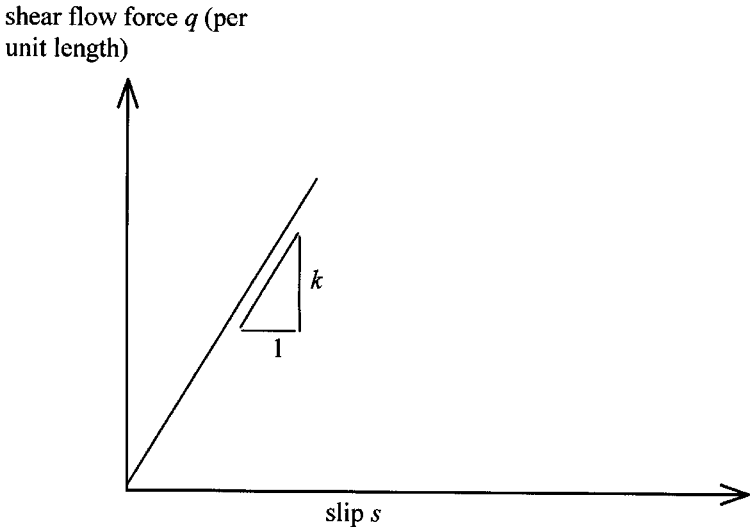


Fig. 3 Response of shear connection

concrete and steel, and that plane sections remain plane (so that the strain diagram is linear), with a slip discontinuity *ds/dz* (the slip strain) at the interface.

2.2. Horizontal equilibrium

The axial forces in the concrete (N_c) and steel (N_s) are

$$N_c = \int_{A_c} \sigma_c dA_c; \qquad N_s = \int_{A_s} \sigma_s dA_s \tag{1}$$

where A_c and A_s are the areas of the concrete and steel respectively, and for each material

$$\sigma_c = E_c \varepsilon_c; \qquad \sigma_s = E_s \varepsilon_s \tag{2}$$

in which E_c and E_s are the elastic moduli of the concrete and steel respectively, and the strains are given by

$$\varepsilon_c = (y - y_c)\rho; \qquad \varepsilon_s = (y - y_s)\rho \tag{3}$$

in which y is the distance below the reference position, y_c is the coordinate of the neutral axis for the concrete and y_s is that for the slab, as shown in Fig. 1.

From Eqs. (1), (2) and (3)

$$N_c = (B_c - y_c A_c) E_c \rho; \qquad N_s = (B_s - y_s A_s) E_s \rho \tag{4}$$

where B_c and B_s are the first moments of area of the concrete and steel respectively below the reference position. Since the beam is subjected to bending only, horizontal equilibrium requires that

$$N_c + N_s = 0 \tag{5}$$

so that

$$\rho[(B_c - y_c A_c)E_c + (B_s - y_s A_s)E_s] = 0 \tag{6}$$

which yields a relationship between the neutral axis depths given by:

$$y_c = \frac{\overline{BE} - y_s A_s E_s}{A_c E_c}; \quad y_s = \frac{\overline{BE} - y_c A_c E_c}{A_s E_s} \tag{7}$$

where $\overline{BE} = B_c E_c + B_s E_s$.

2.3. Slip strain

The slip strain ε_{slip} between the steel and concrete is (Fig. 1)

$$\varepsilon_{slip} = \frac{ds}{dz} = (y_c - y_s)\rho \tag{8}$$

so that

$$y_s = \frac{\overline{BE}}{\overline{AE}} - \frac{A_c E_c}{\overline{AE}\rho}\varepsilon_{slip}; \quad y_c = \frac{\overline{BE}}{\overline{AE}} - \frac{A_c E_c}{\overline{AE}\rho}\varepsilon_{slip} + \frac{\varepsilon_{slip}}{\rho} \tag{9}$$

where $\overline{AE} = A_c E_c + A_s E_s$.

2.4. Internal bending moment

The internal bending moment within a cross-section is given by

$$M_{int} = \int_A \sigma y dA = \int_{A_c} E_c \varepsilon_c y dA_c + \int_{A_s} E_s \varepsilon_s y dA_s \tag{10}$$

Hence, substituting Eq. (3) into Eq. (10) gives

$$M_{int} = \overline{EI}\rho - (B_c E_c y_c + B_s E_s y_s)\rho \tag{11}$$

where $\overline{EI} = E_c I_c + E_s I_s$ in which I_c and I_s are the second moments of area of the concrete and steel about the reference position, respectively. Substituting Eq. (9) into Eq. (11) produces

$$M_{int} = \left(\overline{EI} - \frac{\overline{BE}^2}{\overline{AE}}\right)\rho + \frac{\overline{BE}}{\overline{AE}}\varepsilon_{slip} A_c E_c - B_c E_c \varepsilon_{slip} \tag{12}$$

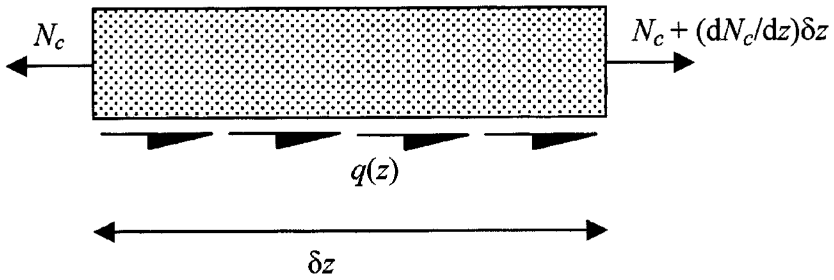


Fig. 4 Free body diagram of slab at interface

2.5. Shear connection

Fig. 4 shows a free body diagram of the concrete slab of length δz with its shear connection at its soffit. By considering horizontal equilibrium,

$$\left(N_c + \frac{dN_c}{dz}\delta z + q\delta z\right) - N_c = 0 \tag{13}$$

so that

$$\frac{dN_c}{dz} + q = 0 \tag{14}$$

which produces

$$\frac{dN_c}{dz} = -ks(z) \tag{15}$$

If Eq. (9) is substituted into Eq. (4) for the concrete slab, then

$$N_c = B_c E_c \rho - A_c E_c \frac{\overline{BE}}{\overline{AE}}\rho - \frac{\overline{\overline{AE}}}{\overline{AE}}\varepsilon_{slip} \tag{16}$$

where $\overline{\overline{AE}} = A_c E_c A_s E_s$, and so

$$\frac{dN_c}{dz} = \left(B_c E_c - \frac{A_c E_c \overline{BE}}{\overline{AE}}\right)\frac{d\rho}{dz} - \frac{\overline{\overline{AE}}}{\overline{AE}}\frac{d^2 s}{dz^2} \tag{17}$$

which when using the equilibrium Eq. (15) gives

$$\left(B_c E_c - \frac{A_c E_c \overline{BE}}{\overline{AE}}\right)\frac{d\rho}{dz} - \frac{\overline{\overline{AE}}}{\overline{AE}}\frac{d^2 s}{dz^2} = -ks \tag{18}$$

The internal moment M_{int} is equal to the applied moment field $M(z)$ (denoted as M for ease of notation), so that from Eq. (12)

$$\rho = \gamma M + \alpha \varepsilon_{slip} \tag{19}$$

and hence

$$\frac{d\rho}{dz} = \gamma \frac{dM}{dz} + \alpha \frac{d^2 s}{dz^2} \tag{20}$$

in which

$$\alpha = \frac{B_c E_c \overline{AE} - A_c E_c \overline{BE}}{\overline{AE}\,\overline{EI} - \overline{BE}^2} \tag{21}$$

and

$$\gamma = \frac{\overline{AE}}{\overline{AE}\,\overline{EI} - \overline{BE}^2} \tag{22}$$

Substituting Eq. (20) into Eq. (18) produces the following generic form of the linear differential equation for partial interaction:

$$\bar{\alpha} \frac{d^2 s}{dz^2} - ks = \alpha \frac{dM}{dz} \tag{23}$$

where

$$\bar{\alpha} = -\frac{(B_c E_c)^2 A_s E_s + (B_s E_s)^2 A_c E_c - \overline{\overline{AE}}\,\overline{EI}}{\overline{AE}\,\overline{EI} - \overline{BE}^2} \tag{24}$$

The generic differential Eq. (23) may be solved routinely to produce the slip and slip strain respectively as

$$s = C_1 e^{vz} + C_2 e^{-vz} - \frac{\alpha}{k} \frac{dM}{dz} \tag{25}$$

and

$$\varepsilon_{slip} = vC_1 e^{vz} - vC_2 e^{-vz} - \frac{\alpha}{k} \frac{d^2 M}{dz^2} \tag{26}$$

in which

$$v^2 = \frac{k}{\bar{\alpha}} \tag{27}$$

and C_1 and C_2 are constants of integration.

Eqs. (25) and (26), which are the solution of the differential equation Eq. (23), assume that the expression of M is not higher than to the second order in z. For higher orders the previous approach is still valid, but the expressions derived for the slip and slip strain need to be modified accordingly.

2.6. Curvature, rotation and deflection

The expression for the curvature can be obtained by substituting Eq. (26) into Eq. (19) as

$$\rho = \gamma M + \alpha \nu C_1 e^{\nu z} - \alpha \nu C_2 e^{-\nu z} - \frac{\alpha^2}{k}\frac{d^2 M}{dz^2} \tag{28}$$

which when integrated with respect to z yields the slope as

$$\theta = \gamma \int M dz + \alpha C_1 e^{\nu z} + \alpha C_2 e^{-\nu z} - \frac{\alpha^2}{k}\int \frac{d^2 M}{dz^2} dz + \hat{C}_1 \tag{29}$$

and which when again integrated with respect to z produces the deflection of the beam as

$$v = \gamma \int\int M dz + \frac{\alpha C_1 e^{\nu z}}{\nu} - \frac{\alpha C_2 e^{-\nu z}}{\nu} - \frac{\alpha^2}{k}\int\int \frac{d^2 M}{dz^2} dz + \hat{C}_1 z + \hat{C}_2 \tag{30}$$

where $\hat{C}_1$ and $\hat{C}_2$ are constants of integration. The axial force in the concrete is given after appropriate substitution into Eq. (16) as

$$N_c = \gamma_1 \rho + \gamma_2 \varepsilon_{slip} \tag{31}$$

in which

$$\gamma_1 = B_c E_c - A_c E_c \frac{\overline{BE}}{\overline{\overline{AE}}} \tag{32}$$

and

$$\gamma_2 = -\frac{\overline{\overline{AE}}}{\overline{\overline{AE}}} \tag{33}$$

3. Applications

Eqs. (25) to (30) form the basis for investigating the behaviour of composite beams under a variety of end conditions. The use of these equations in a generic form is illustrated in this section for a simply supported beam, for a beam encastré at both ends, and for a propped cantilever. In Fig. 2, which depicts a redundant beam under a uniformly distributed load, the bending moment M along the beam is defined as

$$M = -M_0 + R_0 z - \frac{wz^2}{2} \tag{34}$$

where R_0 and M_0 are the vertical reaction and the moment at the left support, and w is the uniformly distributed load. The slip and slip strain can then be expressed using Eqs. (25) and (26) as

$$s = C_1 e^{\nu z} + C_2 e^{-\nu z} - \frac{R_0 - wz}{k}\alpha \tag{35}$$

and

$$\varepsilon_{slip} = \nu C_1 e^{\nu z} - \nu C_2 e^{-\nu z} + \frac{w}{k}\alpha \tag{36}$$

and so the curvature, rotation and deflection are, respectively

$$\rho = \gamma\left(-M_0 + R_0 z - \frac{wz^2}{2}\right) + \alpha\nu C_1 e^{\nu z} - \alpha\nu C_2 e^{-\nu z} + \frac{\alpha^2}{k}w \tag{37}$$

$$\theta = \gamma\left(-M_0 z + \frac{R_0 z^2}{2} - \frac{wz^3}{6}\right) + \alpha C_1 e^{\nu z} + \alpha C_2 e^{-\nu z} + \frac{\alpha^2}{k}wz + \hat{C}_1 \tag{38}$$

$$v = \gamma\left(-\frac{M_0 z^2}{2} + \frac{R_0 z^3}{6} - \frac{wz^4}{24}\right) + \frac{\alpha C_1 e^{\nu z}}{\nu} - \frac{\alpha C_2 e^{-\nu z}}{\nu} + \frac{\alpha^2}{k}\frac{wz^2}{2} + \hat{C}_1 z + \hat{C}_2 \tag{39}$$

Applying the kinematic boundary conditions that $v(z = 0) = 0$ and $v(z = L) = 0$, the constants of integration $\hat{C}_1$ and $\hat{C}_2$ may be determined as

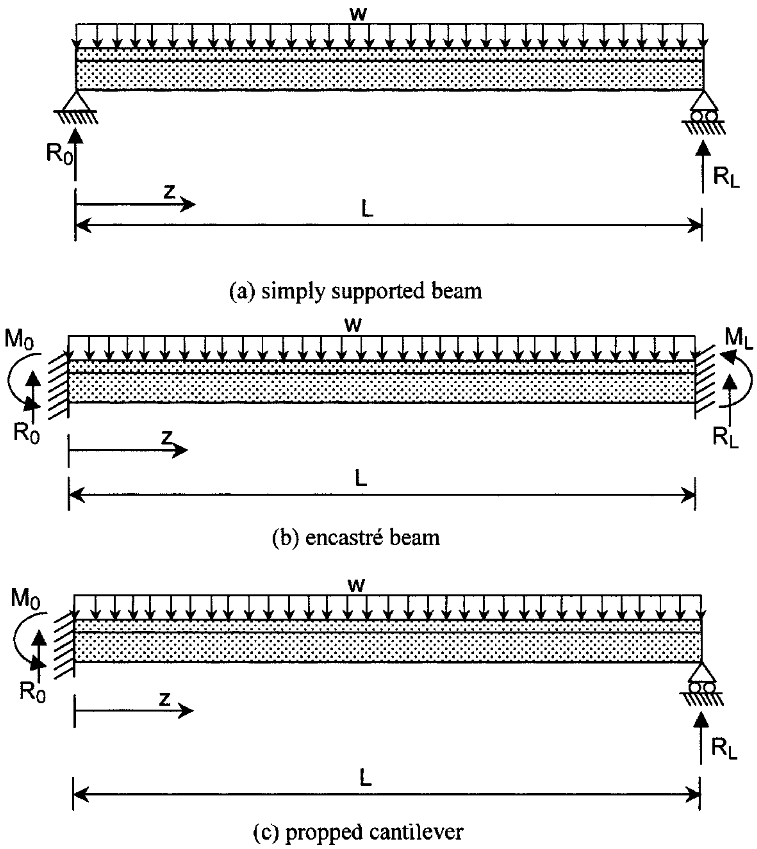


Fig. 5 Composite beams

$$\hat{C}_1 = \gamma\left(\frac{M_0 L}{2} - \frac{R_0 L^2}{6} + \frac{wL^3}{24}\right) - \frac{\alpha C_1 e^{\nu L}}{\nu L} + \frac{\alpha C_2 e^{-\nu L}}{\nu L} - \frac{\alpha^2}{k}\frac{wL}{2} - \frac{\alpha C_2}{\nu L} + \frac{\alpha C_1}{\nu L} \quad (40)$$

$$\hat{C}_2 = \frac{\alpha}{\nu}(C_2 - C_1) \quad (41)$$

In this form, the constants of integration can be prescribed by imposing the static boundary conditions for the relevant beam type, as illustrated in the following sub-sections for a simply supported beam, for a beam encastré at both ends, and for a propped cantilever.

3.1. Simply supported beam subject to a uniformly distributed load

The reactions for the simply supported beam shown in Fig. 5(a) are determined from statics as:

$$R_0 = R_L = \frac{wL}{2}, \qquad M_0 = M_L = 0 \quad (42)$$

The expressions for the slip and slip strain are then determined using Eqs. (35) and (36), and applying the boundary conditions that $\varepsilon_{slip}(z = 0) = 0$ and $\varepsilon_{slip}(z = L) = 0$ to calculate the constants of integration C_1 and C_2. Hence,

$$s = \frac{w\psi_1}{\alpha\nu}\left(-e^{\nu(z-L)} + e^{-\nu z}\right) - \frac{w\alpha}{k}\left(\frac{L}{2} - z\right) \quad (43)$$

$$\varepsilon_{slip} = \frac{w\psi_1}{\alpha}\left(-e^{\nu(z-L)} - e^{-\nu z}\right) + \frac{w\alpha}{k} \quad (44)$$

where $\psi_1 = \dfrac{\alpha^2}{k(e^{-\nu L} + 1)}$

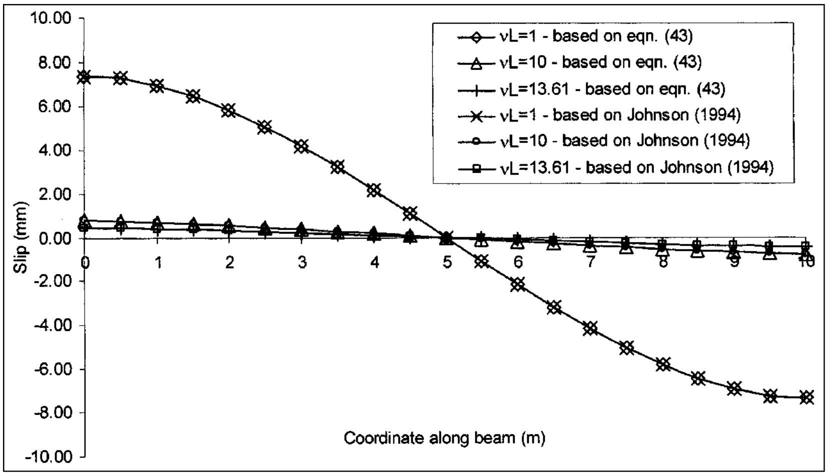
The curvature, rotation and deflection along the beam can be determined from Eqs. (37), (38) and (39), and after simplifying these become

$$\rho = w\left[-\frac{\gamma z^2}{2} + \frac{\gamma L z}{2} + \frac{\alpha^2}{k} - \psi_1\left(e^{-\nu z} + e^{\nu(z-L)}\right)\right] \quad (45)$$

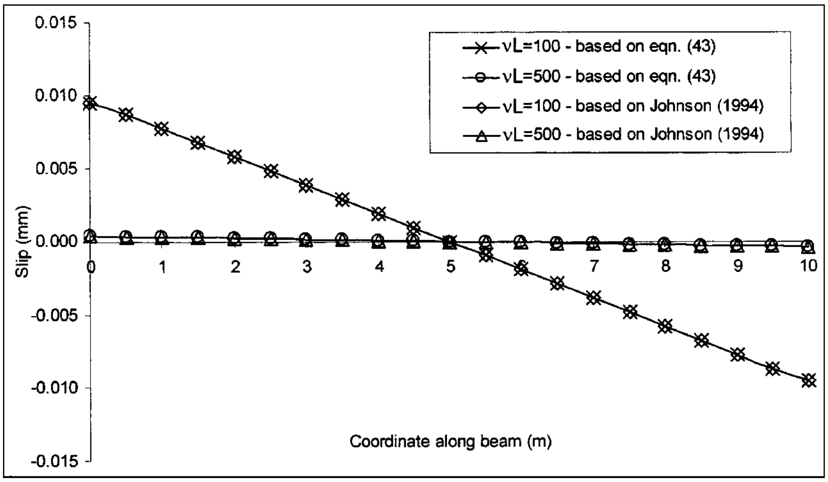
$$\theta = w\left[-\frac{\gamma z^3}{6} + \frac{\gamma L z^2}{4} + \frac{\alpha^2 z}{k} - \frac{\gamma L^3}{24} - \frac{L\alpha^2}{2k} - \frac{\psi_1}{\nu}\left(-e^{-\nu z} + e^{\nu(z-L)}\right)\right] \quad (46)$$

$$v = w\left[-\frac{\gamma z^4}{24} + \frac{\gamma L z^3}{12} + \frac{\alpha^2 z^2}{2k} - \left(\frac{\gamma L^3}{24} + \frac{L\alpha^2}{2k}\right)z + \frac{\alpha^2}{k\nu^2} - \frac{\psi_1}{\nu^2}\left(e^{-\nu z} + e^{\nu(z-L)}\right)\right] \quad (47)$$

Fig. 6 illustrates that the results of the slip calculated using Eq. (43) are identical to those presented in Johnson (1994). This behaviour is illustrated for various values of the dimensionless stiffness νL, where ν is defined in Eq. (27), and in particular for the value of $\nu L = 13.61$ that has been worked as a practical example in Johnson (1994) using a stiffness of 150,000 kN/m and a connector spacing of 180 mm, so that $k = 150{,}000/0.18 = 833\times10^3$ kN/m^2. The example in Johnson is based directly on the closed form solution based on Newmark's approach (Newmark, Siess and Viest 1951). The cross-sectional properties for



(a) Moderate interaction



(b) High interaction

Fig. 6 Slip along a simply supported beam 10 m long subject to a uniformly distributed load

the beam adopted in the comparison are those of the worked example in Johnson (1994) and comprise a rectangular concrete element having a width of 600 mm, a depth of 300 mm and an elastic modulus E_1 = 20 kN/mm^2, and the corresponding values for the rectangular steel element are 60 mm width, 300 mm depth and E_2 = 200 kN/mm^2. The beam is assumed be cast shored (Oehlers and Bradford 1995, 1999), and to support a uniformly distributed load of 35 kN/m.

It is interesting to note that the dimensionless parameter νL derived in this paper is identical to the dimensionless stiffness χL introduced by Girhammar and Pan (1993) as

$$\chi = \left[k\left(\frac{1}{A_1E_1} + \frac{1}{A_2E_2} + \frac{h^2}{I_1E_1 + I_2E_2}\right)\right]^{\frac{1}{2}} = \nu \tag{48}$$

The cross-sectional properties used to determine χ in Eq. (48) are, however, calculated about the centroid of the concrete and steel element respectively and h represents the distance between these centroids, while the cross-sectional properties used to determine v are calculated about the reference position.

3.2. Encastré composite beam subject to a uniformly distributed load

Fig. 2, which is reproduced again in Fig. 5(b), shows a beam encastré at its ends. From the symmetry of the loading and of the support conditions shown in Fig. 5(b)

$$R_0 = R_L = \frac{wL}{2}, \qquad M_0 = -M_L \tag{49}$$

where R_L and M_L are the vertical reaction and the moment at the right hand support.

Applying the static boundary conditions that $s(z = 0) = 0$ and $s(z = L) = 0$, the constants of integration that are related to the slip and the slip strain can be determined as

$$C_1 = \frac{1}{2k}\frac{e^{-\nu L}\alpha wL}{e^{-\nu L} - 1} \tag{50}$$

and

$$C_2 = -\frac{1}{2k}\frac{\alpha wL}{e^{-\nu L} - 1} \tag{51}$$

which when substituted into Eqs. (35) and (36) yield the expressions of the slip and slip strain as

$$s = \frac{wL\psi_2}{\alpha}(e^{\nu(z-L)} - e^{-\nu z}) - \frac{w\alpha}{k}\left(\frac{L}{2} - z\right) \tag{52}$$

$$\varepsilon_{slip} = \frac{wL\nu\psi_2}{\alpha}(e^{\nu(z-L)} + e^{-\nu z}) + \frac{w\alpha}{k} \tag{53}$$

where $\psi_2 = \dfrac{\alpha^2}{2k(e^{-\nu L} - 1)}$.

The value of the moment at the supports is calculated imposing $\theta(z = 0) = 0$, which yields.

$$M_0 = -M_L = \frac{wL^2}{12} \tag{54}$$

Eq. (54) implies that the points of contraflexure for a beam encastré at both ends are independent of the value of the shear connection stiffness and are located at the same position as those for a beam with full interaction.

Using Eqs. (49), (50), (51) and (54) in Eqs. (40) and (41), the distribution of the curvature, slope and deflection for an encastré beam are, respectively

$$\rho = w\left[-\frac{\gamma z^2}{2} + \frac{\gamma Lz}{2} - \frac{\gamma L^2}{12} + \frac{\alpha^2}{k} + \psi_2 \nu L(e^{\nu(z-L)} + e^{-\nu z})\right] \tag{55}$$

$$\theta = w\left[-\frac{\gamma z^3}{6} + \frac{\gamma L z^2}{4} + \left(\frac{\alpha^2}{k} - \frac{\gamma L^2}{12}\right)z - \frac{\alpha^2 L}{2k} + \psi_2 L(e^{\nu(z-L)} - e^{-\nu z})\right] \tag{56}$$

and

$$v = w\left[-\frac{\gamma z^4}{24} + \frac{\gamma L z^3}{12} + \left(\frac{\alpha^2}{k} - \frac{\gamma L^2}{12}\right)\frac{z^2}{2} - \frac{\alpha^2 L z}{2k} + \frac{\psi_2 L}{\nu}(e^{\nu(z-L)} + e^{-\nu z} - e^{-\nu L} - 1)\right] \tag{57}$$

3.3. Propped cantilever subject to a uniformly distributed load

The left hand end of the propped cantilever ($z = 0$) shown in Fig. 5(c) is assumed to be fixed while the right hand end ($z = L$) is assumed to be a roller support. From statics,

$$R_0 = wL - R_L, \qquad M_L = 0, \qquad M_0 = \frac{wL^2}{2} - R_L L \tag{58}$$

and so the expressions for the slip and slip strain are defined once the constants of integration C_1 and C_2 are obtained from the static boundary conditions that $s(z = 0) = 0$ and $\varepsilon_s(z = L) = 0$, as follows

$$C_1 = \frac{\alpha e^{-\nu L}}{k\nu}\frac{\nu R_0 e^{-\nu L} - w}{1 + e^{-2\nu L}} \tag{59}$$

and

$$C_2 = \frac{\alpha}{k\nu}\frac{\nu R_0 + w e^{-\nu L}}{1 + e^{-2\nu L}} \tag{60}$$

and the expressions for the slip and slip strain are then obtained substituting Eqs. (59) and (60) into Eqs. (35) and (36). The longitudinal variation of the curvature, slope and deflection can then be written respectively as

$$\rho = -\frac{w\gamma}{2}z^2 + \gamma R_0 z + \frac{w\gamma L^2}{2} - R_0\gamma L + \frac{\alpha^2 w}{k} + \alpha\nu C_1 e^{\nu z} - \alpha\nu C_2 e^{-\nu z} \tag{61}$$

$$\theta = -\frac{w\gamma}{6}z^3 + \frac{\gamma R_0}{2}z^2 + \left(\frac{w\gamma L^2}{2} - R_0\gamma L + \frac{\alpha^2}{k}w\right)z + \alpha C_1 e^{\nu z} + \alpha C_2 e^{-\nu z} + \hat{C}_1 \tag{62}$$

$$v = -\frac{w\gamma}{24}z^4 + \frac{\gamma R_0}{6}z^3 + \left(\frac{w\gamma L^2}{2} - R_0\gamma L + \frac{\alpha^2 w}{k}\right)\frac{z^2}{2} + \frac{\alpha C_1 e^{\nu z}}{\nu} - \frac{\alpha C_2 e^{-\nu z}}{\nu} + \hat{C}_1 z + \hat{C}_2 \tag{63}$$

where

$$\hat{C}_1 = -\frac{5\gamma L^3 w}{24} + \frac{\gamma L^2 R_0}{3} + \frac{\alpha^2}{L\nu^2 k}(w + \nu R_0) - \frac{\alpha^2 w L}{2k} - \frac{2\alpha^2(we^{-\nu L} + \nu R_0)}{L\nu^2 k(1 + e^{-2\nu L})} \tag{64}$$

and

$$\hat{C}_2 = \frac{\alpha^2(2we^{-\nu L} - \nu R_0 e^{-2\nu L} + \nu R_0)}{\nu^2 k(1 + e^{-2\nu L})} \tag{65}$$

in which the value of the reaction at the fixed support ($z = 0$) is calculated using $\theta(z = 0)$ as

$$R_0 = \frac{5wL}{8} \cdot \left\{ \frac{(5\gamma L^4 v^2 k - 24\alpha^2 + 12\alpha^2 L^2 v^2)(e^{-2vL} + 1) + 48\alpha^2 e^{-vL}}{5v(3\alpha^2 vL^2 + \gamma L^4 vk)(e^{-2vL} + 1) + 15v\alpha^2 L(e^{-2vL} - 1)} \right\} \quad (66)$$

The end reactions are therefore a function of the stiffness of the shear connection. Dimensionless values of the vertical reaction $8\,R_0/5wL$ are shown in Fig. 7 for different values of the dimensionless stiffness vL, plotted logarithmically, while the variation of the location of the point of contraflexure a_cL ($z = a_cL$) is shown in Fig. 8 expressed as a function of the dimensionless coefficient a_c. When the beam has full interaction, its stiffness is sensibly uniform along its length, and the vertical

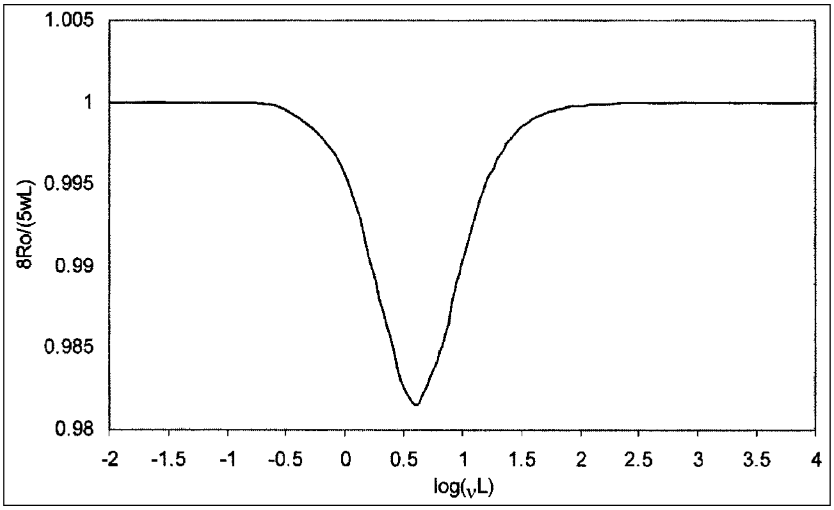


Fig. 7 Relationship between the reaction at the fixed support and the dimensionless stiffness vL for a propped cantilever subject to a uniformly distributed load

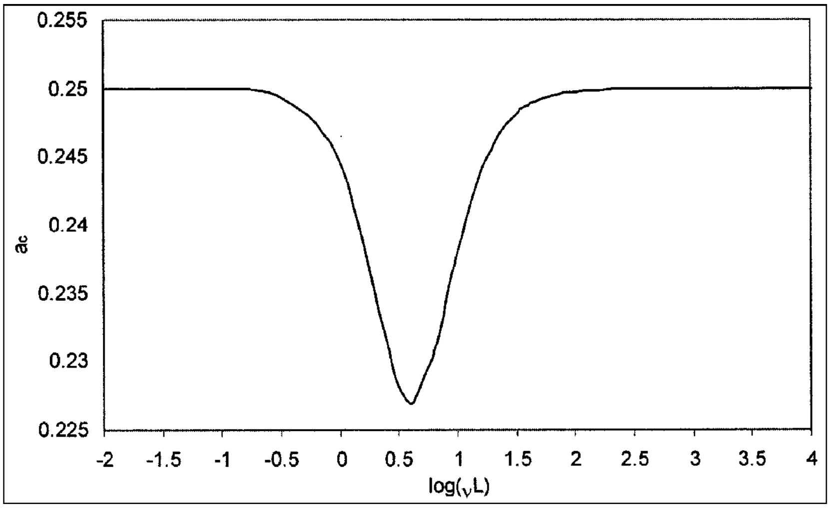


Fig. 8 Relationship between the location of the inflexion point (at $z = a_cL$) and the dimensionless stiffness vL for a propped cantilever subject to a uniformly distributed load

reaction at the support is 5 $wL/8$ (Hall and Kabaila 1986). Similarly, when the beam has a low degree of interaction, its stiffness is again sensibly uniform along its length and acts as a non-composite steel and concrete section with the same curvature at each cross-section, and again the vertical reaction is 5 $wL/8$. However, for partial interaction, the effective flexural rigidity varies along the length of the beam, and within the approximate range $10^{-1} < \nu L < 10^{2.5}$ identified in Fig. 7, the end reaction is less than 5 $wL/8$, with the greatest disparity being only about 2% at $\nu L \approx 10^{0.6}$.

In the present case of the propped cantilever, as well as the previous case of an encastré beam, it has been assumed that the concrete is uncracked in the negative moment region. If this effect is to be included, the technique of this paper may be modified, so as to produce a modelling based on $E_c = 0$ and E_s for a cracked region, and one based on E_c and E_s within the uncracked region. Since the transition from cracked to uncracked takes place at the inflexion point (if it is assumed that the concrete tensile strength is zero), the inflexion point(s) may be located by imposing the kinematic and static boundary conditions at either side of it (them), and solving the constraint equations that result to determine its (their) location. Within the framework of the model herein, this algorithm may be easily formulated and solved numerically, but is outside the subject matter of the present paper.

4. Conclusions

A generic model for partial interaction between two elastic materials, that herein represent the concrete slab and the steel joist in a composite steel-concrete flexural member, has been derived in this paper. The motivation for the form of the derivation is to present a formulation that lends itself to simple application to a number of beam support conditions by invoking the relevant static boundary conditions. The kinematic boundary conditions, known *a priori* for the model, are incorporated within the generic derivation.

The model has been utilised to describe the behaviour of a simply supported beam, whose solution is well documented, and that of statically indeterminate structures, viz. an encastré beam and a propped cantilever subjected to uniformly distributed loading, and closed form solutions of the deformations and reactions for these cases have been given. The influence of the shear connection stiffness on the vertical reactions at the support, as well as on the location of the inflexion point, has been determined for a propped cantilever. Owing to the general non-uniformity of a propped cantilever with partial interaction along its length, there is a range of the dimensionless stiffness parameter νL over which the reactions and inflexion point vary. However, this variation was shown to be only slight.

The generic representation forms the basis for the techniques to investigate such material nonlinear effects as shrinkage, creep, cracking, limit slip capacity of the shear connectors, and combinations of these.

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